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Existence and Uniqueness on Periodic Solutions of Fourth-order Nonlinear Differential Equations*

Ani Jiang[1], Jianying Shao[2,†]

[1]College of Mathematics and Computer Science, Hunan University of Arts and Science,

Changde, Hunan 415000, P.R. China

[2]College of Mathematics, Physics and Information Engineering, Jiaxing University,

Jiaxing, Zhejiang 314001, P.R. China

Abstract: In this paper, we study the problem of periodic solutions for fourth-order nonlinear differential equations. Under proper conditions, we employ a novel proof to establish some criteria to ensure the existence and uniqueness of T-periodic solutions. Moreover, we give two examples to illustrate the effectiveness of our main results.

Keywords: Fourth-order nonlinear differential equation; Periodic solution; Mawhin's continuation theorem.

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†Corresponding author. Tel.:+86 057383643075; fax: +86 057383643075; E-mail:shaojianying2008@yahoo.cn

1. Introduction

The existence of periodic solutions for fourth-order nonlinear differential equations have been widely investigated and are still being investigated due to their applications in many fields such as nonlinear oscillations[1-2], fluid mechanical and nonlinear elastic mechanical phenomena [3-8]. Recently, C. Bereanu [9], C. Zhao et al. [10] and Q. Fan et al. [11] discussed the existence of T-periodic solutions of fourth-order nonlinear differential equation of the type

$$u^{''''}(t) + au^{'''}(t) - pu^{''}(t) + qu'(t) - g(t, u(t)) = e(t) \tag{1.1}$$

where $a, p, q \in R$, $g : R^2 \to R$ is continuous, T-periodic in t, $f : R \to R$ is continuous, and $e : R \to R$ is a continuous T-periodic function.

Moreover, the authors in [9−11] also provided some sufficient conditions for the existence of T-periodic solutions of equation (1.1), which generalized and improved the known results in references [1, 4, 6, 7]. However, to the best of our knowledge, most authors of the bibliographies listed above only consider the existence of periodic solutions of equation (1.1), and there exist few results for the existence and uniqueness of periodic solutions of this equation. Thus, in this case, it is worth to study the problem of existence and uniqueness of periodic solutions of fourth-order nonlinear differential equation (1.1).

The purpose of this article is to investigate the existence and uniqueness of T-periodic solutions of (1.1). By using some inequality techniques and Mawhin's continuation theorem, we establish some sufficient conditions for the existence and uniqueness of T-periodic solutions of (1.1). Moreover, two illustrative examples are given in Section 4.

2. Preliminary Results

Let us introduce some notations. For $m \in \mathbf{N}$, we denote by C_T^m the Banach space

$$C_T^m = \{u \in C^m(R, R) : u(t) = u(t + T) \quad \text{for all } t \in R\}$$

endowed with the norm

$$\|u\|_{(m)} = \sum_{k=0}^{m} \|u^{(k)}\|_\infty \qquad (u \in C_T^m),$$

where, for a function $v \in C_T^0$, we have that

$$\|v\|_\infty = \max_{[0,T]} |v|.$$

For $x \in C_T^0$, we will denote

$$|x|_p = (\int_0^T |x(t)|^p dt)^{\frac{1}{p}} \quad (p > 0).$$

Now, let $\widetilde{f} : \mathbf{R^5} \to \mathbf{R}$ be a continuous function, T-periodic with respect to the first variable and consider the fourth-order differential equation

$$u'''' = \widetilde{f}(t, u, u', u'', u'''). \tag{2.1}$$

Lemma 2.1(see [12]) Assume that the following conditions hold.
(i) There exists $\rho > 0$ such that, for each $\lambda \in (0, 1]$, one has that any possible T-periodic solution u of the problem

$$u'''' = \lambda \widetilde{f}(t, u, u', u'', u''')$$

satisfies the a priori estimation $\|u\|_{(3)} < \rho$.
(ii) The continuous function $F : \mathbf{R} \to \mathbf{R}$ defined by

$$F(x) = \int_0^T \widetilde{f}(t, x, 0, 0, 0)dt \qquad (x \in \mathbf{R}),$$

satisfies $F(-\rho)F(\rho) < 0$.
Then, (2.1) has a least one T-periodic solution u such that $\|u\|_{(3)} < \rho$.

From Lemma 2.2 in [13] and the proof of inequality (10) in [11, pp 124], we obtain

Lemma 2.2. Let $x(t) \in C_T^1$. Suppose that there exists a constant $D \geq 0$ such that

$$|x(\tau_0)| \leq D, \ \tau_0 \in [0, T]. \tag{2.2}$$

Then

$$|x|_2 \leq \frac{T}{\pi}|x'|_2 + \sqrt{T}D, \quad |x|_\infty \leq D + \frac{1}{2}\sqrt{T}(\int_0^T |x'(t)|^2 dt)^{1/2}. \tag{2.3}$$

Lemma 2.3. For any $u \in C_T^2$ one has that

$$\int_0^T |u'(t)|^2 dt \leq (\frac{T}{2\pi})^2 \int_0^T |u''(t)|^2 dt. \tag{2.4}$$

Proof. Lemma 2.3 is a direct consequence of the Wirtinger inequality, and see [14,15] for its proof.

Lemma 2.4.(see [11]) For any $u \in C_T^4$ one has that

$$\|u^{(k)}\|_\infty \le T^{3-k}(\frac{1}{2})^{3-(k-1)} \int_0^T |u^{''''}(t)|dt \quad (k = 1, 2, 3). \tag{2.5}$$

Lemma 2.5. Assume that that one of the following conditions is satisfied:

(A_1) for $t, x_1, x_2 \in R, x_1 \ne x_2$,

$$1 + p(\frac{T}{2\pi})^2 > 0, \ (g(t, x_1) - g(t, x_2))(x_1 - x_2) < 0;$$

(A_2) there exists a nonnegative constant B such that $1 > B(\frac{T}{\pi})^2(\frac{T}{2\pi})^2$, and

$$1 + p(\frac{T}{2\pi})^2 > B(\frac{T}{\pi})^2(\frac{T}{2\pi})^2, \ B(x_1 - x_2)^2 \ge (g(t, x_1) - g(t, x_2))(x_1 - x_2) > 0,$$

where $t, x_1, x_2 \in R, x_1 \ne x_2$.

Then (1.1) has at most one T-periodic solution.

Proof. Suppose that $u_1(t)$ and $u_2(t)$ are two T-periodic solutions of (1.1). Set $Z(t) = u_1(t) - u_2(t)$. Then, we obtain

$$(u_1(t) - u_2(t))^{''''} + a(u_1(t) - u_2(t))^{'''} - p(u_1(t) - u_2(t))^{''} + q(u_1(t) - u_2(t))^{'}$$

$$-(g(t, u_1(t)) - g(t, u_2(t))) = 0. \tag{2.6}$$

Integrating (2.6) from 0 to T, it results that

$$\int_0^T (g(t, u_1(t)) - g(t, u_2(t)))dt = 0,$$

Therefore, in view of integral mean value theorem, it follows that there exists a constant $\gamma \in [0, \ T]$ such that

$$g(\gamma, u_1(\gamma))) - g(\gamma, u_2(\gamma)) = 0. \tag{2.7}$$

Since $g(t, x)$ is a strictly monotone function in x, (2.7) implies that

$$Z(\gamma) = u_1(\gamma) - u_2(\gamma) = 0. \tag{2.8}$$

Then, from (2.3), we have

$$|Z|_2 \leq \frac{T}{\pi}|Z'|_2. \tag{2.9}$$

Multiplying $Z(t)$ and (2.6) and then integrating it from 0 to T, it follows that

$$\begin{aligned}
\int_0^T |Z^{''}(t)|^2 dt + p\int_0^T |Z^{'}(t)|^2 dt &= \int_0^T Z(t)(g(t,u_1(t)) - g(t,u_2(t)))dt \\
&= \int_0^T (u_1(t) - u_2(t))(g(t,u_1(t)) - g(t,u_2(t)))dt. \qquad (2.10)
\end{aligned}$$

Now suppose that (A_1)(or (A_2)) holds, we shall consider two cases as follows.

Case (i) If (A_1) holds , (2.4) and (2.10) yield that

$$\begin{aligned}
0 &\leq (1 + p(\frac{T}{2\pi})^2)\int_0^T |Z^{''}(t)|^2 dt \\
&\leq \int_0^T |Z^{''}(t)|^2 dt + p\int_0^T |Z^{'}(t)|^2 dt \\
&= \int_0^T (u_1(t) - u_2(t))(g(t,u_1(t)) - g(t,u_2(t)))dt \\
&\leq 0, \qquad \text{where } p < 0, \qquad (2.11)
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \int_0^T |Z^{''}(t)|^2 dt \\
&\leq \int_0^T |Z^{''}(t)|^2 dt + p\int_0^T |Z^{'}(t)|^2 dt \\
&= \int_0^T (u_1(t) - u_2(t))(g(t,u_1(t)) - g(t,u_2(t)))dt \\
&\leq 0, \qquad \text{where } p \geq 0, \qquad (2.12)
\end{aligned}$$

which, together with (2.8), implies that

$$Z(t) \equiv Z'(t) \equiv Z^{''}(t) \equiv 0, \quad \text{for all } \ t \in R.$$

Hence, equation (1.1) has at most one T-periodic solution.

Case (ii) If (A_2) holds, using (2.3), (2.4) and (2.9), we obtain

$$\begin{aligned}
(1 + p(\frac{T}{2\pi})^2)\int_0^T |Z^{''}(t)|^2 dt &\leq \int_0^T |Z^{''}(t)|^2 dt + p\int_0^T |Z^{'}(t)|^2 dt \\
&= \int_0^T Z(t)(g(t,u_1(t)) - g(t,u_2(t)))dt
\end{aligned}$$

$$
\begin{aligned}
&\le B\int_0^T |Z(t)|^2dt \\
&\le B(\frac{T}{\pi})^2\int_0^T |Z'(t)|^2dt \\
&\le B(\frac{T}{\pi})^2(\frac{T}{2\pi})^2\int_0^T |Z''(t)|^2dt, \quad \text{where } p<0, \qquad (2.13)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^T |Z''(t)|^2dt &\le \int_0^T |Z''(t)|^2dt + p\int_0^T |Z'(t)|^2dt \\
&\le B(\frac{T}{\pi})^2(\frac{T}{2\pi})^2\int_0^T |Z''(t)|^2dt, \qquad \text{where } p\ge 0, \qquad (2.14)
\end{aligned}
$$

From (2.8) and (A_2), (2.13) and (2.14) yield that

$$Z(t)\equiv Z'(t)\equiv Z''(t)\equiv 0, \quad \text{for all} \quad t\in R.$$

Therefore, equation (1.1) has at most one T-periodic solution. The proof of Lemma 2.5 is now complete.

3. Main Results

Theorem 3.1. Assume that that one of the following conditions is satisfied:

$(A_1)^*$ Let (A_1) hold, and there exists a nonnegative constant d_0 such that

$$(g(t,u)+e(t))u<0, \text{ for all } t\in R, |u|\ge d_0;$$

$(A_2)^*$ There exist nonnegative constants d_0 and B such that (A_2) holds, and

$$(g(t,u)+e(t))u>0, \text{ for all } t\in R, |u|\ge d_0.$$

Then equation (1.1) has a unique T-periodic solution.

Proof. From Lemma 2.5, together with $(A_1)^*$(or $(A_2)^*$), it is easy to see that equation (1.1) has at most one T-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that equation (1.1) has at least one T-periodic solution. To do this, we shall use Lemma 2.1 with the nonlinearity $\widetilde{f}: R^5\to R$ given by

$$\widetilde{f}(t,u,v,w,z)=e(t)-az+pw-qv+g(t,u) \quad ((t,u,v,w,z)\in R^5).$$

For $\lambda \in (0, 1]$, we consider the fourth-order differential equation

$$u^{''''}(t) + \lambda a u^{'''}(t) - \lambda p u^{''}(t) + \lambda q u'(t) - \lambda g(t, u(t)) = \lambda e(t). \tag{3.1}$$

Let us show that (i) in Lemma 2.1 is satisfied, that means, there exists $\rho > 0$ such that any possible T-periodic solution u of (3.1) is such that

$$\|u\|_3 < \rho. \tag{3.2}$$

Let $\lambda \in (0, 1]$ and u be a possible T-periodic solution of (3.1). In what follows C_k denotes a fixed constant independent of λ and u. Integrating in (3.1) from 0 to T, it results that

$$\int_0^T [g(t, u(t)) + e(t)]dt = 0,$$

which together with $(A_1)^*$ (or $(A_2)^*$) imply that

$$\exists \xi \in [0, T] : |u(\xi)| < d_0.$$

It follows from (2.3) that

$$\|u\|_\infty \le d_0 + \frac{1}{2}\sqrt{T}(\int_0^T |u^{'}(t)|^2 dt)^{1/2}. \tag{3.3}$$

In view of (2.4), (3.3) implies that

$$\|u\|_\infty \le d_0 + \frac{T\sqrt{T}}{4\pi}(\int_0^T |u^{''}(t)|^2 dt)^{1/2}. \tag{3.4}$$

It follows that

$$|\int_0^T e(t)u(t)dt| \le T d_0 \|e\|_\infty + \frac{T^2\sqrt{T}}{4\pi}\|e\|_\infty(\int_0^T |u^{''}(t)|^2 dt)^{1/2}, \tag{3.5}$$

and

$$|\int_0^T g(t, 0)u(t)dt| \le T d_0 \|g(t, 0)\|_\infty + \frac{T^2\sqrt{T}}{4\pi}\|g(t, 0)\|_\infty(\int_0^T |u^{''}(t)|^2 dt)^{1/2}. \tag{3.6}$$

On the other hand, multiplying equation (3.1) by u and integrating it from 0 to T, it follows that

$$\int_0^T |u^{''}(t)|^2 dt + \lambda p \int_0^T |u^{'}(t)|^2 dt = \lambda \int_0^T u(t)g(t, u(t))dt + \lambda \int_0^T e(t)u(t)dt \tag{3.7}$$

Now suppose that $(A_1)^*$(or $(A_2)^*$) holds, we shall consider two cases as follows.

Case (1) If $(A_1)^*$ holds, using (2.4), (3.5), (3.6) and (3.7), we have

$$\begin{aligned}
&(1+p(\frac{T}{2\pi})^2)\int_0^T |u''(t)|^2dt \\
\leq\ & \int_0^T |u''(t)|^2dt+\lambda p\int_0^T |u'(t)|^2dt \\
\leq\ & \lambda\int_0^T (u(t)-0)(g(t,u(t))-g(t,0))dt+\lambda\int_0^T u(t)g(t,0)dt \\
&+Td_0\|e\|_\infty+\frac{T^2\sqrt{T}}{4\pi}\|e\|_\infty(\int_0^T |u''(t)|^2dt)^{1/2} \\
\leq\ & Td_0(\|g(t,0)\|_\infty+\|e\|_\infty) \\
&+\frac{T^2\sqrt{T}}{4\pi}(\|g(t,0)\|_\infty+\|e\|_\infty)(\int_0^T |u''(t)|^2dt)^{1/2},\ \text{where } p<0,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T |u''(t)|^2dt \ \leq\ & \int_0^T |u''(t)|^2dt+\lambda p\int_0^T |u'(t)|^2dt \\
\leq\ & Td_0(\|g(t,0)\|_\infty+\|e\|_\infty) \\
&+\frac{T^2\sqrt{T}}{4\pi}(\|g(t,0)\|_\infty+\|e\|_\infty)(\int_0^T |u''(t)|^2dt)^{1/2},\ \text{where } p\geq 0,
\end{aligned}$$

which imply that there exists a positive constant C_1 satisfying

$$\int_0^T |u''(t)|^2dt\leq C_1 \quad\text{and}\quad \int_0^T |u''(t)|dt\leq\sqrt{TC_1} \tag{3.8}$$

Case (2) If $(A_2)^*$ holds, using (2.3), (2.4), (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned}
&(1+p(\frac{T}{2\pi})^2)\int_0^T |u''(t)|^2dt \\
\leq\ & \int_0^T |u''(t)|^2dt+\lambda p\int_0^T |u'(t)|^2dt \\
\leq\ & \lambda\int_0^T (u(t)-0)(g(t,u(t))-g(t,0))dt+\lambda\int_0^T u(t)g(t,0)dt \\
&+Td_0\|e\|_\infty+\frac{T^2\sqrt{T}}{4\pi}\|e\|_\infty(\int_0^T |u''(t)|^2dt)^{1/2} \\
\leq\ & B|u|_2^2+Td_0(\|g(t,0)\|_\infty+\|e\|_\infty) \\
&+\frac{T^2\sqrt{T}}{4\pi}(\|g(t,0)\|_\infty+\|e\|_\infty)(\int_0^T |u''(t)|^2dt)^{1/2} \\
\leq\ & B(\frac{T}{\pi}|u'|_2+\sqrt{T}d_0)^2 \\
&+Td_0(\|g(t,0)\|_\infty+\|e\|_\infty)+\frac{T^2\sqrt{T}}{4\pi}(\|g(t,0)\|_\infty+\|e\|_\infty)(\int_0^T |u''(t)|^2dt)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq B(\frac{T}{\pi}\frac{T}{2\pi}|u''|_2+\sqrt{T}d_0)^2+Td_0(\|g(t,0)\|_\infty+\|e\|_\infty)\\
&\quad+\frac{T^2\sqrt{T}}{4\pi}(\|g(t,0)\|_\infty+\|e\|_\infty)(\int_0^T|u^{''}(t)|^2dt)^{1/2},\ \text{where } p<0,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^T|u^{''}(t)|^2dt &\leq \int_0^T|u^{''}(t)|^2dt+\lambda p\int_0^T|u^{'}(t)|^2dt\\
&\leq B(\frac{T}{\pi}\frac{T}{2\pi}|u''|_2+\sqrt{T}d_0)^2+Td_0(\|g(t,0)\|_\infty+\|e\|_\infty)\\
&\quad+\frac{T^2\sqrt{T}}{4\pi}(\|g(t,0)\|_\infty+\|e\|_\infty)(\int_0^T|u^{''}(t)|^2dt)^{1/2},\ \text{where } p\geq 0,
\end{aligned}
$$

which, together with $1+p(\frac{T}{2\pi})^2>B(\frac{T}{\pi})^2(\frac{T}{2\pi})^2$ and $1>B(\frac{T}{\pi})^2(\frac{T}{2\pi})^2$, yield that (3.8) holds.

Using (3.4) and (3.8) it follows that there exists C_2 such that

$$
\|u\|_\infty\leq C_2 \tag{3.9}
$$

If $a=0$, using (3.1), (3.8) and (3.9) it follows that there exists C_3 such that

$$
\int_0^T|u^{''''}(t)|dt\leq C_3, \tag{3.10}
$$

which together with (2.5) and (3.9) implies the existence of a constant $\rho>d_0$ such that (3.2) holds.

If $a\neq 0$, multiplying equation (3.1) by u''' and integrating it from 0 to T, it follows that

$$
\begin{aligned}
|a|\int_0^T|u^{'''}(t)|^2dt &\leq |q|\int_0^T|u^{'}(t)||u^{'''}(t)|dt\\
&\quad+\sup_{|u|\leq C_2,t\in R}|g(t,u)|\int_0^T|u^{'''}(t)|dt+|e|_\infty\int_0^T|u^{'''}(t)|dt\\
&\leq |q|(\int_0^T|u^{'}(t)|^2dt)^{\frac{1}{2}}(\int_0^T|u^{'''}(t)|^2dt)^{\frac{1}{2}}\\
&\quad+(\sup_{|u|\leq C_2,t\in R}|g(t,u)|+|e|_\infty)(\int_0^T|u^{'''}(t)|^2dt)^{\frac{1}{2}}(\int_0^T1dt)^{\frac{1}{2}}\\
&\leq (|q|\frac{T}{2\pi}\sqrt{C_1}+\sqrt{T}\sup_{|u|\leq C_2,t\in R}|g(t,u)|+\sqrt{T}|e|_\infty)(\int_0^T|u^{'''}(t)|^2dt)^{\frac{1}{2}},
\end{aligned}
$$

which yields that there exists a positive constant C_4 satisfying

$$
\int_0^T|u^{'''}(t)|^2dt\leq C_4\ \ \text{and}\ \ \int_0^T|u^{'''}(t)|dt\leq\sqrt{TC_4} \tag{3.11}
$$

Then, we can also show that (3.10) hold. This implies the existence of a constant $\rho > d_0$ such that (3.2) holds.

Now, to show that (ii) in Lemma 2.1 is satisfied, it suffices to remark that

$$F(x) = \int_0^T [g(t,x) + e(t)]dt \quad (x \in R).$$

Hence, from $(A_1)^*$(or $(A_2)^*$) and $\rho > d_0$ it results that $F(-\rho)F(\rho) < 0$. Then, using Lemma 2.1 it follows that equation (3.1) has at least one T-periodic solution u satisfying (3.2). This completes the proof.

Theorem 3.2. Suppose that $\int_0^T e(t)dt = 0$. Assume that that one of the following conditions is satisfied:

$(A_1)^{**}$ Let (A_1) hold. Moreover, for any continuous T- periodic function u we have

$$\int_0^T g(t,u(t))dt < 0 \ \text{ if } \min_{\mathbf{R}} u \geq d_0,$$

and

$$\int_0^T g(t,u(t))dt > 0 \ \text{ if } \max_{\mathbf{R}} u \leq -d_0;$$

$(A_2)^{**}$ Let (A_2) hold. Moreover, for any continuous T- periodic function u we have

$$\int_0^T g(t,u(t))dt > 0 \ \text{ if } \min_{\mathbf{R}} u \geq d_0,$$

and

$$\int_0^T g(t,u(t))dt < 0 \ \text{ if } \max_{\mathbf{R}} u \leq -d_0.$$

Then equation (1.1) has a unique T-periodic solution.

Proof. Integrating in (3.1) from 0 to T, it results that

$$\int_0^T g(t,u(t))dt = 0,$$

which together with $(A_1)^{**}$ (or $(A_2)^{**}$) imply that

$$\exists \xi \in [0,T] : |u(\xi)| < d_0.$$

Now the proof proceeds in the same way as in Theorem 3.1.

4. Examples and Remarks

Example 4.1 Let $a, b, c : R \to R$ be two continuous, strictly positive and T-periodic function and $e : R \to R$ be continuous, T-periodic. Then the fourth-order differential equation

$$u''''(t) + 200u'''(t) + (\frac{1.5\pi}{T})^2 u''(t) + 500u'(t) + a(t)u(t) + b(t)u^3(t) + c(t)u^5(t) = e(t)$$

has a unique T-periodic solution. For the proof, it suffice to remark that the function $g(t, u) \equiv -a(t)u - b(t)u^3 - c(t)u^5$ satisfies

$$(g(t, u) + e(t))u < 0, \text{ for all } t \in R, |u| \geq d_0,$$

where $d_0 \in \mathbf{R}$ is sufficiently large. Hence, $p = -(\frac{1.5\pi}{T})^2$ and g satisfy $(A_1)^*$ and the result follows from Theorem 3.1.

Example 4.2 Let $\beta : \mathbf{R} \to \mathbf{R}$ be continuous, strictly positive and T-periodic, and $e : \mathbf{R} \to \mathbf{R}$ be continuous, T-periodic with $\int_0^T e(t)dt = 0$. If $\|\beta\|_\infty < 1$, then the fourth-order differential equations

$$u''''(t) + 1200u'''(t) + (\frac{1.5\pi}{T})^2 u''(t) + 100u'(t) - \frac{5}{16}\frac{1}{1 + (\frac{T}{\pi})^2(\frac{T}{2\pi})^2}\beta(t)u(t) = e(t)$$

has a unique T-periodic solution. For the proof, it suffice to remark that the function $g(t, u) \equiv (1 - (\frac{1.5\pi}{T})^2)\beta(t)u(t)$ with $B = \frac{5}{16}\frac{1}{1+(\frac{T}{\pi})^2(\frac{T}{2\pi})^2}$ and $p = -(\frac{1.5\pi}{T})^2$ satisfy $1 > B(\frac{T}{\pi})^2(\frac{T}{2\pi})^2$, and

$$1 + p(T/2\pi)^2 > B(\frac{T}{\pi})^2(\frac{T}{2\pi})^2, \ B(x_1 - x_2)^2 \geq (g(t, x_1) - g(t, x_2))(x_1 - x_2) > 0,$$

where $t, x_1, x_2 \in R, x_1 \neq x_2$. Hence, (4.2) satisfies $(A_2)^{**}$ and the result follows from Theorem 3.2.

Remark 4.1. Obviously, the authors in [9−11] only consider the existence of periodic solutions of fourth-order nonlinear differential equation. Hence, the results obtained in [9−11] and the references cited therein are not applicable to Examples 4.1-4.2.

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