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Application of Pettis integration to delay second order differential inclusions

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Abstract

In this paper some fixed point principle is applied to prove, in a separable Banach space, the existence of solutions for delayed second order differential inclusions with three-point boundary conditions of the form

$$\ddot{u}(t) \in F(t, u(t), u(h(t)), \dot{u}(t)) + H(t, u(t), u(h(t)), \dot{u}(t))\, a.e.\, t \in [0, 1],$$

where F is a convex valued multifunction upper semi continuous on $E \times E \times E$, H is a lower semicontinuous multifunction and h is a bounded and continuous mapping on $[0, 1]$.

The existence of solutions is obtained under the assumptions that $\mathrm{F}(t, x, y, z) \subset \Gamma_1(t)$, $\mathrm{H}(t, x, y, z) \subset \Gamma_2(t)$, where the multifunctions $\Gamma_1, \Gamma_2 : [0, 1] \rightrightarrows E$ are uniformly-Pettis integrable .

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1 Introduction

The present paper deals, in a separable Banach space E, with the existence of solutions for the second order differential inclusion with delay of the form

$$(\mathcal{P}_r) \begin{cases} \ddot{u}(t) \in F(t, u(t), u(h(t)), \dot{u}(t)) + H(t, u(t), u(h(t)), \dot{u}(t)), & a.e.\ t \in [0, 1]; \\ u(t) = \varphi(t), \quad \forall t \in [-r, 0]; \\ u(0) = 0; \quad u(\theta) = u(1), \end{cases}$$

where $r > 0$ and θ is a given number in $[0, 1[$, $F : [0, 1] \times E \times E \times E \rightrightarrows E$, $H : [0, 1] \times E \times E \times E \rightrightarrows E$, $h : [0, 1] \to [-r, 1]$, $t - r \leq h(t) \leq t$, and $\varphi : [-r, 0] \to E$. The given mappings h and φ are continuous, F is a convex closed valued multifunction Lebesgue-measurable on $[0, 1]$ and upper semi-continuous on $E \times E \times E$ and H is a closed valued multifunction measurable and lower semi-continuous on $E \times E \times E$. Furthermore, $F(t, x, y, z) \subset \Gamma_1(t)$ and

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$H(t,x,y,z) \subset \Gamma_2(t)$ for all $(t,x,y,z) \in [0,1]\times E\times E\times E$ where, for $i=\overline{1,2}$, $\Gamma_i : [0,1] \rightrightarrows E$ is Pettis uniformly integrable.

A solution u of $(\mathcal{P}_r)$ is a mapping $u : [-r,1] \to E$ satisfying $\ddot{u}(t) \in F(t,u(t),u(h(t)),\dot{u}(t))+ H(t,u(t),u(h(t)),\dot{u}(t))$ for almost every $t \in [0,1]$, $u(t) = \varphi(t)$ for all $t \in [-r,0]$ and $u(0) = 0$; $u(\theta) = u(1)$, with $u \in \mathbf{X} := \mathbf{C}_E([-r,1]) \cap \mathbf{W}^{2,1}_{P,E}([0,1])$ equipped with the norm

$$\|u\|_{\mathbf{X}} = \max\{\sup_{t\in[-r,1]} \|u(t)\|, \sup_{t\in[0,1]} \|\dot{u}(t)\|\}.$$

Second order differential inclusions with three-point boundary conditions have been studied by serval authors (see [1] [3], [5] and [14]). For example, the authors in [3] studied the existence of solutions for a second order differential inclusion with three-point boundary conditions of the form

$$\ddot{u}(t) \in F(t,u(t),\dot{u}(t)) + H(t,u(t),\dot{u}(t)),$$

where $F : [0,1] \times E \times E \rightrightarrows E$ is a convex compact valued multifunction, Lebesgue-measurable on $[0,1]$ and upper semicontinuous on $E\times E$ and H a nonempty closed valued multifunction, such that H is $\mathcal{L}([0,1])\otimes\mathcal{B}(E)\otimes\mathcal{B}(E)$-measurable and lower semicontinuous on $E \times E$, under the assumptions that $F(t,x,y) \subset \Gamma_1(t), H(t,x,y) \subset \Gamma_2(t)$ in the case where Γ_1 and Γ_2 are integrably bounded. The same differential inclusion has been studied in [1] with the same conditions on F and H where Γ_1, Γ_2 are uniformly Pettis integrable.

The existence of solutions for second order delayed problems has also been discussed in the literature, we cite for example the results given in [4], [7], [8], [9], [12], [14] and [16].

The paper is organized as follows. After we recall some basic notations and preliminary theorems in section 3 we present our main result.

2 Notation and Preliminaries

Let $(E,\|\cdot\|)$ be a separable Banach space and E' is its topological dual, $\overline{\mathbf{B}}(0,\rho)$ is the closed ball of E of center 0 and radius $\rho > 0$ and $\overline{\mathbf{B}}_E$ is the closed unit ball of E; $\mathcal{L}([0,1])$ is the σ-algebra of Lebesgue-measurable sets on $[0,1]$; $\lambda = dt$ is the Lebesgue measure on [0,1]; $\mathcal{B}(E)$ is the σ-algebra of Borel subsets of E. By $\mathbf{L}^1_E([0,1])$ we denote the space of all Lebesgue-Bochner integrable E-valued mappings defined on $[0,1]$. We denote the topology of uniform convergence on weakly compact convex sets by $\mathcal{T}^w_{co}$. Restricted to E', this is the Mackey topology, which is the strongest locally convex topology on E' and we denote it by $\mathcal{T}(E',E)$.

Let $\mathbf{C}_E([0,1])$ be the Banach space of all continuous mappings $u : [0,1] \to E$, endowed with the sup-norm, and $\mathbf{C}^1_E([0,1])$ be the Banach space of all continuous mappings $u : [0,1] \to E$ with continuous derivative, equipped with the norm

$$\|u\|_{\mathbf{C}^1} = \max\{\max_{t\in[0,1]} \|u(t)\|, \max_{t\in[0,1]} \|\dot{u}(t)\|\}.$$

Now, let $f : [0,1] \to E$ be a scalarly integrable mapping, that is, for every $x' \in E'$, the scalar function $t \mapsto \langle x', f(t)\rangle$ is Lebesgue-integrable on $[0,1]$, f is said to be

Pettis integrable if, for every set $A \in \mathcal{L}([0,1])$, the weak integral $\int_A f(t)dt$ defined by $\langle x', \int_A f(t)dt\rangle = \int_A \langle x', f(t)\rangle dt$ for all $x' \in E'$, belongs to E.
We denote by $\mathbf{P}^1_E([0,1])$ the space of all Pettis-integrable $E-$valued mappings defined on $[0,1]$. The Pettis norm of any element $f \in \mathbf{P}^1_E([0,1])$ is defined by

$$\|f\|_{Pe} = \sup_{x' \in \overline{B}_{E'}} \int_{[0,1]} |\langle x', f(t)\rangle| dt.$$

The space $\mathbf{P}^1_E([0,1])$ endowed with $\|.\|_{Pe}$ is a normed space. A subset $\mathcal{K} \subset \mathbf{P}^1_E([0,1])$ is Pettis uniformly integrable ((PUI) for short) if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each measurable subset A of $[0,1]$ we have

$$\lambda(A) \leqslant \delta \Rightarrow \sup_{f \in \mathcal{K}} \|1_A f\|_{Pe} \leqslant \varepsilon.$$

If $f \in \mathbf{P}^1_E([0,1])$, the singleton $\{f\}$ is PUI since the set $\{\langle x', f\rangle : \|x'\| \leqslant 1\}$ is uniformly integrable.
For more details on the theory of the Pettis integration we can refer the reader to [6], [10], [11] and [15].

A mapping $v : [0,1] \to E$ is said to be scalarly derivable when there exists some mapping $\dot{v} : [0,1] \to E$ (called the weak derivative of v) such that, for every $x' \in E'$, the scalar function $\langle x', v(\cdot)\rangle$ is derivable and its derivative is equal to $\langle x', \dot{v}(\cdot)\rangle$. The weak derivative $\ddot{v}$ of $\dot{v}$ when it exists is the weak second derivative.

By $\mathbf{W}^{2,1}_{P,E}([0,1])$ we denote the space of all continuous mappings $u \in \mathbf{C}_E([0,1])$ such that their first usual derivatives $\dot{u}$ are continuous and their second weak derivatives belong to $\mathbf{P}^1_E([0,1])$.

For closed subsets A and B of E, the excess of A over B is defined by

$$e(A,\, B) = \sup_{a \in B} d(a, B) = \sup_{a \in A}(\inf_{b \in B} \|a - b\|),$$

and the support function $\delta^*(\cdot, A)$ associated with A is defined on E' by

$$\delta^*(x', A) = \sup_{a \in A} \langle x', a\rangle.$$

Recall also that a set $K \subset \mathbf{P}^1_E([0,1])$ is said to be decomposable if and only if for every $u,\, v \in K$ and any $A \in \mathcal{L}([0,1])$ we have $u.\mathbf{1}_A + v.(1 - \mathbf{1}_A) \in K$.

In the sequel, we need the following lemma that summarizes some properties of some Green type function, see [1] and [3].

Lemma 2.1 *Let E be a separable Banach space and let $G : [0,1] \times [0,1] \to \mathbb{R}$ be the function defined by*

$$G(t,s) = \begin{cases} -s & if \quad 0 \leq s < t \\ -t & if \quad t < s \leq \theta \\ t\frac{(s-1)}{(1-\theta)} & if \quad \theta < s \leq 1, \end{cases} \tag{2.1}$$

if $0 \le t < \theta$, *and*

$$G(t,s) = \begin{cases} -s & if \quad 0 \le s < \theta \\ \frac{\theta(s-t)+s(t-1)}{(1-\theta)} & if \quad \theta \le s < t \\ t\frac{(s-1)}{(1-\theta)} & if \quad t < s \le 1, \end{cases} \tag{2.2}$$

if $\theta \le t \le 1$.
Then the following assumptions hold.
(i) $G(.,s)$ *is derivable on* $[0,1]$, *for every* $s \in [0,1]$ *except on the diagonal, and its derivative is given by*

$$\frac{\partial G}{\partial t}(t,s) = \begin{cases} 0 & if \quad 0 \le s < t \\ -1 & if \quad t < s \le \theta \\ \frac{(s-1)}{(1-\theta)} & if \quad \theta < s \le 1, \end{cases} \tag{2.3}$$

if $0 \le t < \theta$, *and*

$$\frac{\partial G}{\partial t}(t,s) = \begin{cases} 0 & if \quad 0 \le s < \theta \\ \frac{(s-\theta)}{(1-\theta)} & if \quad \theta \le s < t \\ \frac{(s-1)}{(1-\theta)} & if \quad t < s \le 1, \end{cases} \tag{2.4}$$

if $\theta \le t \le 1$.
(ii) $G(.,.)$ *and* $\frac{\partial G}{\partial t}(.,.)$ *satisfies*

$$\sup_{t,s\in[0,1]} |G(t,s)| \le 1, \quad \sup_{\substack{t,s\in[0,1] \\ t\neq s}} |\frac{\partial G}{\partial t}(t,s)| \le 1. \tag{2.5}$$

(iii) Let $f \in \mathbf{P}^1_E([0,1])$ *and let* $u_f : [0,1] \to E$ *be the mapping defined by*

$$u_f(t) = \int_0^1 G(t,s)f(s)ds \quad ,\forall t \in [0,1], \tag{2.6}$$

then one has
(1) $u_f(0) = 0$ *and* $u_f(\theta) = u_f(1)$.
(2) The mapping $t \longmapsto u_f(t)$ *is continuous, i.e* $u_f \in \mathbf{C}_E([0,1])$.
(3) The mapping u_f *is scalarly derivable, that is, for every* $x' \in E'$, *the scalar function* $\langle x', u_f(.)\rangle$ *is derivable, and its weak derivative* $\dot{u}_f$ *satisfies*

$$\begin{aligned} \lim_{h\to 0}\langle x', \frac{u_f(t+h)-u_f(t)}{h}\rangle &= \langle x', \dot{u_f}(t)\rangle \\ &= \int_0^1 \frac{\partial G}{\partial t}(t,s)\langle x', f(s)\rangle ds \\ &= \langle x', \int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)ds\rangle \end{aligned}$$

for all $t \in [0,1]$ *and for all* $x' \in E'$. *Consequently*

$$\dot{u}_f(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s) ds \quad , \forall\, t \in [0,1], \tag{2.7}$$

and $\dot{u}_f$ *is a continuous mapping from* $[0,1]$ *into* E.
(4) The mapping $\dot{u}_f$ *is scalarly derivable, that is, there exists a mapping* $\ddot{u}_f : [0,1] \to E$ *such tha, for every* $x' \in E'$, *the scalar function* $\langle x', \dot{u}_f(.)\rangle$ *is derivable with* $\frac{d}{dt}\langle x', \dot{u}_f(t)\rangle = \langle x', \ddot{u}_f(t)\rangle$*; furthermore*

$$\ddot{u}_f = f \quad a.e. \, on \; [0,1]. \tag{2.8}$$

Let us mention a useful consequence of Lemma 2.1.

Proposition 2.1 *Let* E *be a separable Banach space and let* $f : [0,1] \to E$ *be a continuous mapping (respectively a mapping in* $\mathbf{P}^1_E([0,1])$*). Then the mapping*

$$u_f(t) = \int_0^1 G(t,s) f(s) ds, \forall t \in [0,1]$$

is the unique $\mathbf{C}^2_E([0,1])$*-solution (respectively* $\mathbf{W}^{2,1}_{P,E}([0,1])$*-solution) to the differential equation*

$$\begin{cases} \ddot{u}(t) = f(t) \;\; \forall t \in [0,1]; \\ u(0) = 0; \; u(\theta) = u(1). \end{cases}$$

Proposition 2.2 *(See [2]) Let* X *be a compact space and* $M : X \rightrightarrows \mathbf{P}^1_E([0,1])$ *be a lower semicontinuous multifunction with closed and decomposable values. Then* M *has a continuous selection.*

For the proof of our main result, we also need the following fixed point theorem which is the multivalued analogue of the Shaefer continuation principle. For more details for the fixed point theory we refer the reader to [13].

Theorem 2.1 *Let* Y *be a normed linear space and* $A : Y \rightrightarrows Y$ *be an upper semicontinuous compact multivalued operator with compact convex values. Suppose that there exists an* $R > 0$ *such that the a priori estimate*

$$x \in \lambda A x \;\; (0 < \lambda \leq 1) \Rightarrow \|x\| \leq R \tag{2.9}$$

holds. Then A *has a fixed point in the ball* $\overline{\mathbf{B}}(0,R)$.

3 Main result

Now, we are able to prove our main existence theorem.

Theorem 3.1 *Let E be a separable Banach space, $F : [0,1] \times E \times E \times E \rightrightarrows E$ be a convex closed valued multifunction, Lebesgue-measurable on $[0,1]$ and upper semicontinuous on $E \times E \times E$. Let $H : [0,1] \times E \times E \times E \rightrightarrows E$ be another multifunction with nonempty closed values such that H is $\mathcal{L}([0,1]) \otimes \mathcal{B}(E) \otimes \mathcal{B}(E) \otimes \mathcal{B}(E)$-measurable and lower semicontinuous on $E \times E \times E$. Assume that, for $i = 1, 2$, there is some convex $\| \cdot \|$-compact valued, and measurable multifunction $\Gamma_i : [0,1] \rightrightarrows E$ which is Pettis uniformly integrable, such that $F(t, x, y, z) \subset \Gamma_1(t)$ and $H(t, x, y, z) \subset \Gamma_2(t)$ for all $(t, x, y, z) \in [0,1] \times E \times E \times E$. Let $h : [0,1] \to [-r, 1]$, with $t - r < h(t) < t$, be a continuous mapping and $\varphi \in \mathbf{C}_E([-r, 0])$ with $\varphi(0) = 0$. Then the boundary value problem $(\mathcal{P}_r)$ has at least one solution in $\mathbf{X} := \mathbf{C}_E([-r, 1]) \cap \mathbf{W}^{2,1}_{P,E}([0,1])$.*

Proof. **Step 1**. Taking $\overline{co}(\{0\} \cup \Gamma_i(t))$ if necessary, we may suppose that $0 \in \Gamma_i(t)$ for all $t \in [0,1]$ and $i = 1, 2$.

For $t \in [0,1]$, let $\Gamma(t) = \Gamma_1(t) + \Gamma_2(t)$, and observe that the multifunction Γ inherits all the properties of Γ_1 and Γ_2, that is, Γ is convex $\| \cdot \|$-compact valued, and measurable multifunction, further, it is Pettis uniformly integrable.

Let us consider the differential inclusion

$$\begin{cases} \ddot{u}(t) \in \Gamma(t), & \text{a.e. } t \in [0,1]; \\ u(t) = \varphi(t), & \forall t \in [-r, 0]; \\ u(0) = 0; \quad u(\theta) = u(1). \end{cases} \tag{3.1}$$

We wish to show that the $\mathbf{X}$-solutions set $\mathbf{X}_\Gamma$ of (3.1) is nonempty and convex compact in the Banach space $\mathbf{X}$ endowed with the norm $\|.\|_{\mathbf{X}}$.

Let us recall (see [10]) that the set $\mathbf{S}^{Pe}_\Gamma$ of all Pettis integrable selections of Γ is nonempty, convex and sequentially compact for the topology of pointwise convergence on $\mathbf{L}^\infty_\mathbb{R} \otimes E'$ and that the multivalued integral

$$\int_0^1 \Gamma(t)dt = \{ \int_0^1 f(t)dt; f \in \mathbf{S}^{Pe}_\Gamma \}$$

is convex and norm compact in E.

In view of Lemma 2.1 and Proposition 2.2, the solutions set $\mathbf{X}_\Gamma$ of (3.1) is characterized by

$$\mathbf{X}_\Gamma = \{ u \in \mathbf{X} : \ u = \varphi \text{ on } [-r, 0] \text{ and } u(t) = \int_0^1 G(t,s) f(s) ds, \forall t \in [0,1]; f \in \mathbf{S}^{Pe}_\Gamma \}.$$

Clearly $\mathbf{X}_\Gamma$ is convex. Furthermore, for all $u \in \mathbf{X}_\Gamma$ there is $f \in \mathbf{S}_\Gamma^{Pe}$ such that for $t, t' \in [0,1]$

$$\begin{aligned}
\|u(t) - u(t')\| &= \sup_{x' \in \overline{\mathbf{B}}_{E'}} |\langle x', u(t) - u(t')\rangle| \\
&= \sup_{x' \in \overline{\mathbf{B}}_{E'}} |\langle x', \int_0^1 G(t,s)f(s)ds - \int_0^1 G(t',s)f(s)ds\rangle| \\
&\leq \sup_{x' \in \overline{\mathbf{B}}_{E'}} \int_0^1 |G(t,s) - G(t',s)||\langle x', f(s)\rangle| ds \\
&\leq \sup_{x' \in \overline{\mathbf{B}}_{E'}} \int_0^1 |G(t,s) - G(t',s)||\delta^*(x', \Gamma(s))| ds
\end{aligned}$$

and by Lemma 2.1,

$$\|\dot{u}(t) - \dot{u}(t')\| \leq \sup_{x' \in \overline{\mathbf{B}}_{E'}} \int_0^1 |\frac{\partial G}{\partial t}(t,s) - \frac{\partial G}{\partial t}(t',s)||\delta^*(x', \Gamma(s))| ds.$$

The function G is continuous on the compact set $[0,1] \times [0,1]$, so it is uniformly continuous there. In addition, the set $\{|\delta^*(x', \Gamma(.))| : x' \in \overline{\mathbf{B}}_{E'}\}$ is uniformly integrable in $\mathbf{L}^1_{\mathbb{R}}([0,1])$. Then, the right-hand side of the above inequalities tends to 0 as $t \to t'$. We conclude that the sets $\mathbf{X}_\Gamma$ and $\{\dot{u} : u \in \mathbf{X}_\Gamma\}$ are equicontinuous in $\mathbf{C}_E([0,1])$. Since $\varphi \in \mathbf{C}_E([-r,0])$ we get the equicontinuity of $\mathbf{X}_\Gamma$ in $\mathbf{X}$. On the other hand, for each $t \in [-r,1]$ and each $\tau \in [0,1]$, the sets $\mathbf{X}_\Gamma(t) = \{u(t) : u \in \mathbf{X}_\Gamma\}$ and $\{\dot{u}_f(\tau) : u \in \mathbf{X}_\Gamma\}$ are relatively compact in E because they are included in the norm compact sets $\int_0^1 G(t,s)\Gamma(s)ds$ and $\int_0^1 \frac{\partial G}{\partial t}(t,s)\Gamma(s)ds$ respectively. The Ascoli-Arzelà theorem yields that $\mathbf{X}_\Gamma$ is relatively compact in $\mathbf{X}$ with respect to $\|.\|_\mathbf{X}$. We claim that $\mathbf{X}_\Gamma$ is closed in $(\mathbf{X}, \|.\|_\mathbf{X})$. Let (u_n) be a sequence in $\mathbf{X}_\Gamma$ converging to $\xi \in \mathbf{X}$ with respect to $\|.\|_\mathbf{X}$. Then, for each n, there exists $f_n \in \mathbf{S}_\Gamma^{Pe}$ such that

$$u_n(t) = \int_0^1 G(t,s)f_n(s)ds, \quad \forall t \in [0,1]$$

and $u_n(t) = \varphi(t)$ for all $t \in [-r,0]$. As $\mathbf{S}_\Gamma^{Pe}$ is sequentially compact for the topology of pointwize convergence on $\mathbf{L}^\infty_{\mathbb{R}} \otimes E'$, we extract from (f_n) a subsequence that we do not relabel and which converges $\sigma(\mathbf{P}^1_E, \mathbf{L}^\infty_{\mathbb{R}} \otimes E')$ to a mapping $f \in \mathbf{S}_\Gamma^{Pe}$. In particular, for each $x' \in E'$ and for every $t \in [0,1]$, we have

$$\begin{aligned}
\lim_{n\to\infty} \langle x', \int_0^1 G(t,s)f_n(s)ds\rangle &= \lim_{n\to\infty} \int_0^1 \langle G(t,s)x', f_n(s)\rangle ds \\
&= \int_0^1 \langle G(t,s)x', f(s)\rangle ds \qquad (3.2) \\
&= \langle x', \int_0^1 G(t,s)f(s)ds\rangle,
\end{aligned}$$

and

$$
\begin{aligned}
\lim_{n\to\infty}\langle x', \int_0^1 \frac{\partial G}{\partial t}(t,s) f_n(s)ds\rangle &= \lim_{n\to\infty}\int_0^1 \langle \frac{\partial G}{\partial t}(t,s)x', f_n(s)\rangle ds \\
&= \int_0^1 \langle \frac{\partial G}{\partial t}(t,s)x', f(s)\rangle ds \qquad (3.3)\\
&= \langle x', \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s)ds\rangle .
\end{aligned}
$$

As the set valued integral $\int_0^1 G(t,s)\Gamma(s)ds$ and $\int_0^1 \frac{\partial G}{\partial t}(t,s)\Gamma(s)ds$ ($t\in[0,1]$) are norm-compacts, (3.2) and (3.3) show that the sequences $(u_n(\cdot)) = (\int_0^1 G(.,s)f_n(s)ds)$ and $(\dot{u}_n(\cdot)) = (\int_0^1 \frac{\partial G}{\partial t}(.,s)f_n(s)ds)$ converge pointwise to $u(.)$ and $\dot{u}(.)$ respectively, for E endowed with the strong topology, where

$$
u(t) = \int_0^1 G(t,s)f(s)ds, \ \ \forall t\in[0,1],
$$

$$
\dot{u}(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)ds, \ \ \forall t\in[0,1],
$$

and $u(t)=\varphi(t)$ for all $t\in[-r,0]$. Thus we get $\xi = u$. This shows the compactness of $\mathbf{X}_\Gamma$ in $\mathbf{X}$.

Step 2. Let $\Phi : \mathbf{X}_\Gamma \rightrightarrows \mathbf{P}^1_E([0,1])$ be the multifunction given by

$$
\Phi(u) = \{v\in \mathbf{P}^1_E([0,1]) : v(t)\in H(t,u(t),u(h(t)),\dot{u}(t)), \text{ a.e. on } [0,1]\}.
$$

We will prove that, for $\mathbf{X}_\Gamma$ endowed with the norm $\|\cdot\|_{\mathbf{X}}$, the multifunction Φ admits a continuous selection. It is clear that Φ has nonempty closed decomposable values. According to Proposition 2.2, it sufficient to prove that Φ is lower semicontinuous. Let $u_0\in \mathbf{X}_\Gamma$, $v_0\in\Phi(u_0)$ and let (u_n) be a sequence in $\mathbf{X}_\Gamma$ converging to u_0 in $(\mathbf{X},\|\cdot\|_{\mathbf{X}})$. Since $u_0\in\mathbf{X}_\Gamma$, there exists $f_0\in \mathbf{S}^{Pe}_\Gamma$ such that

$$
u_0(t) = \int_0^1 G(t,s)f_0(s)ds, \ \ \forall t\in[0,1]
$$

and $u_0(t)=\varphi(t)$ for all $t\in[-r,0]$, and since $(u_n)\subset\mathbf{X}_\Gamma$, for each n, there exists $f_n\in\mathbf{S}^{Pe}_\Gamma$ such that

$$
u_n(t) = \int_0^1 G(t,s)f_n(s)ds, \ \ \forall t\in[0,1]
$$

and $u_n(t)=\varphi(t)$ for all $t\in[-r,0]$.

For any $n\in\mathbb{N}$, $H(.,u_n(.),u_n(h(.)),\dot{u}_n(.))$ is measurable with nonempty closed values, so according to [10, Theorem III. 41], the multifunction Λ_n defined from $[0,1]$ into E by

$$
\Lambda_n(t) = \{w\in H(t,u_n(t),u_n(h(t)),\dot{u}_n(t)) : \|w - v_0(t)\| = d(v_0(t), H(t,u_n(t),u_n(h(t)),\dot{u}_n(t)))\},
$$

is also measurable with closed values, and since $H(., u_n(.), u_n(h(.)), \dot{u}_n(.))$ has compact values, Λ_n has nonempty values. In view of the existence theorem of measurable selections (see [10]), there is a measurable mapping $v_n : [0,1] \to E$ such that $v_n(t) \in \Lambda_n(t)$, for all $t \in [0,1]$. This yields $v_n(t) \in H(t, u_n(t), u_n(h(t)), \dot{u}_n(t))$ and

$$\begin{aligned}\lim_{n\to\infty} \|v_n(t) - v_0(t)\| &= \lim_{n\to\infty} d(v_0(t), H(t, u_n(t), u_n(h(t)), \dot{u}_n(t)))\\ &\leq \lim_{n\to\infty} e(H(t, u_0(t), u_0(h(t)), \dot{u}_0(t)), H(t, u_n(t), u_n(h(t)), \dot{u}_n(t)))\\ &= 0,\end{aligned}$$

the last equality follows from the fact that H is lower semicontinuous with compact values and hence it is h-lower semicontinuous. This shows that (v_n) converges pointwise to v_0 and since $H(t, x, y, z) \subset \Gamma_2(t)$ for all $(t, x, y, z) \in [0,1] \times E \times E \times E$, the convergence also holds strongly in $\mathbf{P}^1_E([1,0])$. Indeed,

$$\begin{aligned}\lim_{n\to\infty} \|v_n - v_0\|_{Pe} &= \lim_{n\to\infty} \sup_{x'\in\overline{\mathbf{B}}_{E'}} \int_0^1 |\langle x', v_n(t) - v_0(t)\rangle| dt\\ &= \lim_{n\to\infty} \sup_{x'\in\overline{\mathbf{B}}_{E'}} \int_0^1 |\langle x', v_n(t)\rangle - \langle x', v_0(t)\rangle| dt.\end{aligned}$$

As $v_n(t) \in \Gamma_2(t)$ for all $n \in \mathbb{N}$ and as Γ_2 is scalarly uniformly integrable and hence the set $\{\langle x', v_n(.)\rangle : \|x'\| \leq 1\}$ is uniformly integrable in $\mathbf{L}^1_E([0,1])$, we get

$$\lim_{n\to\infty} \|v_n - v_0\|_{Pe} = \sup_{x'\in\overline{\mathbf{B}}_{E'}} \int_0^1 \lim_{n\to\infty} |\langle x', v_n(t)\rangle - \langle x', v_0(t)\rangle| dt = 0.$$

Therefore Φ is lower semicontinous. An application of Proposition 2.2 implies that, for $\mathbf{X}_\Gamma$ endowed with the norm $\|\cdot\|_{\mathbf{X}}$, there exists a continuous mapping $K : \mathbf{X}_\Gamma \to \mathbf{P}^1_E([0,1])$ such that $K(u) \in \Phi(u)$ for all $u \in \mathbf{X}_\Gamma$, or equivalently, for each $u \in \mathbf{X}_\Gamma$ the inclusion $K(u)(t) \in H(t, u(t), u(h(t)), \dot{u}(t))$ holds for a.e. $t \in [0,1]$.

Step 3. We transform the problem

$$(\mathcal{P}) \begin{cases} \ddot{u}(t) \in F(t, u(t), u(h(t)), \dot{u}(t)) + K(u)(t), & a.e.\ t \in [0,1];\\ u(t) = \varphi(t), \quad \forall t \in [-r, 0];\\ u(0) = 0; \quad u(\theta) = u(1), \end{cases}$$

into a fixed point inclusion in the Banach space $\mathbf{X}_\Gamma$. By Lemma 2.1 and Proposition 2.2, the existence of solutions of $(\mathcal{P})$ is equivalent to the problem of finding $u \in \mathbf{X}_\Gamma$ such that

$$\begin{cases} u(t) \in \displaystyle\int_0^1 G(t,s)(F(s, u(s), u(h(s)), \dot{u}(s)) + K(u)(s)) ds, \quad \forall t \in [0,1];\\ u(t) = \varphi(t), \quad \forall t \in [-r, 0]. \end{cases} \tag{3.4}$$

Define the operator $\mathcal{A}$ on $\mathbf{X}_\Gamma$ by

$$\mathcal{A}u = \{v \in \mathbf{X}/\ v = \varphi\ on\ [-r, 0]\ and\ v(t) = \int_0^1 G(t,s)g(s)ds,\ \forall t \in [0,1], g = f + K(u), \quad f \in \mathbf{S}^{Pe}_F(u)\} \tag{3.5}$$

where

$$\mathbf{S}_F^{Pe}(u) = \{\vartheta \in \mathbf{P}_E^1([0,1]) / \ \vartheta(t) \in F(t, u(t), u(h(t)), \dot{u}(t)), \ a.e. t \in [0,1]\}. \tag{3.6}$$

Then, the integral inclusion (3.4) is equivalent to the operator inclusion

$$u(t) \in \mathcal{A}u(t), \ \ \forall t \in [-r, 1]. \tag{3.7}$$

Let us show that $\mathbf{S}_F^{Pe}$ has nonempty values. Indeed, for any Lebesgue measurable mappings $u, w : [0,1] \to E$ and $v : [-r, 1] \to E$, there is a Lebesgue-measurable selection $s \in \mathbf{S}_{\Gamma_1}^{Pe}$ such that $s(t) \in F(t, u(t), v(h(t)), w(t))$ a.e. Indeed, there exist sequences (u_n), (v_n) and (w_n) of simple E-valued mappings which converge pointwise to u, v and w respectively, for E endowed with the norm topology. Notice that the multifunctions $F(., u_n(.), v_n(h(.)), w_n(.))$ are Lebesgue-measurable. In view of the existence theorem of measurable selection (see [10]), for each n, there is a Lebesgue-measurable selection s_n of $F(., u_n(.), v_n(h(.)), w_n(.))$. As $s_n(t) \in F(t, u_n(t), v_n(h(t)), w_n(t)) \subset \Gamma_1(t)$, for all $t \in [0,1]$ and as $\mathbf{S}_{\Gamma_1}^{Pe}$ is sequentially weakly compact in $\mathbf{P}_E^1([0,1])$, by Eberlein-Šmulian theorem, we may extract from (s_n) a subsequence (s'_n) which converges $\sigma(\mathbf{P}_E^1, \mathbf{L}_{\mathbb{R}}^\infty \otimes E')$ to a mapping $s \in \mathbf{S}_{\Gamma_1}^{Pe}$. That is, for each $x' \in E'$ and each $\zeta \in \mathbf{L}_{\mathbb{R}}^\infty$ we have

$$\lim_{n\to\infty} \langle \zeta(\cdot)x', s'_n(\cdot)\rangle = \langle \zeta(\cdot)x', s(\cdot)\rangle$$

or equivalently

$$\lim_{n\to\infty} \int_0^1 \langle \zeta(t)x', s'_n(t)\rangle dt = \int_0^1 \langle \zeta(t)x', s(t)\rangle dt,$$

i.e.,

$$\lim_{n\to\infty} \int_0^1 \zeta(t)\langle x', s'_n(t)\rangle dt = \int_0^1 \zeta(t)\langle x', s(t)\rangle dt.$$

This last equality shows that for each $x' \in E'$, the sequence $(\langle x', s'_n(\cdot)\rangle)_n$ $\sigma(\mathbf{L}_{\mathbb{R}}^1, \mathbf{L}_{\mathbb{R}}^\infty)$-converges to $\langle x', s(\cdot)\rangle$. Let $(e_k^*)_{k\in\mathbb{N}}$ be a dense sequence for the Mackey topology $\tau(E', E)$. Let $k \in \mathbb{N}$ be fixed. Applying the Banach-Mazur's theorem trick to $(\langle e_k^*, s'_n(.)\rangle)_n$ provides a sequence (z_n), $z_n \in co\{\langle e_k^*, s'_m(.)\rangle : m \geqslant n\}$ such that (z_n) converges pointwise $a.e$ to $\langle e_k^*, s(.)\rangle$. Using this fact and the pointwise convergence of the sequences (u_n), (v_n) and (w_n), the upper semicontinuity of F$(t, . \ , . \ , .)$ and the compacity of its values, it is not difficult to check that $s(t) \in F(t, u(t), v(h(t)), w(t))$ $a.e.$ Indeed, for almost every $t \in [0,1]$ we have

$$\begin{aligned}
\langle e_k^*, s(t)\rangle &\in \bigcap_n \overline{co}(\bigcup_{m\geq n} \langle e_k^*, s'_m(t)\rangle) \\
&\subset \bigcap_n \overline{co}(\bigcup_{m\geq n} (e_k^* \circ F(t, u_m(t), v_m(h(t)), w_m(t)))) \\
&= \overline{co}(\limsup_{n\to\infty} (e_k^* \circ F(t, u_m(t), v_m(h(t)), w_m(t)))) \\
&= \overline{co}(e_k^* \circ F(t, u(t), v(h(t)), w(t))) = (e_k^* \circ F(t, u(t), v(h(t)), w(t)))
\end{aligned}$$

since F has closed convex values. This implies that $s(t) \in F(t, u(t), v(h(t)), w(t))$ $a.e$, and then the operator $\mathcal{A}$ is well defined. Using Lemma 2.1 and the assumption $\varphi(0) = 0$, it is clear that $\mathcal{A}$ has its values in $\mathbf{X}_\Gamma$.

Now, we will show that the multivalued operator $\mathcal{A}$ satisfy all the conditions of Theorem 2.1. Clearly $\mathcal{A}u$ is convex for each $u \in \mathbf{X}_\Gamma$. First, we prove that $\mathcal{A}$ has compact values in $\mathbf{X}_\Gamma$. Since $\mathbf{X}_\Gamma$ is compact, it suffices to see that $\mathcal{A}$ has closed values in $\mathbf{X}_\Gamma$. For each $u \in \mathbf{X}_\Gamma$, let (v_n) be a sequence in $\mathcal{A}u$ converging to $v \in \mathbf{X}_\Gamma$. Then by (3.5), for every n there exists $f_n \in \mathbf{S}^{Pe}_F(u) \subset \mathbf{S}^{Pe}_{\Gamma_1}$ such that

$$v_n(t) = \int_0^1 G(t, s) g_n(s) ds, \ \ \forall t \in [0, 1],$$

where $g_n = f_n + K(u) \in \mathbf{S}^{Pe}_\Gamma$ and $v_n(t) = \varphi(t)$ for all $t \in [-r, 0]$. Since $\mathbf{S}^{Pe}_{\Gamma_1}$ is sequentially $\sigma(\mathbf{P}^1_E, \mathbf{L}^\infty_\mathbb{R} \otimes E')$-compact, we may extract from (f_n) a subsequence (that we do not relabel) converging $\sigma(\mathbf{P}^1_E, \mathbf{L}^\infty_\mathbb{R} \otimes E')$ to a mapping $f \in \mathbf{S}^{Pe}_{\Gamma_1}$. Since $F(t, \cdot, \cdot, \cdot)$ is upper semicontinuous and has convex compact values, by repeating the arguments given above, we get $f(t) \in F(t, u(t), u(h(t)), \dot{u}(t))$ a.e. $t \in [0, 1]$. Hence (g_n) converges $\sigma(\mathbf{P}^1_E, \mathbf{L}^\infty_\mathbb{R} \otimes E')$ to the mapping $g = f + K(u) \in \mathbf{S}^{Pe}_\Gamma$. In particular, for every $x' \in E'$ and for every $t \in [0, 1]$, we have

$$\begin{aligned}
\lim_{n\to\infty} \langle x', \int_0^1 G(t, s) g_n(s) ds\rangle &= \lim_{n\to\infty} \int_0^1 \langle G(t, s) x', g_n(s)\rangle ds \\
&= \int_0^1 \langle G(t, s) x', g(s)\rangle ds \\
&= \langle x', \int_0^1 G(t, s) g(s) ds\rangle.
\end{aligned}$$

As the set-valued integral $\int_0^1 G(t, s)\Gamma(s) ds$ $(t \in [0, 1])$ is norm compact the last equality shows that the sequence $(v_n(.)) = (\int_0^1 G(\cdot, s) g_n(s) ds)$ converges pointwise to $\int_0^1 G(., s) g(s) ds$, for E endowed with the strong topology. At this point, it is worth to mention that the sequence $(\dot{v}_n(.)) = (\int_0^1 \frac{\partial G}{\partial t}(., s) g_n(s) ds)$ converges pointwise to $\int_0^1 \frac{\partial G}{\partial t}(., s) g(s) ds$, for E endowed with the strong topology using as above, the weak convergence of (g_n) and the norm compactness of the set-valued integral $\int_0^1 \frac{\partial G}{\partial t}(t, s)\Gamma(s) ds$. As (v_n) converges in $\mathbf{X}_\Gamma$ to the mapping v, then

$$v(t) = \int_0^1 G(t, s) g(s) ds, \ \ \forall t \in [0, 1]$$

and $v(t) = \varphi(t)$ for all $t \in [-r, 0]$. Since $g = f + K(u)$ and $f \in \mathbf{S}^{Pe}_F(u)$, we get $v \in \mathcal{A}$. This says that $\mathcal{A}u$ is compact in $\mathbf{X}_\Gamma$.

Next, we show that $\mathcal{A}$ is a compact operator, that is, $\mathcal{A}$ maps bounded sets into relatively compact sets in $\mathbf{X}_\Gamma$. Let S be a bounded set in $\mathbf{X}_\Gamma$. We have $\mathcal{A}(S) \subset \mathbf{X}_\Gamma$. But $\mathbf{X}_\Gamma$ is compact in $\mathbf{X}$, then $\mathcal{A}(S)$ is relatively compact in $\mathbf{X}$ and hence $\mathcal{A}$ is compact.

Now, we show that the graph of $\mathcal{A}$, $\text{gph}(\mathcal{A}) = \{(u,v) \in \mathbf{X}_\Gamma \times \mathbf{X}_\Gamma / \ v \in \mathcal{A}u\}$ is closed. Let (u_n, v_n) be a sequence of $\text{gph}(\mathcal{A})$ converging uniformly to $(u,v) \in \mathbf{X}_\Gamma \times \mathbf{X}_\Gamma$ with respect to $\| \cdot \|_{\mathbf{X}}$. Since $v_n \in \mathcal{A}u_n$, for each n, there exists $f_n \in \mathbf{S}_F^{Pe}(u_n) \subset \mathbf{S}_{\Gamma_1}^{Pe}$ such that

$$v_n(t) = \int_0^1 G(t,s) g_n(s) ds, \quad \forall t \in [0,1],$$

where $g_n = f_n + K(u_n)$ and $v_n(t) = \varphi(t)$ for all $t \in [-r, 0]$. As $\mathbf{S}_\Gamma^{Pe}$ is sequentially compact for the topology of pointwise convergence on $\mathbf{L}_{\mathbb{R}}^\infty \otimes E'$, we may extract from (g_n) a subsequence (that we do note relabel) converging $\sigma(\mathbf{P}_E^1, \mathbf{L}_{\mathbb{R}}^\infty \otimes E')$ to a mapping $g \in \mathbf{S}_\Gamma^{Pe}$. Observing that $f_n(t) = g_n(t) - K(u_n)(t) \in F(t, u_n(t), u_n(h(t)), \dot{u}_n(t))$. Since $\|u_n - u\|_{\mathbf{X}} \to 0$ and $F(t,.,.,.)$ is upper semicontinuous on $E \times E \times E$ with convex compact values, repeating the arguments given above, we conclude that $f(t) = g(t) - K(u)(t) \in F(t, u(t), u(h(t)), \dot{u}(t))$. Equivalently, $f \in \mathbf{S}_F^{Pe}(u)$. On the other hand, it is not difficult to see that the sequence $(v_n(.)) = (\int_0^1 G(.,s) g_n(s) ds)$ converges pointwise to $\int_0^1 G(.,s) g(s) ds$ and that the sequence $(\dot{v}_n(.)) = (\int_0^1 \frac{\partial G}{\partial t}(.,s) g_n(s) ds)$ converges pointwise to $\int_0^1 \frac{\partial G}{\partial t}(.,s) g(s) ds$, for E endowed with the strong topology. As (v_n) converges to v in $(\mathbf{X}_\Gamma, \| \cdot \|_{\mathbf{X}})$ we get

$$v(t) = \int_0^1 G(t,s) g(s) ds, \quad \forall t \in [0,1],$$

where $g = f + K(u)$ and $v(t) = \varphi(t)$ for all $t \in [-r, 0]$. This shows that $\mathcal{A}$ has a closed graph and hence it is an upper semicontinuous operator on $\mathbf{X}_\Gamma$.

Finally, we show that there exists an $R > 0$ such that the a priori estimate

$$u \in \lambda \mathcal{A}u \ \ (0 < \lambda \leq 1) \Rightarrow \|u\| \leq R$$

holds. We have

$$u \in \lambda \mathcal{A}u \quad \Leftrightarrow \quad \text{there exists } f \in \mathbf{S}_F^{Pe}(u) \subset \mathbf{S}_{\Gamma_1}^{Pe}$$

such that

$$\begin{cases} u(t) = \lambda \displaystyle\int_0^1 G(t,s) g(s) ds, \quad \forall t \in [0,1]; \\ u(t) = \lambda \varphi(t), \quad \forall t \in [-r, 0], \end{cases}$$

where $g = f + K(u) \in \mathbf{S}_{\Gamma}^{Pe}$. For each $t \in [0,1]$, using relation (2.5) and the assumption over Γ, we have

$$\begin{aligned}
\|u(t)\| &= \sup_{x' \in \overline{\mathbf{B}}_{E'}} |\langle x', u(t)\rangle| \\
&= \sup_{x' \in \overline{\mathbf{B}}_{E'}} |\langle x', \int_0^1 G(t,s)g(s)ds\rangle| \\
&= \sup_{x' \in \overline{\mathbf{B}}_{E'}} |\int_0^1 G(t,s)\langle x', g(s)\rangle ds| \\
&\leq \sup_{x' \in \overline{\mathbf{B}}_{E'}} \int_0^1 |G(t,s)||\langle x', g(s)\rangle| ds \\
&\leq \sup_{x' \in \overline{\mathbf{B}}_{E'}} \int_0^1 |\delta^*(x', \Gamma(s))| ds
\end{aligned}$$

and

$$\|\dot{u}(t)\| \leq \sup_{x' \in \overline{\mathbf{B}}_{E'}} \int_0^1 |\frac{\partial G}{\partial t}(t,s)||\langle x', g(s)\rangle| ds \leq \sup_{x' \in \overline{\mathbf{B}}_{E'}} \int_0^1 |\delta^*(x', \Gamma(s))| ds.$$

Since the set $\{|\delta^*(x', \Gamma(s))| : x' \in \overline{\mathbf{B}}_{E'}\}$ is uniformly integrable in $\mathbf{L}_{\mathbb{R}}^1([0,1])$, there exists a function $k \in \mathbf{L}_{\mathbb{R}}^1([0,1])$ such that, for all $x' \in \overline{\mathbf{B}}_{E'}$ and for all $s \in [0,1]$ we have

$$|\delta^*(x', \Gamma(s))| \leq |k(s)|.$$

We get

$$\|u(t)\| \leq \int_0^1 |k(s)| ds = \|k\|_{\mathbf{L}_{\mathbb{R}}^1}$$

and

$$\|\dot{u}(t)\| \leq \|k\|_{\mathbf{L}_{\mathbb{R}}^1}.$$

On the other hand, for each $t \in [-r, 0]$ we have

$$\|u(t)\| = \|\lambda\varphi(t)\| \leq \|\varphi\|_{\mathbf{C}_E([-r,0])}.$$

Taking the above inequalities into account, we obtain

$$\|u\|_{\mathbf{X}} \leq \max(\|k\|_{\mathbf{L}_{\mathbb{R}}^1}, \|\varphi\|_{\mathbf{C}_E([-r,0])}) = R.$$

Hence by Theorem 2.1, we conclude that $\mathcal{A}$ has a fixed point u in the ball $\overline{\mathbf{B}}(0,R)$, what, in turn, means that this point is a solution in $\mathbf{X}_\Gamma$ to the problem $(\mathcal{P})$. That is, $\ddot{u}(t) \in F(t, u(t), u(h(t)), \dot{u}(t)) + K(u)(t)$, a.e. $t \in [0,1]$ and $u(t) = \varphi(t)$ for all $t \in [-r, 0]$. Since $K(u)(t) \in H(t, u(t), u(h(t)), \dot{u}(t))$, we get that u is a solution in $\mathbf{X}_\Gamma$ to our boundary value problem $(\mathcal{P}_r)$ and the proof of the theorem is complete. ■

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