A MEMORY TYPE BOUNDARY STABILIZATION OF A MIDLY DAMPED WAVE EQUATION

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ABSTRACT: We consider the wave equation with a mild internal dissipation. It is proved that any small dissipation inside the domain is sufficient to uniformly stabilize the solution of this equation by means of a nonlinear feedback of memory type acting on a part of the boundary. This is established without any restriction on the space dimension and without geometrical conditions on the domain or its boundary.

Mathematics Subject Classification: 35L20, 35B40
Key words and Phrases: wave equations, internal dissipation, stabilization

1 INTRODUCTION

In this paper we are concerned with the uniform stability of the solution to the following mixed problem:

$$\begin{cases} u_{tt}(t,x) + \alpha u_t(t,x) = \triangle u(t,x) + g(t,x), \qquad t>0,\ x\in\Omega, \\ \frac{\partial u}{\partial \nu}(t,x) + \int_0^t k(t-s,x)u_s(s,x)ds = h(t,x), \qquad t>0, \qquad x\in\Gamma_0, \\ u(t,x)=0, \qquad t>0,\ x\in\Gamma_1, \\ u(0,x)=u_0(x),\ u_t(x)=u_1(x), \qquad x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n (the n-dimensional Euclidean space, $n \geq 1$) with a boundary $\Gamma = \partial\Omega$ of class C^2; (Γ_0, Γ_1) is a partition of Γ such that $\mathrm{int}(\Gamma_1) \neq \emptyset$; $\nu(x)$ denotes the outward normal vector to Γ at $x \in \Gamma$; $\frac{\partial}{\partial \nu}$ is the normal derivative on Γ; α is a positive number and g, h, u_0, u_1 are given functions; $\triangle$ is the Laplacian with respect to the spatial variable x and the subscript t denotes differentiation with respect to the variable t.

Problem (1.1) models, for instance, the evolution of sound in a compressible fluid with reflection of sound at the surface of the material. The boundary condition in (1.1) is general and covers a fairly large variety of different physical configurations. The physical meaning of this boundary condition as well as the following three particular cases

$$\frac{\partial p}{\partial \nu}(t,x) + \zeta(x) p_t(t,x) = 0, \qquad t > 0, \qquad x \in \Gamma, \tag{1.2}$$

$$\frac{\partial p}{\partial \nu}(t,x) + \beta(x) p_t(t,x) + \alpha(x) p(t,x) = 0, \qquad t > 0, \qquad x \in \Gamma, \tag{1.3}$$

$$\begin{array}{l} m(x)\delta_{tt}(t,x) + d(x)\delta_t(t,x) + K(x)\delta(t,x) = -p(t,x), \\ \frac{\partial p}{\partial \nu}(t,x) + \delta_{tt}(t,x) = 0, \qquad t > 0, \qquad x \in \Gamma, \end{array} \tag{1.4}$$

is discussed in [4]. See also references therein for questions of existence, uniqueness, regularity and asymptotic behavior. In [1] the exponential decay of the energy of problem (1.1) with the boundary condition (1.2) in the case $\zeta(x) \equiv C$ a positive constant, $g \equiv h \equiv 0$ and $\alpha < 0$ on the n-dimensional open unit cube was established. More delicate is the same problem with boundary condition (1.2), $g \equiv h \equiv 0$ without internal damping *i.e* $\alpha = 0$. This is discussed in Komornik and Zuazua [2] and Zuazua [6].

Inspired by the method developed in [2], we shall prove exponential decay for solutions of problem (1.1) ($h \equiv 0$) using an appropriately chosen energy functional. In fact, we shall uniformly stabilize the solution of the wave equation by a nonlinear feedback of memory type acting on a part of the boundary provided the equation contains a mild damping (however small it is) in the interior of the domain.

Let R_+ denote the set of nonnegative real numbers and

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u\,|_{\Gamma_1} = 0\} \tag{1.5}$$

where $H^1(\Omega)$ is the usual Sobolev space.

By a real function $a(t,x) \in L^1_{loc}(R_+; L^\infty(\Gamma_0))$ of positive type we mean a function satisfying

$$\int_0^T \int_{\Gamma_0} v(t) \int_0^t a(t-s) v(s) ds \, d\sigma \, dt \geq 0 \tag{1.6}$$

for all $v \in C(R_+; H^1_{\Gamma_1}(\Omega))$ and for every $T > 0$. See [3] for more information on functions of positive type.

In [4], Propst and Prüss have reformulated problem (1.1) (with $\alpha = 0$) as an integral equation of variational type and then have used results and methods developed in the second author's monograph [5] to derive, among others the following theorem:

Theorem 1.1 *Suppose that Γ_0 and Γ_1 are closed in Γ. Let $u_0 \in H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega)$, $u_1 \in H^1_{\Gamma_1}(\Omega)$, $g \in W^{1,1}_{loc}(R_+; L^2(\Omega))$, $h \in W^{2,1}_{loc}(R_+; L^2(\Gamma_0))$ and $h \in C(R_+; H^{1/2}(\Gamma_0))$, $k \in BV_{loc}(R_+; C^1(\Gamma_0))$ of positive type, either $u_1 = 0$ on Γ_0 or k is locally absolutely continuous in t, uniformly with respect to $x \in \Gamma_0$ and k' (the derivative of k with respect to t) is in $BV_{loc}(R_+; L^\infty(\Gamma_0))$, then there is a unique solution $u \in C(R_+; H^2(\Omega)) \cap C^1(R_+; H^1_{\Gamma_1}(\Omega)) \cap C^2(R_+; L^2(\Omega))$ and $u(t,x)$ satisfies (1.1) for all $t \geq 0$ and almost all x.*

$W^{m,p}$ and C^m are the usual Sobolev space and the space of continuously differentiable functions up to order m respectively. BV is the space of functions of bounded variation.

2 Exponential decay

In this section we assume the existence of a regular strong solution to problem (1.1) in the sense of the preceding theorem with $h \equiv 0$.

Note that the Poincaré inequality holds in $H^1_{\Gamma_1}(\Omega)$ *i.e*

$$\exists \beta > 0, \ \ \|v\|_2^2 \leq \beta \|\nabla v\|_2^2, \quad \text{for all} \quad v \in H^1_{\Gamma_1}(\Omega). \tag{2.7}$$

Combined with the trace inequality the preceding inequality (2.7) yields

$$\exists \gamma > 0, \ \ \int_{\Gamma_0} v^2 d\sigma \leq \gamma \int_\Omega |\nabla v|^2 \, d\sigma, \quad \text{for all} \quad v \in H^1_{\Gamma_1}(\Omega). \tag{2.8}$$

We suppose that our boundary material is characterized by the function

$$k(t,x) = p(x) e^{-t}, \ \ t \geq 0, \qquad x \in \Gamma_0, \tag{2.9}$$

with $0 \leq p(x) \in C^1(\Gamma_0)$ and $\|p(x)\|_\infty = M$.
Let us introduce the energy functional

$$E(u;\, t) = \frac{1}{2} \int_\Omega \left(|u_t|^2 + |\nabla u|^2 \right) dx + \frac{1}{2} \int_{\Gamma_0} p(x) \left(\int_0^t e^{-(t-s)} u_s(s) ds \right)^2 d\sigma. \tag{2.10}$$

Differentiating the energy functional (2.10) using (1.1) we obtain

$$\frac{d}{dt}E(u;\,t) = -\alpha \int_\Omega u_t^2 dx - \int_{\Gamma_0} p(x) \left(\int_0^t e^{-(t-s)} u_s(s) ds \right)^2 d\sigma + \int_\Omega u_t g dx. \tag{2.11}$$

Remark 2.1 *Note that if $g \equiv 0$, then it is readily seen that the energy is decreasing.*

Next, for $\varepsilon > 0$ we will define

$$E_\varepsilon(u;\,t) = E(u;\,t) + \varepsilon\varphi(u;\,t), \qquad t \geq 0, \tag{2.12}$$

where

$$\varphi(u;\,t) = \int_\Omega u_t u dx. \tag{2.13}$$

For the sake of brevity, we will write $E_\varepsilon(t)$ for $E_\varepsilon(u;\,t)$ and $\varphi(t)$ for $\varphi(u;\,t)$. Using the Poincaré inequality (2.7) we have

$$|\varphi(t)| \leq \frac{1}{2}\int_\Omega u_t^2 dx + \frac{1}{2}\beta \int_\Omega |\nabla u|^2 \, dx \leq (1+\beta)E(t). \tag{2.14}$$

It then follows that

$$|E_\varepsilon(t) - E(t)| \leq \varepsilon(1+\beta)E(t), \;\; t \geq 0. \tag{2.15}$$

We are now ready to prove our main theorem.

Theorem 2.2 *Assume that h and k are as above. Let $u_0 \in H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega)$, $u_1 \in H^1_{\Gamma_1}(\Omega)$ and $g \in W^{1,1}_{loc}(R_+; L^2(\Omega))$. If*

$$\int_0^t e^{\varepsilon\omega s} \left(\int_\Omega g^2 dx \right) ds$$

grows no faster than a polynomial as $t \to \infty$ for some ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{2\alpha}{5 + 4\alpha^2\beta}, \frac{2}{1 + 2M\gamma}, \frac{1}{1+\beta} \right\}$$

where β and γ are the constants in (2.7) and (2.8) and $1/2 < \omega < 1$, then there exists a positive constant C such that

$$E(t) \leq Ce^{-\varepsilon\omega t}, \qquad t \geq 0.$$

Proof: Differentiating the functional $E_\varepsilon(t)$ we find

$$\begin{aligned} E_\varepsilon^{'}(t) &= E^{'}(t) + \varepsilon\varphi^{'}(t) \\ &= -\alpha \int_\Omega u_t^2 dx - \int_{\Gamma_0} p(x) \left(\int_0^t e^{-(t-s)} u_s(s) ds \right)^2 d\sigma \\ &\quad + \int_\Omega u_t g dx + \varepsilon \int_\Omega u_{tt} u dx + \varepsilon \int_\Omega u_t^2 dx. \end{aligned} \tag{2.16}$$

Using problem (1.1) we get

$$\begin{aligned} \int_\Omega u_{tt} u dx &= -\alpha \int_\Omega u_t u dx + \int_\Omega (\triangle u) u dx + \int_\Omega u g dx \\ &= -\alpha \int_\Omega u_t u dx - \int_{\Gamma_0} p(x) u \int_0^t e^{-(t-s)} u_s(s) ds d\sigma \\ &\quad - \int_\Omega |\nabla u|^2 dx + \int_\Omega u g dx. \end{aligned} \tag{2.17}$$

Making use of the Hölder inequality and the algebraic inequality

$$ab \le \lambda a^2 + \frac{1}{4\lambda} b^2, \qquad a,\ b \in R,\ \lambda > 0, \tag{2.18}$$

we have the following estimates

$$\int_\Omega u_t u dx \le c_1 \int_\Omega u_t^2 dx + \frac{1}{4c_1} \int_\Omega u^2 dx \tag{2.19}$$

$$\int_\Omega u_t g dx \le c_2 \int_\Omega u_t^2 dx + \frac{1}{4c_2} \int_\Omega g^2 dx \tag{2.20}$$

$$\int_\Omega u g dx \le c_3 \int_\Omega u^2 dx + \frac{1}{4c_3} \int_\Omega g^2 dx \tag{2.21}$$

$$\begin{aligned} &\int_{\Gamma_0} p(x) u \int_0^t e^{-(t-s)} u_s(s) ds d\sigma \le \\ &\quad c_4 M \int_{\Gamma_0} u^2 dx + \tfrac{1}{4c_4} \int_{\Gamma_0} p(x) \left(\int_0^t e^{-(t-s)} u_s(s) ds \right)^2 d\sigma \end{aligned} \tag{2.22}$$

Replacing the expression (2.17) into (2.16) and taking into account the estimates (2.19)-(2.22) we obtain

$$\begin{aligned} E_\varepsilon^{'}(t) &\le -2\varepsilon E(t) - (\alpha - 2\varepsilon) \int_\Omega u_t^2 dx + (\alpha c_1 \varepsilon + c_2) \int_\Omega u_t^2 dx \\ &\quad + \beta \left(\tfrac{\alpha\varepsilon}{4c_1} + c_3 \varepsilon \right) \int_\Omega |\nabla u|^2 dx + 2M c_4 \gamma \varepsilon \int_\Omega |\nabla u|^2 dx \\ &\quad - \left(1 - \varepsilon - \tfrac{\varepsilon}{4c_4} \right) \int_{\Gamma_0} p(x) \left(\int_0^t e^{-(t-s)} u_s(s) ds \right)^2 d\sigma \\ &\quad + \left(\tfrac{\varepsilon}{4c_3} + \tfrac{1}{4c_2} \right) \int_\Omega g^2 dx. \end{aligned} \tag{2.23}$$

Note that we have used the inequalities (2.7) and (2.8) in (2.23). Let us choose $c_1 = 2\alpha\beta$, $c_2 = \varepsilon$, $c_3 = 1/8\beta$ and $c_4 = 1/4M\gamma$, then (2.23) yields

$$\begin{aligned} E_\varepsilon^{'}(t) &\le -\varepsilon E(t) - \left\{ \alpha - (\tfrac{5}{2} + 2\alpha^2\beta)\varepsilon \right\} \int_\Omega u_t^2 dx + \left(2\beta\varepsilon + \tfrac{1}{4\varepsilon} \right) \int_\Omega g^2 dx \\ &\quad - \left\{ 1 - (M\gamma + \tfrac{1}{2})\varepsilon \right\} \int_{\Gamma_0} p(x)^2 \left(\int_0^t e^{-(t-s)} u_t(s) ds \right)^2 d\sigma. \end{aligned} \tag{2.24}$$

Now we choose $\varepsilon > 0$ so that $\alpha - (\frac{5}{2} + 2\alpha^2\beta)\varepsilon \geq 0$ and $1 - (M\gamma + \frac{1}{2})\varepsilon \geq 0$, i.e.,

$$\varepsilon \leq \min\left\{\frac{2\alpha}{5 + 4\alpha^2\beta}, \frac{2}{1 + 2M\gamma}\right\}. \tag{2.25}$$

Hence,

$$E_\varepsilon'(t) \leq -\varepsilon E(t) + K(\varepsilon, \beta)\int_\Omega g^2 dx. \tag{2.26}$$

It follows from (2.15) that

$$(1 - (1+\beta)\varepsilon)E(t) \leq E_\varepsilon(t) \leq (1 + (1+\beta)\varepsilon)E(t),\ \ t \geq 0. \tag{2.27}$$

If moreover $\varepsilon > 0$ satisfies $\varepsilon < 1/(1+\beta)$, let a be any real number such that $0 < a \leq 1 - (1+\beta)\varepsilon$, then

$$aE(t) \leq E_\varepsilon(t) \leq (2-a)E(t),\ \ t \geq 0. \tag{2.28}$$

Using (2.28) in (2.16) we deduce

$$E_\varepsilon'(t) \leq -\frac{\varepsilon}{2-a}E_\varepsilon(t) + K(\varepsilon, \beta)\int_\Omega g^2 dx. \tag{2.29}$$

Consequently,

$$E_\varepsilon(t) \leq \left\{E_\varepsilon(0) + K(\varepsilon, \beta)\int_0^t e^{\varepsilon\omega s}\left(\int_\Omega g^2 dx\right) ds\right\} e^{-\varepsilon\omega t}, \qquad t \geq 0, \tag{2.30}$$

where $\omega = 1/(2-a)$. Once again in view of (2.28) we infer from (2.30) that

$$E(t) \leq \left\{\frac{2-a}{a}E(0) + \frac{K(\varepsilon, \beta)}{a}\int_0^t e^{\varepsilon\omega s}\left(\int_\Omega g^2 dx\right) ds\right\} e^{-\varepsilon\omega t},\ \ t \geq 0. \tag{2.31}$$

The proof is now complete.

Remark 2.2 *It is clear from the proof that α may depend on the spatial variable x.*

Remark 2.3 *If the mild damping is in the boundary instead of the equation, i.e,*

$$\begin{cases} u_{tt}(t,x) = \triangle u(t,x) + g(t,x),\ \ t > 0,\ x \in \Omega, \\ \frac{\partial u}{\partial \nu}(t,x) + \alpha u_t(t,x) + \int_0^t k(t-s,x)u_t(s,x)ds = h(t,x),\ \ t > 0,\ x \in \Gamma_0, \\ u(t,x) = 0,\ \ t > 0,\ x \in \Gamma_1, \\ u(0,x) = u_0(x),\ u_t(x) = u_1(x),\ x \in \Omega, \end{cases}$$

then considering the energy functional (2.10) we proceed as in [2] with $\alpha(x) = m(x).\nu(x)$ *where* $m(x) = x - x^0$, $x^0 \in R^n$, *and*

$$\Gamma_0 = \{x \in \Gamma : m(x).\nu(x) > 0\},$$
$$\Gamma_1 = \{x \in \Gamma : m(x).\nu(x) \leq 0\}.$$

The appropriate perturbed energy functional is

$$E_\varepsilon(u;\, t) = E(u;\, t) + \varepsilon \int_\Omega u_t \left\{(n-1)u + 2(m(x).\nabla u)\right\} dx, \;\; t \geq 0.$$

In this case we do not impose to the function h *(and* g*) to vanish identically, we are restricted however to the space dimension condition* $n \leq 3$ *when* $cl(\Gamma_0) \cap cl(\Gamma_1) \neq \emptyset$ *because of the limited validity of Grisvard's inequality (see [2] and [6]).*

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