Electronic Journal of Qualitative Theory of Differential Equations
2003, No. 18, 1-7; http://www.math.u-szeged.hu/ejqtde/

ON THE UNIFORMLY CONTINUITY OF THE SOLUTION MAP FOR TWO DIMENSIONAL WAVE MAPS

Svetlin Georgiev Georgiev
Penka Vasileva Georgieva

Abstract

University of Sofia, Faculty of Mathematics and Informatics, Department of Differential Equations, 1000 Sofia, Bulgaria
e-mail: *sgg2000bg@yahoo.com*, *p_g@abv.bg*

Abstract

Abstract. The aim of this paper is to analyze the properties of the solution map to the Cauchy problem for the wave map equation with a source term, when the target is the hyperboloid $\mathcal{H}^2$ that is embedded in $\mathcal{R}^3$. The initial data are in $\dot{H}^1 \times L^2$. We prove that the solution map is not uniformly continuous.

Abstract

Subject classification: Primary 35L10, Secondary 35L50.

In this paper we study the properties of the solution map $(u_\circ, u_1, g) \longrightarrow u(t, x)$ to the Cauchy problem

$$u_{tt} - \triangle u - (|u_t|^2 - |\nabla_x u|^2)u = g(t, x), \tag{1}$$

$$u(0, x) = u_\circ(x) \in \dot{H}^1(\mathcal{R}^2), \quad u_t(0, x) = u_1(x) \in L^2(\mathcal{R}^2) \tag{2}$$

in the case when $x \in \mathcal{R}^2$ and the target is the hyperboloid $\mathcal{H}^2 : u_1^2 + u_2^2 - u_3^2 = -1$, $\mathcal{H}^2 \hookrightarrow \mathcal{R}^3$. Here

$$|u_t|^2 = u_{1t}^2 + u_{2t}^2 - u_{3t}^2,$$

$$|\nabla_x u|^2 = |\nabla_{x_1} u|^2 + |\nabla_{x_2} u|^2,$$

$$|\nabla_{x_i} u|^2 = u_{1x_i}^2 + u_{2x_i}^2 - u_{3x_i}^2, \quad i = 1, 2.$$

More precisely, we prove that the solution map $(u_\circ, u_1, g) \longrightarrow u(t, x)$ to the Cauchy problem (1), (2) is not uniformly continuous.

In [1] is proved that the solution map isn't uniformly continuous in the case when $g \equiv 0$.

When we say that the solution map $(u_\circ, u_1, g) \longrightarrow u(t, x)$ is uniformly continuous we understand: *for every positive constant ϵ there exist positive constants δ and R such that for any two solutions $u, v : \mathcal{R} \times \mathcal{R}^2 \longrightarrow \mathcal{H}^2$ of (1), (2), with right hands $g = g_1$, $g = g_2$ of (1), so that*

(3)

$$E(0, u - v) \le \delta, \quad ||g_1||_{L^1([0,1]L^2(\mathcal{R}^2))} \le R, \quad ||g_2||_{L^1([0,1]L^2(\mathcal{R}^2))} \le R, \quad ||g_1 - g_2||_{L^1([0,1]L^2(\mathcal{R}^2))} \le R,$$

the following inequality holds

$$E(t, u - v) \leq \epsilon \quad for \quad \forall t \in [0, 1], \tag{4}$$

where

$$E(t, u) := ||\partial_t u(t, \cdot)||^2_{L^2(\mathcal{R}^2)} + ||\nabla_x u(t, \cdot)||^2_{L^2(\mathcal{R}^2)}.$$

Here we prove

Theorem 1. *There exist constant $\epsilon > 0$ such that for every pair of positive constants δ and R there exists smooth solutions u, v: $\mathcal{R} \times \mathcal{R}^2 \longrightarrow \mathcal{H}^2$ of (1), (2), with right hands $g = g_1$, $g = g_2$ of (1), so that*

$$E(0, u - v) \leq \delta, \quad ||g_1||_{L^1([0,1]L^2(\mathcal{R}^2))} \leq R, \quad ||g_2||_{L^1([0,1]L^2(\mathcal{R}^2))} \leq R, \quad ||g_1 - g_2||_{L^1([0,1]L^2(\mathcal{R}^2))} \leq \delta,$$

and

$$E(1, u - v) \geq \epsilon.$$

Proof. We suppose that the solution map $(u_\circ, u_1, g) \longrightarrow u(t, x)$ to the Cauchy problem (1), (2) is uniformly continuous. Then for every $\epsilon > 0$ there exist positive constants δ and R such that for any solution u of (1), (2) with right hand g of (1) for which

$$E(0, u) \leq \delta, \quad ||g||_{L^1([0,1]L^2(\mathcal{R}^2))} \leq R \tag{5}$$

and the inequality

$$E(t, u) \leq \epsilon \tag{6}$$

holds for every $t \in [0, 1]$ (in this case $v = 0$, which is solution of (1) with right hand $g = 0$). Let

$$\begin{aligned} & u = (u_1, u_2, u_3), \\ & u_1 = \sinh \chi \cos \phi_1, \\ & u_2 = \sinh \chi \sin \phi_1, \\ & u_3 = \cosh \chi, \quad \chi \geq 0, \quad \phi_1 \in [0, 2\pi], \end{aligned} \tag{$\star$}$$

$\chi = Y^2$, where Y is solution to the Cauchy problem

$$Y_{tt} - \triangle Y = 0, \tag{7}$$

$$Y(0, x) = 0, \quad Y_t(0, x) = q(x), \tag{8}$$

$$q(x) = \int_{\mathcal{R}^2} sin(x\xi)\phi(\xi)d\xi,$$

$$\phi(\xi) \equiv \phi_N(\xi) = H(A_N)\frac{1}{\sqrt{|\xi|}},$$

$H(\cdot)$ is the characteristic function of correspond set, $x\xi = x_1\xi_1 + x_2\xi_2$,

$$A_N = \{\xi \in \mathcal{R}^2, \xi_1 = rcos\phi, \xi_2 = rsin\phi, N_\circ \leq |\xi| \leq N, \phi \in \Big(\frac{\pi}{6}, \frac{\pi}{4}\Big)\},$$

$N > N_\circ > 0$ are fixed such that $N_\circ$ is close enough to N, $sin(\xi\eta) \geq a_1$, $cos(|\eta|) \geq a_2$, $sin(|\xi|) \geq a_4$ for $\xi \in A_N$, $\eta \in A_N$, where $0 < a_1 < 1$, $0 < a_2 < 1$, $0 < a_4 < 1$ (for instance $N_\circ = 1 - p$, $N = 1$, p is close enough to zero), $g = (g_1, g_2, g_3)$,

$$g_1 = cosh\chi cos\phi_1(\chi_{tt} - \Delta\chi) + \frac{1}{r^2} sinh\chi cos\phi_1 - \frac{2x_2}{r^2} cosh\chi sin\phi_1 \chi_{x_1} +$$

$$+\frac{2x_1}{r^2} cosh\chi \chi_{x_2} sin\phi_1 + \frac{sinh^3\chi cos\phi_1}{r^2},$$

$$g_2 = cosh\chi sin\phi_1(\chi_{tt} - \Delta\chi) + \frac{1}{r^2} sinh\chi sin\phi_1 + \frac{2x_2}{r^2} cosh\chi cos\phi_1 \chi_{x_1} -$$

$$\frac{2x_1}{r^2} cosh\chi \chi_{x_2} cos\phi_1 + \frac{sinh^3\chi sin\phi_1}{r^2},$$

$$g_3 = sinh\chi f,$$

$$f = 2Y_t^2 - 2Y_r^2 + \frac{sinh(2Y^2)}{2r^2},$$

$x_1 = rcos\phi_1$, $x_2 = rsin\phi_1$, $r > 0$. Then the function u which is defined with $(\star)$ is a solution to (1).

We can to write the solution of the problem (7), (8) in the form

$$Y(t,x) = \int_{\mathcal{R}^2} sin(t|\xi|) sin(x\xi) \frac{\phi_N(\xi)}{|\xi|} d\xi. \tag{9}$$

For the function Y, which is defined with (9), we have the following estimates

$$||Y||_{L^2(\mathcal{R}^2)} \leq \Big|\Big| sin(t|x|) \frac{\phi(x)}{|x|} \Big|\Big|_{L^2(\mathcal{R}^2)} = \tag{10}$$

$$= \Big(\int_{A_N} \Big(sin(t|x|) \frac{\phi_N(x)}{|x|} \Big)^2 dx \Big)^{\frac{1}{2}} \leq \sqrt{\frac{\pi}{12}} \Big(\int_{N_\circ}^{N} \frac{1}{\rho^2} d\rho \Big)^{\frac{1}{2}} = \sqrt{\frac{\pi(N - N_\circ)}{12 N N_\circ}},$$

$$|Y(t,x)| \leq \int_{\mathcal{R}^2} \Big| sin(t|\xi|) sin(x\xi) \frac{\phi_N(\xi)}{|\xi|} \Big| d\xi \leq \int_{A_N} \frac{1}{|\xi|^{\frac{3}{2}}} d\xi = \tag{11}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{N_\circ}^{N} \frac{\rho}{\rho^{\frac{3}{2}}} d\rho d\phi = \frac{\pi}{6}(\sqrt{N} - \sqrt{N_\circ}),$$

$$|Y| \leq |x| \frac{\pi}{18} (N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}}). \tag{12}$$

$$||Y_t||_{L^2(\mathcal{R}^2)} \leq ||cos(t|x|)\phi(x)||_{L^2(\mathcal{R}^2)} \leq \Big(\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{N_\circ}^{N} \frac{1}{\rho} \rho d\rho d\phi \Big)^{\frac{1}{2}} = \sqrt{\frac{\pi}{12}(N - N_\circ)}. \tag{13}$$

$$|Y_t(t,x)| = \Big| \int_{\mathcal{R}^2} cos(t|\xi|) sin(x\xi) \phi_N(\xi) d\xi \Big| \leq \int_{\mathcal{R}^2} \phi_N(\xi) d\xi = \frac{\pi}{18}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}}), \tag{14}$$

Similarly, we have

$$|Y_{x_i}| \leq \frac{\pi}{18}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}}), \tag{15}$$

$$||Y_{x_i}||_{L^2(\mathcal{R}^2)} \leq \sqrt{\frac{\pi}{12}(N - N_\circ)}. \tag{16}$$

On the other hand

$$||f||_{L^2(\mathcal{R}^2)} \leq ||2Y_t^2||_{L^2(\mathcal{R}^2)} + ||2Y_r^2||_{L^2(\mathcal{R}^2)} + \Big|\Big|\frac{sinh(2Y^2)}{2r^2}\Big|\Big|_{L^2(\mathcal{R}^2)}. \tag{17}$$

Now we use (13), (14). Then

$$||2Y_t^2||_{L^2(\mathcal{R}^2)} \leq 2\frac{\pi}{18}\Big(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})||Y_t||_{L^2(\mathcal{R}^2)} \leq \tag{18}$$

$$\frac{\pi}{9}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})\sqrt{\frac{\pi(N - N_\circ)}{12}}.$$

Similarly

$$||2Y_r^2||_{L^2(\mathcal{R}^2)} \leq \frac{\pi}{9}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})\sqrt{\frac{\pi(N - N_\circ)}{12}}. \tag{18'}$$

Let $\Omega = \{x \in \mathcal{R}^2 : |x| \leq 1\}$. Then

$$\Big|\Big|\frac{sinh(2Y^2)}{2r^2}\Big|\Big|_{L^2(\mathcal{R}^2)} \leq \Big|\Big|\frac{sinh(2Y^2)}{2r^2}\Big|\Big|_{L^2(\Omega)} + \Big|\Big|\frac{sinh(2Y^2)}{2r^2}\Big|\Big|_{L^2(\mathcal{R}^2\setminus\Omega)} \tag{19}$$

Since (12) holds, we have that there exists constant c_1 such that $|sinh(2Y^2)| \leq c_1(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})^2|x|^2$ and

$$\Big|\Big|\frac{sinh(2Y^2)}{2r^2}\Big|\Big|_{L^2(\Omega)} = \Big(\int_\Omega\Big(\frac{sinh(2Y^2)}{2r^2}\Big)^2 dx\Big)^{\frac{1}{2}} \leq \tag{20}$$

$$\leq c_1(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})^2\Big(\int_\Omega\Big(\frac{|x|^2}{2|x|^2}\Big)^2 dx\Big)^{\frac{1}{2}} = \sqrt{2\pi}\frac{c_1}{2}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})^2.$$

On the other hand (here we use (11) and the fact that $sinhx$ increases for every x)

$$\Big|\Big|\frac{sinh(2Y^2)}{2r^2}\Big|\Big|_{L^2(\mathcal{R}^2\setminus\Omega)} \leq sinh(\frac{\pi^2}{18}(\sqrt{N} - \sqrt{N_\circ})^2)\frac{\sqrt{\pi}}{2}. \tag{21}$$

From (19), (20), (21) we get

$$\Big|\Big|\frac{sinh(2Y^2)}{2r^2}\Big|\Big|_{L^2(\mathcal{R}^2)} \leq sinh(\frac{\pi^2}{18}(\sqrt{N} - \sqrt{N_\circ})^2)\frac{\sqrt{\pi}}{2} + \sqrt{2\pi}\frac{c_1}{2}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})^2 \tag{22}$$

and from (17), (18), (18'), (22)

$$||f||_{L^2(\mathcal{R}^2)} \leq \tag{22'}$$

$$\le 2\frac{\pi}{9}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})\sqrt{\frac{\pi(N-N_\circ)}{12}} + sinh(\frac{\pi^2}{18}(\sqrt{N}-\sqrt{N_\circ})^2)\frac{\sqrt{\pi}}{2} + \sqrt{2\pi}\frac{c_1}{2}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})^2,$$

(23)
$$||g_3||_{L^2(\mathcal{R}^2)} = ||sinh\chi f||_{L^2(\mathcal{R}^2)} \le sinh(\frac{\pi^2}{36}(\sqrt{N}-\sqrt{N_\circ})^2)||f||_{L^2(\mathcal{R}^2)}.$$

We note that when $N_\circ$ is close enough to N $||g_3||_{L^1([0,1]L^2(\mathcal{R}^2))}$ is close enough to zero .
From third equation of (1) we get that χ is solution to the equation

$$\chi_{tt} - \Delta\chi + \frac{sinh(2\chi)}{2r^2} = f,$$

i.e.

$$\chi_{tt} - \Delta\chi = -\frac{sinh(2\chi)}{2r^2} + f.$$

Then(here we use (11) and the fact that the functions $sinhx$, $coshx$ are increasing for every $x \ge 0$)

(24)
$$||g_1||_{L^2(\mathcal{R}^2)} \le cosh(\frac{\pi^2}{36}(\sqrt{N}-\sqrt{N_\circ})^2)\Big|\Big|f - \frac{sinh2\chi}{2r^2}\Big|\Big|_{L^2(\mathcal{R}^2)} +$$
$$cosh(\frac{\pi^2}{36}(\sqrt{N}-\sqrt{N_\circ})^2)\Big|\Big|\frac{2x_2}{r^2}\chi_{x_1}\Big|\Big|_{L^2(\mathcal{R}^2)} +$$
$$cosh(\frac{\pi^2}{36}(\sqrt{N}-\sqrt{N_\circ})^2)\Big|\Big|\frac{2x_1}{r^2}\chi_{x_2}\Big|\Big|_{L^2(\mathcal{R}^2)} +$$
$$+\Big|\Big|\frac{sinh\chi}{r^2}\Big|\Big|_{L^2(\mathcal{R}^2)} + sinh^2(\frac{\pi^2}{36}(\sqrt{N}-\sqrt{N_\circ})^2)\Big|\Big|\frac{sinh\chi}{r^2}\Big|\Big|_{L^2(\mathcal{R}^2)}.$$

Since $\chi_{x_1} = 2YY_{x_1}$,

$$\Big|\frac{2x_2}{r^2}\chi_{x_1}\Big| \le 2\frac{|x_2|}{r^2}|Y||Y_{x_1}| \le$$

(from (12))

$$\le \frac{\pi}{9}\frac{|x_2||x|}{r^2}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})|Y_{x_i}|,$$

(25)
$$\Big|\Big|\frac{2x_2}{r^2}\chi_{x_1}\Big|\Big|_{L^2(\mathcal{R}^2)} \le \frac{\pi}{9}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})||Y_{x_1}||_{L^2(\mathcal{R}^2)} \le$$

(here we use (16))

$$\le \frac{\pi}{9}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})\sqrt{\frac{\pi(N-N_\circ)}{12}}.$$

Similarly

(26).
$$\Big|\Big|\frac{2x_1}{r^2}\chi_{x_2}\Big|\Big|_{L^2(\mathcal{R}^2)} \le \frac{\pi}{9}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})\sqrt{\frac{\pi(N-N_\circ)}{12}}$$

From (17), (22'), (22), (24), (25), (26) we get

(27)
$$||g_1||_{L^2(\mathcal{R}^2)} \le C_1,$$

where C_1 is close enough to zero when $N_\circ$ is close enough to N. Similarly,

$$||g_2||_{L^2(\mathcal{R}^2)} \leq C_2, \tag{28}$$

where C_2 is close enough to zero when $N_\circ$ is close enough to N. From (23), (27), (28) we have

$$||g||_{L^2(\mathcal{R}^2)} \leq C,$$

where C is close enough to zero when $N_\circ$ is close enough to N. From here the second inequality of (5) is hold for every $R > 0$ when $N_\circ$ is close enough to N.
Sinse

$$Y(0,x) = 0, \quad \chi(0,x) = 0, \quad \chi_t(0,x) = 2Y(0,x)Y_t(0,x) = 0,$$
$$\chi_{x_i}(0,x) = 2Y(0,x)Y_{x_i}(0,x) = 0, \quad i = 1,2,$$
$$u_{1t}(0,x) = cosh\chi(0,x)cos\phi_1\chi_t(0,x) = 0,$$
$$u_{2t}(0,x) = cosh\chi(0,x)sin\phi_1\chi_t(0,x) = 0,$$
$$u_{3t}(0,x) = sinh\chi\chi_t(0,x) = 0,$$
$$u_{1x_1}(0,x) = cosh\chi(0,x)cos\phi_1\chi_{x_1}(0,x) + sinh\chi(0,x)sin\phi_1\frac{x_2}{r^2} = 0,$$
$$u_{2x_1}(0,x) = cosh\chi(0,x)sin\phi_1\chi_{x_1}(0,x) - sinh\chi(0,x)cos\phi_1\frac{x_2}{r^2} = 0,$$
$$u_{3x_1}(0,x) = sinh\chi(0,x)\chi_{x_1}(0,x) = 0,$$
$$u_{1x_2}(0,x) = cosh\chi(0,x)cos\phi_1\chi_{x_2}(0,x) - sinh\chi(0,x)sin\phi_1\frac{x_1}{r^2} = 0,$$
$$u_{2x_2}(0,x) = cosh\chi(0,x)sin\phi_1\chi_{x_2}(0,x) + sinh\chi(0,x)cos\phi_1\frac{x_1}{r^2} = 0,$$
$$u_{3x_2}(0,x) = sinh\chi(0,x)\chi_{x_2}(0,x) = 0,$$

we have

$$E(0,u) = 0,$$

i.e. the first inequality of (5) holds for every $\delta > 0$.
From (6) we get that

$$||\partial_t u||^2_{L^2(\mathcal{R}^2)} \leq \epsilon \quad \forall \quad t \in [0,1]. \tag{29}$$

On the other hand

$$||\partial_t u||^2_{L^2(\mathcal{R}^2)} = ||\partial_t u_1||^2_{L^2(\mathcal{R}^2)} + ||\partial_t u_2||^2_{L^2(\mathcal{R}^2)} - ||\partial_t u_3||^2_{L^2(\mathcal{R}^2)} =$$
$$= ||cosh\chi cos\phi_1\chi_t||^2_{L^2(\mathcal{R}^2)} + ||cosh\chi sin\phi_1\chi_t||^2_{L^2(\mathcal{R}^2)} - ||sinh\chi\chi_t||^2_{L^2(\mathcal{R}^2)} =$$
$$= \int_{\mathcal{R}^2} \chi_t^2(cosh^2\chi - sinh^2\chi)dx = \int_{\mathcal{R}^2} \chi_t^2 dx = ||\chi_t||^2_{L^2(\mathcal{R}^2)}.$$

Therefore, using (29), we get

$$||\chi_t||_{L^2(\mathcal{R}^2)} \leq \epsilon^{\frac{1}{2}}$$

or

$$2||YY_t||_{L^2(\mathcal{R}^2)} \leq \epsilon^{\frac{1}{2}}.$$

From here

$$2\int_{\mathcal{R}^2} \psi Y Y_t dx \le ||\psi||_{L^2(\mathcal{R}^2)} \epsilon^{\frac{1}{2}} \tag{30}$$

for any function $\psi \in L^2(\mathcal{R}^2)$. Let

$$B := a_1^4 a_2^2 a_4^2 \frac{\pi^5}{18.126^2}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})(N^{\frac{7}{4}} - N_\circ^{\frac{7}{4}})^2(\sqrt{N} - \sqrt{N_\circ})^2 \tag{31}$$

and $\epsilon = \frac{B}{2}$,

$$\psi \equiv \psi_N(\xi) = H(A_N)\frac{1}{|\xi|^{\frac{1}{4}}}.$$

For $t = 1$ and $x \in A_N$ we have

$$Y \ge a_1 a_4 \frac{\pi}{6}(\sqrt{N} - \sqrt{N_\circ}) > 0, \tag{32}$$

$$Y_t \ge a_1 a_2 \frac{\pi}{18}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}}) > 0. \tag{33}$$

$$\int_{A_N} \psi dx = \frac{\pi}{21}(N^{\frac{7}{4}} - N_\circ^{\frac{7}{4}}), \tag{34}$$

$$||\psi||_{L^2(A_N)} = \sqrt{\frac{\pi}{18}}(\sqrt{N^{\frac{3}{2}} - N_\circ{}^{\frac{3}{2}}}). \tag{35}$$

From (30), (32), (33), (34), (35) we have for $t = 1$

$$a_1^2 a_2 a_4 \frac{\pi^{\frac{5}{2}}}{126\sqrt{18}}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})^{\frac{1}{2}}(N^{\frac{7}{4}} - N_\circ^{\frac{7}{4}})(\sqrt{N} - \sqrt{N_\circ}) \le \epsilon^{\frac{1}{2}},$$

i. e. $\epsilon \ge B$ which is contradiction with $\epsilon = \frac{B}{2}$. Therefore the solution map is not uniformly continuous.

References

[1] D'Ancona, P., V. Georgiev. On the continuity of the solution operator the wave map system. Preprint

[2] D'Ancona, P., V. Georgiev. On Lipshitz continuity of the solution map for two dimensional wave maps. Preprint.

[3] Shatah, J., M. Struwe. Geometric wave equation. Courant lecture notes in mathematics 2(1998).

[4] Struwe, M. Radial symmetric wave maps from 1+2 - dimensional Minkowski space to the sphere, preprint 2000.

[5] Klainerman, S., S. Selberg. Remarks on the optimal regularity of equations of wave maps type, CPDE, 22(1997), 901-918.

[6] Tatary, D. Local and global results for wave maps I, CPDE, 23(1998), 1781-1793.

Appeared on 2003-10-10.