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Stationary solutions for a generalized Kadomtsev-Petviashvili equation in bounded domain*

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Abstract

In this work, we are mainly concerned with the existence of stationary solutions for the generalized Kadomtsev-Petviashvili equation in bounded domain in $\mathbb{R}^n$

$$\begin{cases} \dfrac{\partial^3}{\partial x^3}u(x,y) + \dfrac{\partial}{\partial x}f(u(x,y)) = D_x^{-1}\Delta_y u(x,y), \text{ in } \Omega, \\ D_x^{-1}u|_{\partial\Omega} = 0, \ u|_{\partial\Omega} = 0, \end{cases}$$

where $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. We utilize critical point theory to establish our main results.

Keywords: Generalized Kadomtsev-Petviashvili equation; Stationary solution; Critical point theory

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1 Introduction

In this work, we shall investigate the stationary solutions for the generalized Kadomtsev-Petviashvili equation in bounded domain in $\mathbb{R}^n$

$$\begin{cases} \dfrac{\partial^3}{\partial x^3}u(x,y) + \dfrac{\partial}{\partial x}f(u(x,y)) = D_x^{-1}\Delta_y u(x,y), \text{ in } \Omega, \\ D_x^{-1}u|_{\partial\Omega} = 0, \ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $D_x^{-1}h(x,y) := \int_{-\infty}^{x} h(s,y)\mathrm{d}s$ denotes the inverse operator, $(x,y) := (x, y_1, \ldots, y_{n-1}) \in \mathbb{R} \times \mathbb{R}^{n-1}, n \geq 2$, $\Delta_y := \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \cdots + \frac{\partial^2}{\partial y_{n-1}^2}$. In this paper, we utilize variational methods and some critical point theroems to study the stationary solutions for the generalized Kadomtsev-Petviashvili equation (1.1).

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Kadomtsev-Petviashvili equation and its generalization appear in many physical progress, for example, see [1–11] and references therein. Generally, it reads

$$\frac{\partial}{\partial t}w(t,x,y)+\frac{\partial^3}{\partial x^3}w(t,x,y)+\frac{\partial}{\partial x}f(w(t,x,y))=D_x^{-1}\Delta_y w(t,x,y), \tag{1.2}$$

where $(t,x,y):=(t,x,y_1,\ldots,y_{n-1})\in\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^{n-1}$, $n\geq 2$, D_x^{-1} and Δ_y are as in (1.1). A solitary wave is a solution of the form

$$w(t,x,y)=u(x-ct,y),$$

where $c>0$ is fixed. Substituting in (1.2), we have,

$$-cu_x+u_{xxx}+(f(u))_x=D_x^{-1}\Delta_y u,$$

or

$$(-u_{xx}+D_x^{-2}u_{yy}+cu-f(u))_x=0. \tag{1.3}$$

In [1] and [2], by virtue of the constrained minimization method, De Bouard and Saut obtained the existence and nonexistence of solitary waves in the cases where power nonlinearities $f(u)=u^p,\ p=m/n,\ m,n$ are relatively prime, n is odd. In [3, 4], Zou et al. established the existence of nontrivial solitary waves of problem (1.3) by a linking theorem. Wang and Willem [5] obtained multiple solitary waves for the generalized Kadomtsev-Petviashvili equation (1.2) in one-dimensional spaces by the Lyusternik-Schnirelman category theory. In [6], Liang and Su considered that the case that the non-constant weight function for generalized Kadomtsev-Petviashvili equation, see [6, the problem $(\mathcal{P})$ and the assumption (Q)]. In [7–9], Xuan dealt with the case where $N\geq 2$ and $f(u)$ satisfies some superlinear conditions. Their main tool in [6–9] is the famous mountain pass theorem.

We also note Fountain and Dual Fountain theorems were established by Bartsch and Willem [12, 13], and both theorems are effective tools for studying the existence of infinitely many large energy solutions and small energy solutions. For more details of recent development in the direction, we refer the reader to [14–19] and references cited therein. Meanwhile, Zou [20] established some variant fountain theorems and many people utilized these theorems to study nonlinear problems, for instance, see [21–27] and references therein.

It should be remarked that Chen and Tang [14] investigated the fractional boundary value problem of the following form

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}u'(t)+\dfrac{1}{2}\,{}_tD_T^{-\beta}u'(t)\right)+\nabla F(t,u(t))=0,\ \text{a.e. } t\in[0,T],\\ u(0)=u(T)=0.\end{cases}$$

In this paper, they adopted Fountain and Dual Fountain theorems to obtain the existence of infinite solutions under some adequate conditions. It is no doubt that the results in the literature are significantly improved.

In [21], by virtue of the variant fountain theorem established in [20], Sun considered the sublinear Schrödinger-Maxwell equations

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, \ \lim_{|x|\to+\infty} \phi(x) = 0, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.4}$$

In that paper, his aim is to study the existence of infinitely many solutions for (1.4) when $f(x,u)$ satisfies sublinear in u at infinity, see [21, (H_2) of Theorem 1.1]. Motivated by paper [21], we also utilize the variant fountain theorem by [20] to investigate that the existence of infinitely many solutions for (1.1) with the nonlinearity f growing sublinearly in u, see Theorem 3.4 in Section 3.

2 Preliminaries

For $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ on $Y := \{g_x : g \in C_0^\infty(\Omega)\}$, we define the inner product

$$(u,v) := \int_\Omega \left[u_x v_x + D_x^{-1}\nabla_y u \cdot D_x^{-1}\nabla_y v\right] \mathrm{d}V, \tag{2.1}$$

where $\nabla_y := \left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{n-1}}\right)$, $\mathrm{d}V = \mathrm{d}x\mathrm{d}y$, and the corresponding norm

$$\|u\| := \left(\int_\Omega \left[u_x^2 + |D_x^{-1}\nabla_y u|^2\right] \mathrm{d}V\right)^{\frac{1}{2}}. \tag{2.2}$$

A function $u : \Omega \to \mathbb{R}$ belongs to X, if there exists $\{u_m\}_{m=1}^\infty \subset Y$ such that

(1) $u_m \to u$ a.e. on Ω, (2) $\|u_j - u_k\| \to 0$, as $j,k \to \infty$.

Note that the space X with inner product (2.1) and norm (2.2) is a Hilbert space, see [6, **Definition**] and [7, P12 and P13]. We know the exponent $\overline{p} := \frac{2(2n-1)}{2n-3} > 2$ is as critical as the critical Sobolev exponent $p^* := \frac{np}{n-p}$, i.e., there exists a constant $C > 0$ such that the estimate

$$\|u\|_{L^{\overline{p}}(\mathbb{R}^n)} \le C \left(\int_{\mathbb{R}^n} \left[u_x^2 + |D_x^{-1}\nabla_y u|^2\right] \mathrm{d}V\right)^{\frac{1}{2}} \tag{2.3}$$

holds for all $u \in C_0^\infty(\mathbb{R}^n)$.

From the interpolation theorem, the boundedness of Ω and estimate (2.3), there is an embedding theorem about X as follows

Lemma 2.1(see [7, Lemma 1]) The embedding from the space $(X, \|\cdot\|)$ into the space $(L^p(\Omega), \|\cdot\|_p)$ is compact for $1 \le p < \overline{p}$.

By Lemma 2.1, there exists $\tau_p > 0$ such that

$$\|u\|_p \leq \tau_p \|u\|,\ p \in [1, \overline{p}),\ \forall u \in X, \tag{2.4}$$

where $\|u\|_p := \left(\int_\Omega |u|^p \mathrm{d}V\right)^{\frac{1}{p}}$.

In what follows, we shall establish the energy functional for problem (1.1). Note that we can rewrite (1.1) in the following form (see [7, (3) of Page 12]):

$$\begin{cases} -\dfrac{\partial^2}{\partial x^2} u(x,y) + D_x^{-2} \Delta_y u(x,y) = f(u(x,y)),\ \text{in } \Omega, \\ D_x^{-1} u|_{\partial\Omega} = 0,\ u|_{\partial\Omega} = 0. \end{cases} \tag{2.5}$$

For each $v \in X$, multiply the both sides of the above equation in (2.5) by $v(x,y)$ and integrate over Ω to obtain

$$\int_\Omega \left(-\frac{\partial^2}{\partial x^2} u(x,y)\right) v(x,y)\mathrm{d}V + \int_\Omega \left(D_x^{-2}\Delta_y u(x,y)\right) v(x,y)\mathrm{d}V = \int_\Omega f(u(x,y))v(x,y)\mathrm{d}V,$$

and then we obtain by Green formula and integration by parts,

$$\int_\Omega \frac{\partial}{\partial x} u(x,y) \cdot \frac{\partial}{\partial x} v(x,y)\mathrm{d}V + \int_\Omega D_x^{-1}\nabla_y u(x,y) \cdot D_x^{-1}\nabla_y v(x,y)\mathrm{d}V = \int_\Omega f(u(x,y))v(x,y)\mathrm{d}V.$$

Therefore, on X, define a functional φ as

$$\varphi(u) := \frac{1}{2}\int_\Omega \left[u_x^2 + |D_x^{-1}\nabla_y u|^2\right]\mathrm{d}V - \int_\Omega F(u)\mathrm{d}V = \frac{1}{2}\|u\|^2 - \Psi(u), \tag{2.6}$$

where $F(u) := \int_0^u f(s)\mathrm{d}s$, $\Psi(u) := \int_\Omega F(u)\mathrm{d}V$.

For the nonlinearity f, we always assume that it satisfies the following conditions:

(H1) $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, and for some $1 < p < \overline{p} = \frac{2(2n-1)}{2n-3}$, $c_0 > 0$, there holds

$$|f(u)| \leq c_0(1 + |u|^{p-1}). \tag{2.7}$$

Lemma 2.2 Let (H1) holds. Then $\varphi \in C^1(X, \mathbb{R})$. Moreover,

$$(\Psi'(u), v) = \int_\Omega f(u)v\mathrm{d}V, \tag{2.8}$$

$$(\varphi'(u), v) = (u, v) - (\Psi'(u), v) = (u, v) - \int_\Omega f(u)v\mathrm{d}V, \tag{2.9}$$

for all $u, v \in X$, and critical points of φ on X are weak solutions of (1.1).

Proof. We first verify (2.8) by definition. For any given $u \in X$, define an associated linear operator $J(u) : X \to \mathbb{R}$ as follows:

$$(J(u), v) = \int_\Omega f(u)v\mathrm{d}V, \quad \forall v \in X.$$

Note that (H1) leads to $f \in L^q(\Omega)$, where $p^{-1}+q^{-1}=1$. Indeed, for any $u \in X$, Lemma 2.1 enables us to find $u \in L^p(\Omega)$, i.e., $\int_\Omega |u|^p \mathrm{d}V < \infty$. Hence,

$$\int_\Omega |f|^q \mathrm{d}V \le \int_\Omega |c_0(1+|u|^{p-1})|^{\frac{p}{p-1}} \mathrm{d}V \le (2c_0^p)^{\frac{1}{p-1}} \int_\Omega (1+|u|^p)\mathrm{d}V < \infty. \tag{2.10}$$

By (H1), (2.10), Lemma 2.1 and the Hölder inequality, there holds

$$|(J(u),v)| \le \int_\Omega |f(u)v|\mathrm{d}V \le \left(\int_\Omega |f|^q \mathrm{d}V\right)^{\frac{1}{q}} \left(\int_\Omega |v|^p \mathrm{d}V\right)^{\frac{1}{p}} \le \tau_p \left(\int_\Omega |f|^q \mathrm{d}V\right)^{\frac{1}{q}} \|v\| < \infty, \quad \forall v \in X.$$

This shows that $J(u)$ is bounded. Combining (H1), (2.6), (2.8), Lemma 2.1 and the mean value theorem, by Hölder inequality, we have,

$$\begin{aligned} |\Psi(u+v)-\Psi(u)-(J(u),v)| &= \left|\int_\Omega [F(u+v)-F(u)-f(u)v]\mathrm{d}V\right| \\ &= \left|\int_\Omega [f(u+\theta v)-f(u)]v\mathrm{d}V\right| \le \left(\int_\Omega |f(u+\theta v)-f(u)|^q \mathrm{d}V\right)^{\frac{1}{q}} \left(\int_\Omega |v|^p \mathrm{d}V\right)^{\frac{1}{p}} \\ &\le \tau_p \left(\int_\Omega |f(u+\theta v)-f(u)|^q \mathrm{d}V\right)^{\frac{1}{q}} \|v\|, \end{aligned} \tag{2.11}$$

where $\theta = \theta(x,y) \in (0,1)$. Note that

$$\left(\int_\Omega |f(u+\theta v)-f(u)|^q \mathrm{d}V\right)^{\frac{1}{q}} \le \left(\int_\Omega [|f(u+\theta v)|^q + |f(u)|^q]\mathrm{d}V\right)^{\frac{1}{q}} < \infty, \quad \forall u,\ v \in X \tag{2.12}$$

by (H1) and (2.10). Together with (2.11) and (2.12), Lebesgue's dominated convergence theorem implies that

$$\frac{|\Psi(u+v)-\Psi(u)-(J(u),v)|}{\|v\|} \to 0, \quad \text{as } v \to 0.$$

Then by the definition of Fréchet derivatives, (2.8) holds.

Next we prove that Ψ' is weakly continuous. Suppose that $u_n \rightharpoonup u_0$ in X, then $f(u_n) \to f(u_0)$ in $L^q(\Omega)$ by (H1), (2.10) and (2.12). By Hölder inequality and Lemma 2.1, we get,

$$\begin{aligned} \|\Psi'(u_n)-\Psi'(u_0)\|_{X^*} &= \sup_{\|v\|=1} \|(\Psi'(u_n)-\Psi'(u_0),v)\| \\ &\le \sup_{\|v\|=1} \left(\int_\Omega |f(u_n)-f(u_0)|^q \mathrm{d}V\right)^{\frac{1}{q}} \left(\int_\Omega |v|^p \mathrm{d}V\right)^{\frac{1}{p}} \\ &\le \tau_p \left(\int_\Omega |f(u_n)-f(u_0)|^q \mathrm{d}V\right)^{\frac{1}{q}} \to 0, \text{ as } n \to \infty. \end{aligned}$$

This shows that Ψ' is weakly continuous. Consequently, Ψ' is continuous. Therefore $\Psi \in C^1(X,\mathbb{R})$. Due to the form of φ' in (2.9), φ' is also continuous and hence $\varphi \in C^1(X,\mathbb{R})$. Furthermore, Ψ' is compact by the weak continuity of Ψ' since X is a Hilbert space.

As we have mentioned, we will utilize the critical point theory to prove our main results. Let us collect some definitions and lemmas that will be used below. One can refer to [10, 28, 29] for more details.

Definition 2.1 Let X be a real Banach space, D an open subset of X. Suppose that a functional $\varphi : D \to \mathbb{R}$ is Fréchet differentiable on D. If $u_0 \in D$ and the Fréchet derivative satisfies $\varphi'(u_0) = 0$, then we say that u_0 is a critical point of the functional φ and $\varphi(u_0)$ is a critical value of φ.

Let $C^1(X, \mathbb{R})$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on X.

Definition 2.2 Let X be a real Banach space and $\varphi \in C^1(X, \mathbb{R})$. We say that φ satisfies the Palais-Smale condition ((PS) condition for short) if for every sequence $\{u_m\} \subset X$ such that $\varphi(u_m)$ is bounded and $\varphi'(u_m) \to 0$ as $m \to \infty$, there exists a subsequence of $\{u_m\}$ which is convergent in X.

Definition 2.3 Let X be a real Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We say that φ satisfies $(\mathrm{PS})_c$ condition if the existence of a sequence $\{u_m\} \subset X$ such that $\varphi(u_m) \to c$ and $\varphi'(u_m) \to 0$ as $m \to \infty$ lead to c is a critical value of φ.

Remark 2.1 It is clear that the (PS) condition implies the $(\mathrm{PS})_c$ condition for each $c \in \mathbb{R}$.

Lemma 2.3(see [28, Theorem 1.2]) Suppose X is a reflexive Banach space with norm $\|\cdot\|$, and let $M \subset X$ be a weakly closed subset of X. Assume $\varphi : M \to \mathbb{R} \cup \{+\infty\}$ is coercive and weak (sequentially) lower semi-continuous on M with respect to X, i.e., suppose the following conditions are fulfilled:

(1) $\varphi(u) \to \infty$ as $\|u\| \to \infty$, $u \in M$.

(2) For any $u \in M$, any sequence $\{u_m\}$ in M such that $u_m \rightharpoonup u$ weakly in X, there holds:

$$\varphi(u) \leq \liminf_{m \to \infty} \varphi(u_m).$$

Then φ is bounded from below on M and attains its infimum in M.

Lemma 2.4(see [29, Theorem 9.12]) Let X be an infinite dimensional real Banach space. Let $\varphi \in C^1(X, \mathbb{R})$ be an even functional which satisfies the (PS) condition, and $\varphi(0) = 0$. Suppose that $X = Q_1 \bigoplus Q_2$, where Q_1 is infinite dimensional, and φ satisfies that

(i) there exists $\alpha > 0$ and $\rho > 0$ such that $\varphi(u) \geq \alpha$ for all $u \in Q_2$ with $\|u\| = \rho$,

(ii) For any finite dimensional subspace $W \subset X$, there is $R = R(W)$ such that $\varphi(u) \leq 0$ on $W \backslash B_{R(W)}$.

Then φ has an unbounded sequence of critical values.

As X is a separable Hilbert space, there exist (see [30]) $\{e_n\}_{n=1}^{\infty} \subset X$ and $\{f_n\}_{n=1}^{\infty} \subset X^*$ such that $f_n(e_m) = \delta_{n,m}$, $X = \overline{\mathrm{span}}\{e_n : n = 1, 2 \ldots\}$ and $X^* = \overline{\mathrm{span}}^{W^*}\{f_n : n = 1, 2 \ldots\}$. For $j, k \in \mathbb{N}$, denote $X_j := \mathrm{span}\{e_j\}$, $Y_k := \bigoplus_{j=1}^{k} X_j$ and $Z_k := \overline{\bigoplus_{j=k+1}^{\infty} X_j}$. Clearly, $X = \overline{\bigoplus_{j \in \mathbb{N}} X_j}$ with $\dim X_j < \infty$ for all $j \in \mathbb{N}$.

Lemma 2.5(see [12]) Let X be defined above. Suppose that

(A1) $\varphi \in C^1(X, \mathbb{R})$ is an even functional.

If for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that

(A2) $a_k := \max_{u \in Y_k, \|u\|=\rho_k} \varphi(u) \le 0$.

(A3) $b_k := \inf_{u \in Z_k, \|u\|=r_k} \varphi(u) \to \infty$ as $k \to \infty$.

(A4) φ satisfies the $(\mathrm{PS})_c$ condition for all $c > 0$.

Then φ has an unbounded sequence of critical values.

In the following, we shall introduce variant fountain theorems by Zou [20]. Let X and the subspace X_k, Y_k and Z_k are defined above. Consider the following C^1-functional $\varphi_\lambda : X \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) := A(u) - \lambda B(u),\ \ \lambda \in [1,2]. \tag{2.13}$$

The following variant fountain theorem was established in [20].

Lemma 2.6 If the functional φ_λ satisfies

(T1) φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Moreover, $\varphi_\lambda(-u) = \varphi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times X$,

(T2) $B(u) \ge 0$; $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of X,

(T3) There exist $\rho_k > r_k > 0$ such that

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\|=\rho_k} \varphi_\lambda(u) \ge 0 > b_k(\lambda) := \max_{u \in Y_k, \|u\|=r_k} \varphi_\lambda(u),\ \ \forall \lambda \in [1,2],$$

$$d_k(\lambda) := \inf_{u \in Z_k, \|u\| \le \rho_k} \varphi_\lambda(u) \to 0 \text{ as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

Then there exist $\lambda_n \to 1, u_{\lambda_n} \in Y_n$ such that

$$\varphi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0,\ \ \varphi_{\lambda_n}(u_{\lambda_n}) \to c_k \in [d_k(2), b_k(1)] \text{ as } n \to \infty.$$

Particularly, if $\{u_{\lambda_n}\}$ has a convergent subsequence for every k, then φ_1 has infinitely many nontrivial critical points $\{u_k\} \in X \backslash \{0\}$ satisfying $\varphi_1(u_k) \to 0^-$ as $k \to \infty$.

3 Main Results

Theorem 3.1 Let $p \in (1,2)$ and (2.7) hold. Then (1.1) has a weak solution.

We adopt Lemma 2.3 to prove Theorem 3.1. We first offer a lemma involving weak lower semi-continuity.

Lemma 3.1 Let (H1) hold. Then the functional φ determined by (2.6) is weak lower semi-continuous on X.

Proof. We first prove $\|\cdot\|$ defined by (2.2) is weak lower semi-continuous on X. Indeed, if the claim is false, there exists a sequence $\{u_n\}$ such that

$$u_n \rightharpoonup u,\ \text{weakly in } X,\ \ \|u\| > \liminf_{n\to\infty} \|u_n\|. \tag{3.1}$$

Hence, there is a constant c such that $\|u\| > c > \liminf_{n\to\infty} \|u_n\|$. Consequently, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $c > \|u_{n_k}\|,\ k = 1, 2, \ldots$. From Hahn-Banach theorem, we know there exists $f_0 \in X^*$(the dual space of X) such that $\|f_0\| = 1$ and $f_0(u) = \|u\|$. Therefore,

$$f_0(u_{n_k}) \le \|f_0\|\|u_{n_k}\| = \|u_{n_k}\| < c,\ k = 1, 2, \ldots, \tag{3.2}$$

on the other hand, note that $u_{n_k} \rightharpoonup u$, and thus

$$\|u\| = f_0(u) = \lim_{k\to\infty} f_0(u_{n_k}) \le c. \tag{3.3}$$

That is a contradiction. Secondly, we will discuss Ψ. By Lemma 2.1, there exists $u \in X$ such that

$$u_k \rightharpoonup u,\ \text{weakly in } X, \quad u_k \to u,\ \text{strongly in } L^p(\Omega), \quad \text{as } k \to \infty. \tag{3.4}$$

By integral mean value theorem, there is a number $\xi = \xi(u_m, u)$ between u_m and u, we have

$$|F(u_m) - F(u)| = \left| \int_0^{u_m} f(s)\mathrm{d}s - \int_0^{u} f(s)\mathrm{d}s \right| = \left| \int_u^{u_m} f(s)\mathrm{d}s \right| = |f(\xi)(u_m - u)|.$$

Combining this and Hölder inequality, note that (2.10), we arrive at

$$\begin{aligned} |\Psi(u_m) - \Psi(u)| &\le \int_\Omega |F(u_m) - F(u)|\mathrm{d}V = \int_\Omega |f(\xi)(u_m - u)|\mathrm{d}V \\ &\le \left(\int_\Omega |f(\xi)|^q \mathrm{d}V\right)^{\frac{1}{q}} \left(\int_\Omega |u_m - u|^p \mathrm{d}V\right)^{\frac{1}{p}} \to 0. \end{aligned} \tag{3.5}$$

Therefore, $\Psi(u_m) \to \Psi(u)$ strongly in X. Hence,

$$\liminf_{m\to\infty} \varphi(u_m) = \liminf_{m\to\infty} \left(\frac{1}{2}\|u_m\|^2 - \Psi(u_m)\right) \ge \frac{1}{2}\|u\|^2 - \Psi(u) = \varphi(u).$$

This completes the proof.

Proof of Theorem 3.1 The energy space X is a Hilbert space, so is reflexive. We easily verify the assumptions of Lemma 2.3 are true with $M = X$. Lemma 3.1 leads to φ is weak lower semi-continuous on X. Next, we will show φ is coercive on X. Indeed, by (2.7) and Lemma 2.1, there exists a constant c_1 such that

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega F(u)\mathrm{d}V \ge \frac{1}{2}\|u\|^2 - \int_\Omega (c_0|u| + c_1|u|^p)\mathrm{d}V \\ &\ge \frac{1}{2}\|u\|^2 - c_0\tau_p\|u\| - c_1\tau_p^p\|u\|^p \end{aligned}$$

and thus $\varphi(u) \to \infty$ as $\|u\| \to \infty$. Lemma 2.3 implies φ can attain its infimum in X, i.e., (1.1) has at least a weak solution. This completes the proof.

Theorem 3.2 Suppose that (H1) and the following two conditions are satisfied.

(H2) There exists $\alpha > 2$ such that, for $u \in \mathbb{R}\backslash\{0\}$, there holds $0 < \alpha F(u) \le uf(u)$.

(H3) $f(u)$ is odd about u, i.e., $f(u) + f(-u) = 0$.

Then (1.1) has infinitely many weak solutions.

In order to obtain that φ satisfies (PS) condition, we need the following lemma. Note that if φ satisfies (PS) condition, then φ satisfies $(\mathrm{PS})_c$ condition for all $c \in \mathbb{R}$ by Remark 2.1.

Lemma 3.2 Assume that (H1) and (H2) hold. Then $\varphi(u)$ satisfies (PS) condition.

Proof. Let $\{u_k\}$ be a sequence in X such that $\{\varphi(u_k)\}$ is bounded and $\varphi(u_k) \to 0$ as $k \to \infty$. We first prove $\{u_k\}$ is bounded. From the definition of functional φ, there exists $C > 0$ such that

$$C \geq \varphi(u_k) = \frac{1}{2}\int_\Omega \left[u_{kx}^2 + |D_x^{-1}\nabla_y u_k|^2\right] \mathrm{d}V - \int_\Omega F(u_k)\mathrm{d}V$$

and

$$(\varphi'(u_k), u_k) = \int_\Omega \left[u_{kx}^2 + |D_x^{-1}\nabla_y u_k|^2\right] \mathrm{d}V - \int_\Omega f(u_k)u_k \mathrm{d}V = o(1)\|u_k\|.$$

Consequently, by (H2), there holds

$$C - \frac{1}{\alpha}o(1)\|u_k\| \geq \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_k\|^2 + \int_\Omega \left(\frac{1}{\alpha}f(u_k)u_k - F(u_k)\right) \mathrm{d}V \geq \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_k\|^2,$$

which implies $\{u_k\}$ is bounded in X. Going if necessary to a subsequence, we can assume that there exists $u \in X$ such that

$$u_k \rightharpoonup u, \text{ weakly in } X, \quad u_k \to u, \text{ strongly in } L^p(\Omega), \quad \text{as } k \to \infty.$$

Hence, $(\varphi'(u_k) - \varphi'(u))(u_k - u) \to 0$, and note that (H1) leads to $f \in L^q(\Omega)$ (see (2.10)), where $p^{-1} + q^{-1} = 1$, hence, by Hölder inequality, we get

$$\begin{aligned}\int_\Omega (f(u_k) - f(u))(u_k - u)\mathrm{d}V &\leq \left(\int_\Omega |f(u_k) - f(u)|^q \mathrm{d}V\right)^{\frac{1}{q}} \left(\int_\Omega |u_k - u|^p \mathrm{d}V\right)^{\frac{1}{p}} \\ &\leq \left(\int_\Omega [|f(u_k)|^q + |f(u)|^q]\mathrm{d}V\right)^{\frac{1}{q}} \left(\int_\Omega |u_k - u|^p \mathrm{d}V\right)^{\frac{1}{p}} \to 0.\end{aligned}$$

Therefore,

$$(\varphi'(u_k) - \varphi'(u))(u_k - u) = \|u_k - u\|^2 - \int_\Omega (f(u_k) - f(u))(u_k - u)\mathrm{d}V.$$

So $\|u_k - u\| \to 0$ as $k \to \infty$, i.e., $\{u_k\}$ converges strongly to u in X. Therefore, φ satisfies (PS) condition.

Proof of Theorem 3.2 If (H2) is satisfied, then we know the following inequalities holds (see [31, Lemma 3.2] and [32, Lemma 2.2]):

$$F(u) \leq F\left(\frac{u}{|u|}\right)|u|^\alpha, \text{ if } 0 < |u| \leq 1, \qquad F(u) \geq F\left(\frac{u}{|u|}\right)|u|^\alpha, \text{ if } |u| \geq 1.$$

It is easy to see that

$$F(u) \leq M|u|^\alpha, \text{ if } |u| \leq 1, \quad F(u) \geq m|u|^\alpha, \text{ if } |u| \geq 1, \tag{3.6}$$

where $M := \max_{|u|=1} F(u) > 0,\ m := \min_{|u|=1} F(u) > 0$ by (H1). Since $F(u) - m|u|^\alpha$ is continuous on $[-1, 1]$, there exists a constant C_2 such that

$$F(u) \geq m|u|^\alpha - C_2, u \in [-1, 1]. \tag{3.7}$$

Consequently, the second inequality of (3.6) and (3.7), we get

$$F(u) \geq m|u|^\alpha - C_2, u \in \mathbb{R}. \tag{3.8}$$

Choosing $Q_1 = Y_k, Q_2 = Z_k$ in Lemma 2.4, we easily find $X = Q_1 \bigoplus Q_2$ and $\dim Q_1 < \infty$. In view of (H1) and (H3), it is obvious $\varphi(u)$ is even and $\varphi(0) = 0$. By Lemma 3.2, $\varphi(u)$ satisfies (PS) condition. We first prove φ satisfies (i) of Lemma 2.4. For any $u \in X$ and $\|u\| \leq \tau_p^{-1} \sqrt[\alpha]{\text{meas}(\Omega)} \Longrightarrow |u| \leq 1$, we have by the first inequality of (3.6) and Lemma 2.1

$$\int_\Omega F(u)\text{d}V \leq M \int_\Omega |u|^\alpha \text{d}V = M\|u\|_\alpha^\alpha \leq M\tau_p^\alpha \|u\|^\alpha.$$

Therefore,

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(u)\text{d}V \geq \frac{1}{2}\|u\|^2 - M\tau_p^\alpha \|u\|^\alpha, \|u\| \leq \tau_p^{-1} \sqrt[\alpha]{\text{meas}(\Omega)}.$$

Therefore, we can choose $\rho > 0$ small enough such that $\varphi(u) \geq \beta > 0$ with $\|u\| = \rho$.

Finally, we show φ satisfies (ii) in Lemma 2.4. Let $W \subset X$ is a finite dimensional subspace. For every $r \in \mathbb{R} \setminus \{0\}$ and $u \in W \setminus \{0\}$ with $\|u\| = 1$, we obtain by (3.8) and Lemma 2.1

$$\varphi(ru) = \frac{r^2}{2} - \int_\Omega F(ru)\text{d}V \leq \frac{r^2}{2} - \int_\Omega (m|ru|^\alpha - C_2)\text{d}V \leq \frac{r^2}{2} - m\tau_p^\alpha r^\alpha + C_2\text{meas}(\Omega).$$

Note $\alpha > 2$, the above inequality leads to there exists r_0 such that $\|ru\| > \rho$ and $\varphi(ru) < 0$ for each $r \geq r_0 > 0$. Since W is a finite dimensional subspace, there exists $R(W) > 0$ such that $\varphi(u) \leq 0$ on $W \backslash B_{R(W)}$.

Lemma 2.4 yields that $\varphi(u)$ has infinitely many critical points, i.e., (1.1) has infinitely many weak solutions. This completes the proof.

Theorem 3.3 Let $p \in (2, \overline{p})$, (2.7), (H2) and (H3) hold. Moreover, the following two conditions are satisfies:

(H4) there exists positive constants $\mu \in [p, \overline{p})$, $\zeta_1 > 0$ and $c_2 > 0$ such that

$$F(u) \leq \zeta_1 |u|^\mu + c_2,\ \forall u \in \mathbb{R}.$$

(H5) there exists positive constants $\mu' \in (2, p]$, $\zeta_2 > 0$ and $c_3 > 0$ such that

$$F(u) \geq \zeta_2 |u|^{\mu'} - c_3,\ \forall u \in \mathbb{R}.$$

Then (1.1) has infinitely many weak solutions $\{u_n\}$ on X for all positive integer n such that $\|u_n\| \to \infty$ as $n \to \infty$.

Proof. (H3), Lemma 2.2 and Lemma 3.2 enable us to obtain that (A1) and (A4) are satisfied.

For any $u \in Y_k$, let

$$\|u\|_* := \left(\int_\Omega |u|^{\mu'} \mathrm{d}V\right)^{\frac{1}{\mu'}} \tag{3.9}$$

and it is easy to verify that $\|\cdot\|$ defined by (3.9) is a norm of Y_k. Since all the norms of a finite dimensional normed space are equivalent, so there exists positive constant c_4 such that $c_4\|u\| \le \|u\|_*$. In view of (H5),

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega F(u)\mathrm{d}V \le \frac{1}{2}\|u\|^2 - \int_\Omega (\zeta_2|u|^{\mu'} - c_3)\mathrm{d}V \\ &\le \frac{1}{2}\|u\|^2 - \zeta_2 c_4^{\mu'}\|u\|^{\mu'} + c_3 \cdot \mathrm{meas}\Omega.\end{aligned}$$

Since $\mu' > 2$, then there exists positive constants d_k such that

$$\varphi(u) \le 0, \text{ for each } u \in Y_k \text{ and } \|u\| \ge d_k. \tag{3.10}$$

For any $u \in Z_k$, let

$$\|u\|_\mu := \left(\int_\Omega |u|^\mu \mathrm{d}V\right)^{\frac{1}{\mu}} \text{ and } \beta_k := \sup_{u\in Z_k, \|u\|=1} \|u\|_\mu \tag{3.11}$$

Since X is compactly embedded into $L^\mu(\Omega)$, there holds (see [10, Lemma 3.8]), $\beta_k \to 0$, as $k \to \infty$. In view of (H4), we find

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega F(u)\mathrm{d}V \ge \frac{1}{2}\|u\|^2 - \int_\Omega (\zeta_1|u|^{\mu} + c_2)\mathrm{d}V \\ &\ge \frac{1}{2}\|u\|^2 - \zeta_1 \beta_k^{\mu}\|u\|^{\mu} - c_2 \cdot \mathrm{meas}\Omega.\end{aligned}$$

Choosing $r_k := 1/\beta_k$, we easily $r_k \to \infty$ as $k \to \infty$, then

$$\varphi(u) \ge \frac{1}{2}r_k^2 - \zeta_1 - c_2 \cdot \mathrm{meas}\Omega \to \infty, \text{ as } k \to \infty.$$

Hence, $b_k := \inf_{u\in Z_k, \|u\|=r_k} \varphi(u) \to \infty$ as $k \to \infty$. Combining this and (3.10), we can take $\rho_k := \max\{d_k, r_k + 1\}$, and thus $a_k := \max_{u\in Y_k, \|u\|=\rho_k} \varphi(u) \le 0$.

Up until now, we have proved the functional φ satisfies all the conditions of Lemma 2.5, then φ has an unbounded sequence of critical values $c_n = \varphi(u_n)$. We only need to show $\|u_n\| \to \infty$ as $n \to \infty$. Indeed, going to a subsequence if necessary, we may assume that there is a constant $\mathcal{M} > 0$ such that $\|u_n\| \le \mathcal{M}$. By this, there exist ξ_n between u_n and 0, integral mean value theorem and the definition of $\varphi(u_n)$ enable us to obtain

$$F(u_n) = \int_0^{u_n} f(s)\mathrm{d}s = f(\xi_n)u_n, \quad (\varphi'(u_n), u_n) = (u_n, u_n) - \int_\Omega f(u_n)u_n \mathrm{d}V = 0. \tag{3.12}$$

Consequently,

$$c_n = \varphi(u_n) = \frac{1}{2}\|u_n\|^2 - \int_\Omega F(u_n)\mathrm{d}V = \frac{1}{2}\|u_n\|^2 - \int_\Omega f(\xi_n)u_n\mathrm{d}V$$
$$= \frac{1}{2}\int_\Omega f(u_n)u_n\mathrm{d}V - \int_\Omega f(\xi_n)u_n\mathrm{d}V.$$

Note Ω is a bounded domain, we easily know $\int_\Omega f(u_n)u_n\mathrm{d}V$ and $\int_\Omega f(\xi_n)u_n\mathrm{d}V$ are bounded from the continuity of f and the boundedness of u_n and ξ_n. This contradicts the unboundness of c_n. This completes the proof.

In the following, we shall prove (1.1) has infinitely many weak solutions by variant fountain theorems by Zou [20]. To facilitate computations for the following proof, without loss of generality, we only consider a special case of (H1), i.e., f satisfies the following condition:

(H6) $f(u) = \mu|u|^{\mu-1}$, where $1 < \mu < 2$ is a constant.

Theorem 3.4 Let (H6) holds. Then (1.1) possesses infinitely many nontrivial solutions.

In order to apply Lemma 2.6 to prove the result, we first define the functionals A, B and φ_λ on our working space X by

$$A(u) := \frac{1}{2}\|u\|^2,\ B(u) := \int_\Omega F(u)\mathrm{d}V \tag{3.13}$$

$$\varphi_\lambda(u) = A(u) - \lambda B(u) = \frac{1}{2}\|u\|^2 - \lambda\int_\Omega F(u)\mathrm{d}V \tag{3.14}$$

for all $u \in X$ and $\lambda \in [1, 2]$. By Lemma 2.2, we know $\varphi_\lambda \in C^1(X, \mathbb{R}),\ \forall\lambda \in [1, 2]$. Note $\varphi_1 = \varphi$, where φ is determined by (2.6).

The following three lemmas play some important roles in our Theorem 3.4.

Lemma 3.3 Let (H6) holds. Then $B(u) \geq 0$. Furthermore, $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of X.

Proof. By simple computation, we have

$$F(u) = \int_0^u f(s)\mathrm{d}s = |u|^\mu,\ \forall u \in X.$$

It yields $B(u) \geq 0$. We will prove there exists $\varepsilon > 0$ such that

$$\mathrm{meas}(|u|^\mu \geq \varepsilon\|u\|^\mu) \geq \varepsilon,\ \forall u \in \mathscr{X}\backslash\{0\}, \forall\mathscr{X} \subset X \text{ and } \dim\mathscr{X} < \infty. \tag{3.15}$$

There exists otherwise a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{X}\backslash\{0\}$ such that

$$\mathrm{meas}\left(|u_n|^\mu \geq \frac{\|u_n\|^\mu}{n}\right) < \frac{1}{n},\ \forall n \in \mathbb{N}. \tag{3.16}$$

For each $n \in \mathbb{N}$, let $v_n := \frac{u_n}{\|u_n\|} \in \mathscr{X} \Longrightarrow \|v_n\| = 1, \forall n \in \mathbb{N}$ and

$$\mathrm{meas}\left(|v_n|^\mu \geq \frac{1}{n}\right) < \frac{1}{n},\ \forall n \in \mathbb{N}. \tag{3.17}$$

Passing to a subsequence if necessary, we may assume $v_n \to v_0$ in X for some $v_0 \in \mathscr{X}$ since $\mathscr{X}$ is of finite dimension. We easily find $\|v_0\| = 1$. Consequently, there exists a constant $\sigma_0 > 0$ such that

$$\mathrm{meas}(|v_0|^\mu \geq \sigma_0) \geq \sigma_0. \tag{3.18}$$

Indeed, if not, then we have

$$\mathrm{meas}\left(|v_0|^\mu \geq \frac{1}{n}\right) = 0,\ \forall n \in \mathbb{N}, \tag{3.19}$$

which implies

$$0 \leq \int_\Omega |v_0|^{\mu+2}\mathrm{d}V \leq \frac{\|v_0\|_2^2}{n} \leq \frac{\tau_p^2\|v_0\|^2}{n} = \frac{\tau_p^2}{n} \to 0,\ \text{as } n \to \infty.$$

This leads to $v_0 = 0$, contradicting to $\|v_0\| = 1$. In view of Lemma 2.1 and the equivalence of any two norms on $\mathscr{X}$, we have

$$\int_\Omega |v_n - v_0|^2\mathrm{d}V \to 0,\ \text{as } n \to \infty. \tag{3.20}$$

For every $n \in \mathbb{N}$, denote

$$\mathscr{N} := \left\{|v_n|^\mu < \frac{1}{n}\right\} \text{ and } \mathscr{N}^c := \left\{|v_n|^\mu \geq \frac{1}{n}\right\},$$

and $\mathscr{N}_0 := \{|v_0|^\mu \geq \sigma_0\}$, where σ_0 is defined by (3.18). Then for n large enough, by (3.18), we see

$$\mathrm{meas}(\mathscr{N} \cap \mathscr{N}_0) \geq \mathrm{meas}(\mathscr{N}_0) - \mathrm{meas}(\mathscr{N}^c) \geq \sigma_0 - \frac{1}{n} \geq \frac{\sigma_0}{2}.$$

Consequently, for n large enough, we arrive at immediately

$$\begin{aligned}\int_\Omega |v_n - v_0|^\mu\mathrm{d}V &\geq \int_{\mathscr{N}\cap\mathscr{N}_0} |v_n - v_0|^\mu\mathrm{d}V \\ &\geq \frac{1}{2^\mu}\int_{\mathscr{N}\cap\mathscr{N}_0} |v_0|^\mu\mathrm{d}V - \int_{\mathscr{N}\cap\mathscr{N}_0} |v_n|^\mu\mathrm{d}V \\ &\geq \left(\frac{\sigma_0}{2^\mu} - \frac{1}{n}\right)\mathrm{meas}(\mathscr{N} \cap \mathscr{N}_0) \geq \frac{\sigma_0^2}{2^{\mu+2}} > 0.\end{aligned}$$

This contradicts to (3.20). Therefore, (3.15) holds. For the ε given in (3.15), we let

$$\mathscr{N}_u := \{|u|^\mu \geq \varepsilon\|u\|^\mu\},\ \forall u \in \mathscr{X}\backslash\{0\}.$$

Then by (3.15), we find

$$\mathrm{meas}(\mathscr{N}_u) \geq \varepsilon,\ \forall u \in \mathscr{X}\backslash\{0\}. \tag{3.21}$$

Consequently, for any $u \in \mathscr{X}\backslash\{0\}$, we see

$$B(u) = \int_\Omega |u|^\mu\mathrm{d}V \geq \int_{\mathscr{N}_u} \varepsilon\|u\|^\mu\mathrm{d}V \geq \varepsilon^2\|u\|^\mu,$$

which implies $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace $\mathscr{X} \subset X$.

Lemma 3.4 Let (H6) holds. Then there exists a sequence $\rho_k \to 0^+$ as $k \to \infty$ such that

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \varphi_\lambda(u) > 0, \forall k \in \mathbb{N},$$

$$d_k(\lambda) := \inf_{u \in Z_k, \|u\| \le \rho_k} \varphi_\lambda(u) \to 0 \text{ as } k \to \infty \text{ uniformly for } \lambda \in [1,2],$$

where $Z_k := \overline{\bigoplus_{j=k}^{\infty} X_j} = \overline{\text{span}\{e_k, \dots\}}$ for any $k \in \mathbb{N}$.

Proof. By the definition of φ_λ and $\lambda \in [1,2]$, we have

$$\varphi_\lambda(u) \ge \frac{1}{2}\|u\|^2 - 2\int_\Omega |u|^\mu \mathrm{d}V = \frac{1}{2}\|u\|^2 - 2\|u\|_\mu^\mu, \quad \forall (\lambda, u) \in [1,2] \times X. \tag{3.22}$$

Let

$$l_k := \sup_{u \in Z_k, \|u\| = 1} \|u\|_\mu, \forall k \in \mathbb{N}. \tag{3.23}$$

Since X is compactly embedded into $L^\mu(\Omega)$, there holds (see [10, Lemma 3.8]),

$$l_k \to 0, \text{ as } k \to \infty.$$

Combining (3.22) and (3.23), we get

$$\varphi_\lambda(u) \ge \frac{1}{2}\|u\|^2 - 2\|u\|_\mu^\mu \ge \frac{1}{2}\|u\|^2 - 2l_k^\mu \|u\|^\mu, \quad \forall (\lambda, u) \in [1,2] \times Z_k. \tag{3.24}$$

For every $k \in \mathbb{N}$, we can choose

$$\rho_k := 8^{\frac{1}{2-\mu}} l_k^{\frac{\mu}{2-\mu}},$$

then $\rho_k \to 0^+$, as $k \to \infty$. Since $\mu \in (1,2)$, we have by direct computation

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \varphi_\lambda(u) \ge \frac{\rho_k^2}{4} > 0, \forall k \in \mathbb{N}.$$

Besides, for each $k \in \mathbb{N}$, (3.24) enables us to obtain

$$\varphi_\lambda(u) \ge -2l_k^\mu \rho_k^\mu, \ \forall \lambda \in [1,2] \text{ and } u \in Z_k \text{ with } \|u\| \le \rho_k.$$

Therefore,

$$-2l_k^\mu \rho_k^\mu \le \inf_{u \in Z_k, \|u\| \le \rho_k} \varphi_\lambda(u) \le 0, \ \forall \lambda \in [1,2] \text{ and } k \in \mathbb{N}.$$

Since $l_k \to 0, \rho_k \to 0^+$, as $k \to \infty$, we find

$$d_k(\lambda) := \inf_{u \in Z_k, \|u\| \le \rho_k} \varphi_\lambda(u) \to 0 \text{ as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

Lemma 3.5 Let (H6) holds. Then for the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ obtained in Lemma 2.5, there exists $0 < r_k < \rho_k$ for each $k \in \mathbb{N}$ such that

$$b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \varphi_\lambda(u) < 0, \ \forall \lambda \in [1,2], k \in \mathbb{N},$$

where $Y_k := \bigoplus_{j=1}^{k} X_j = \mathrm{span}\{e_1, \ldots, e_k\}$.

Proof. Note Y_k is finite dimensional for all $k \in \mathbb{N}$. Then by (3.15), there exists a constant $\varepsilon_k > 0$ such that

$$\mathrm{meas}(\mathscr{N}_u^k) \geq \varepsilon_k, \ \forall u \in Y_k \backslash \{0\}, \tag{3.25}$$

where $\mathscr{N}_u^k := \{|u|^\mu \geq \varepsilon_k \|u\|^\mu\}$, $\forall k \in \mathbb{N}$ and $u \in Y_k \backslash \{0\}$. Combining this, we have

$$\begin{aligned} \varphi_\lambda(u) &\leq \frac{1}{2}\|u\|^2 - \int_\Omega |u|^\mu \mathrm{d}V \leq \frac{1}{2}\|u\|^2 - \int_{\mathscr{N}_u^k} \varepsilon_k \|u\|^\mu \mathrm{d}V \\ &\leq \frac{1}{2}\|u\|^2 - \varepsilon_k \|u\|^\mu \cdot \mathrm{meas}(\mathscr{N}_u^k) \leq \frac{1}{2}\|u\|^2 - \varepsilon_k^2 \|u\|^\mu \leq -\frac{1}{2}\|u\|^2 \end{aligned} \tag{3.26}$$

for all $u \in Y_k$ with $\|u\| \leq \varepsilon_k^{\frac{2}{2-\mu}}$. If we take

$$0 < r_k < \min\{\rho_k, \varepsilon_k^{\frac{2}{2-\mu}}\}, \forall k \in \mathbb{N},$$

(3.26) leads to

$$b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \varphi_\lambda(u) \leq -\frac{r_k^2}{2} < 0, \ \forall \lambda \in [1, 2], k \in \mathbb{N}.$$

Proof of Theorem 3.4 Clearly, $\varphi_\lambda(u)$ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$ and $\varphi_\lambda(-u) = \varphi_\lambda(u)$. Thus (T1) of Lemma 2.6 holds. Lemma 3.3-Lemma 3.5 imply (T2) and (T3) of Lemma 2.6 are satisfied. Therefore, by Lemma 2.6, for each $k \in \mathbb{N}$, there exists $\lambda_n \to 1, u_{\lambda_n} \in Y_n$ such that

$$\varphi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0, \ \varphi_{\lambda_n}(u_{\lambda_n}) \to c_k \in [d_k(2), b_k(1)] \text{ as } n \to \infty. \tag{3.27}$$

We claim $\{u_{\lambda_n}\}$ obtained in (3.27) has a strong convergent subsequence in X. For the sake of notational simplicity, in what follows, we always set $u_n = u_{\lambda_n}$. Indeed,

$$\|u_n\|^2 = 2\varphi_{\lambda_n}(u_n) + 2\lambda_n \int_\Omega |u_n|^\mu \mathrm{d}V \leq C_0 + 4\|u\|_\mu^\mu \leq C_0 + 4\tau_p^\mu \|u\|^\mu$$

for some $C_0 > 0$. This implies $\{u_n\}$ is bounded in X since $\mu \in (1, 2)$. Next, We show $\{u_n\}$ has a strong convergent subsequence in X. Consequently, without loss of generality, we may assume

$$u_n \rightharpoonup u_0, \text{ as } n \to \infty \tag{3.28}$$

for some $u_0 \in X$. In view of Lemma 2.1, $u_n \to u_0$ in $L^\mu(\Omega)$. By (2.6) and (3.14), we find

$$\|u_n - u_0\|^2 = (\varphi'_{\lambda_n}(u_n) - \varphi'_1(u_0), u_n - u_0) + \int_\Omega (\lambda_n f(u_n) - f(u_0))(u_n - u_0) \mathrm{d}V.$$

It is clear that

$$(\varphi'_{\lambda_n}(u_n) - \varphi'_1(u_0), u_n - u_0) \to 0, \text{ as } n \to \infty.$$

On the other hand, Hölder inequality and $u_n \to u_0$ in $L^\mu(\Omega)$ enable us to find

$$\begin{aligned}\int_\Omega (\lambda_n f(u_n) - f(u_0))(u_n - u_0)\mathrm{d}V &\le \mu \int_\Omega (\lambda_n |u_n|^{\mu-1} + |u_0|^{\mu-1})|u_n - u_0|\mathrm{d}V \\ &\le \mu \left[2\left(\int_\Omega |u_n|^\mu \mathrm{d}V\right)^{\frac{\mu-1}{\mu}} + \left(\int_\Omega |u_0|^\mu \mathrm{d}V\right)^{\frac{\mu-1}{\mu}} \right] \left(\int_\Omega |u_n - u_0|^\mu \mathrm{d}V\right)^{\frac{1}{\mu}} \to 0, \text{ as } n \to \infty.\end{aligned}$$

Hence $u_n \to u_0$ in X. Therefore, the claim above is true.

Nowadays, from the last assertion of Lemma 2.6, we know $\varphi = \varphi_1$ has infinitely many nontrivial critical points. Therefore, (1.1) has infinitely many nontrivial solutions.

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