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Rate of approach to the steady state for a diffusion-convection equation on annular domains*

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Abstract

In this paper, we study the asymptotic behavior of global solutions of the equation $u_t = \Delta u + e^{|\nabla u|}$ in the annulus $B_{r,R}$, $u(x,t) = 0$ on ∂B_r and $u(x,t) = M \geq 0$ on ∂B_R. It is proved that there exists a constant $M_c > 0$ such that the problem admits a unique steady state if and only if $M \leq M_c$. When $M < M_c$, the global solution converges in $C^1(\overline{B_{r,R}})$ to the unique regular steady state. When $M = M_c$, the global solution converges in $C(\overline{B_{r,R}})$ to the unique singular steady state, and the blowup rate in infinite time is obtained.

Keywords: Convergence, Steady state, Gradient blowup.

1 Introduction and main results

In this paper we consider the problem

$$\begin{cases} u_t = \Delta u + e^{|\nabla u|}, & x \in B_{r,R},\ t > 0, \\ u(x,t) = 0, & x \in \partial B_r,\ t > 0, \\ u(x,t) = M, & x \in \partial B_R,\ t > 0, \\ u(x,0) = u_0(x), & x \in B_{r,R}. \end{cases} \tag{1.1}$$

Here $r > 0$, $B_{r,R} = \{x \in \mathbb{R}^N; r < |x| < R\}$, $\partial B_r = \{x \in \mathbb{R}^N; |x| = r\}$, $M \geq 0$, and $u_0(x) \in X$, where $X = \{v \in C^1(\overline{B_{r,R}}); v|_{\partial B_r} = 0, v|_{\partial B_R} = M\}$, endowed with the C^1 norm. Problem (1.1) admits a unique maximal classical solution

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$u(x,t)$, whose existence time will be denoted by $T = T(u_0) > 0$, such that $u \in C^{2,1}(\overline{B_{r,R}} \times (0,T)) \cap C^{1,0}(\overline{B_{r,R}} \times [0,T))$.

The differential equation in (1.1) possesses both mathematical and physical interest. This equation arises in the viscosity approximation of Hamilton-Jacobi type equations from stochastic control theory [2] and in some physical models of surface growth [4].

On the other hand, it can serve as a typical model-case in the theory of parabolic PDEs. Indeed, it is the one of the simplest examples (along with Burger's equation) of a parabolic equation with a nonlinearity depending on the first-order spatial derivatives of u.

A basic fact about (1.1) is that the solutions satisfy a maximum principle:

$$\min_{\overline{B_{r,R}}} u_0 \le u(x,t) \le \max_{\overline{B_{r,R}}} u_0, \quad x \in \overline{B_{r,R}}, \quad 0 \le t < T. \tag{1.2}$$

Since Problem (1.1) is well-posed in C^1 locally in time, only three possibilities can occur:

(I) u exists globally and is bounded in C^1:

$$T = \infty \quad \text{and} \quad \sup_{t \ge 0} \|\nabla u(t)\|_\infty < \infty;$$

(II) u blows up in finite time in C^1 norm (finite time gradient blowup):

$$T < \infty \quad \text{and} \quad \lim_{t \to T} \|\nabla u(t)\|_\infty = \infty;$$

(III) u exists globally but is unbounded in C^1 (infinite time gradient blowup):

$$T = \infty \quad \text{and} \quad \limsup_{t \to \infty} \|\nabla u(t)\|_\infty = \infty.$$

For $M = 0$ and $\|u_0\|_{C^1}$ sufficiently small, it is known that (I) occurs and u converges to the unique steady state $S_0 \equiv 0$. On the contrary, if u_0 suitably large, (II) occurs (see [5] and [8]).

For $M > 0$, the situation is slightly more complicated. There exists a critical value M_c (see Section 2 below for its existence) such that (1.1) has a unique, regular and radial ($S_M(x) = S_M(\rho)$ with $\rho = |x|$) steady state S_M if $M < M_c$ and no steady state if $M > M_c$. For the critical case $M = M_c$, there still exists a radial steady state S_{M_c}, but it is singular, satisfying $S_{M_c} \in C([r,R]) \cap C^\infty((r,R])$ with $S_{M_c,\rho} = \infty$.

For one dimensional case (see [8]), it was proved among other things that, if $M > M_c$, then all solutions of (1.1) satisfy (II), and if $0 < M < M_c$, then both

(I) and (II) can occur. Moreover, in [9], it was shown that if $0 \le M < M_c$, then all global solutions of (1.1) are bounded in C^1, and they converge to S_M in C^1. If (II) occurs, with the assumption on the initial data so that the solution is monotonically increasing both in time and in space, Zhang and Hu in [8] studied the blowup estimate and obtained that the blowup rate is close to $\ln \frac{1}{T-t}$ but not exactly equal to $\ln \frac{1}{T-t}$, which is very interesting because the blowup estimate can not be predicted by the usual self-similar transformations. For $N(>1)$ dimensional and zero-Dirichlet problem, in [10], Zhang and Li considered the gradient estimate near the boundary and the blowup rate of the radial case.

The purpose of this paper is to extend the results of [5, 8, 9, 10] to Problem (1.1), i.e., if $M = M_c$ and $u_0 \le S_{M_c}$, then (III) occurs and, u converges in $C(\overline{B_{r,R}})$ exponentially to S_{M_c} , as well as $u_\rho(r,t)$ grows up exponentially to infinity. Therefore, we provide a classification of large time behavior of the solutions of (1.1) for arbitrary spatial dimension. Our main results are as follows:

Theorem 1.1 *(1) If $0 \le M < M_c$, then all global solutions of (1.1) converges in $C(\overline{B_{r,R}})$ to S_M. Moreover, if $u_0 \le S_M$, then the solution of (1.1) is global in time and converges in $C^1(\overline{B_{r,R}})$ to S_M, and we have the uniform exponential convergence*

$$\lim_{t\to\infty} \frac{\ln |U(\cdot) - u(\cdot,t)|}{t} = -\lambda_1,$$

where λ_1 is the first eigenvalue of (3.2) (see Section 3 below).

(2) If $M = M_c$, then all global solutions of (1.1) converge in $C(\overline{B_{r,R}})$ to S_M. Moreover, if $u_0 \le S_M$, then the solution of (1.1) is global in time and converges in $C^1(\overline{B_{r,R}})$ to S_M, and we have the uniform exponential convergence

$$\lim_{t\to\infty} \frac{\ln |U(\cdot) - u(\cdot,t)|}{t} = -\lambda_1,$$

as well as the blowup estimate

$$\lim_{t\to\infty} \frac{u_\nu(x,t)}{t} = \lambda_1, \ x \in \partial B_r,$$

where λ_1 is the first eigenvalue of (4.1) (see Section 4 below).

2 Stationary states and global existence

From the maximum principle, if Problem (1.1) admits a steady state $S_M(x)$, then it is unique and radial, and if $M_1 > M_2$, then $S_{M_1} > S_{M_2}$ in $(r,R]$. So the stationary state satisfies

$$\begin{cases} -S_{M,\rho\rho} - \dfrac{N-1}{\rho} S_{M,\rho} = e^{S_{M,\rho}}, & r < \rho < R, \\ S_M(r) = 0, \ S_M(R) = M. \end{cases} \tag{2.1}$$

For $M > 0$, from the existence theory of ODEs, we know that $S_{M,\rho} > 0$ in $(r, R]$. Then $S_{M,\rho}$ satisfies $e^{S_{M,\rho}} \le -S_{M,\rho\rho} \le ce^{S_{M,\rho}}$ in $(r, R]$, where $c > 1$ is some constant. We consider a special case where $S_{M,\rho}(r) = \infty$, so we have

$$\ln \frac{1}{c(\rho - r)} \le S_{M,\rho}(\rho) \le \ln \frac{1}{\rho - r},$$

from which we get

$$(\rho - r)\Big(1 + \ln \frac{1}{c(\rho - r)}\Big) \le S_M(\rho) \le (\rho - r)\Big(1 + \ln \frac{1}{\rho - r}\Big). \tag{2.2}$$

So we can deduce that there exists $M_c > 0$ such that if $M > M_c$, then Problem (1.1) does not admit a steady state, if $0 < M < M_c$, then Problem (1.1) admits a unique regular steady state $S_M \in C^2([r, R])$, and if $M = M_c$, then Problem (1.1) still admits a steady state $S_{M_c} \in C([r, R]) \cap C^2((r, R])$, which is singular in the sense that it has infinite derivative on the boundary ∂B_r.

Theorem 2.1 *Assume that $M \ge 0$. If u is a global solution of Problem (1.1), then*
(1) Problem (1.1) admits a steady state S_M satisfying (2.1);
(2) $u(\cdot, t) \to S_M(\cdot)$ in $C(\overline{B_{r,R}})$ as $t \to \infty$.

Proof. (1) Let $\chi(\rho)$ be the solution of

$$-\Delta\chi = 1, \quad r < \rho < R; \quad \chi(r) = 0,\ \chi(R) = M, \tag{2.3}$$

and $\kappa(\rho)$ be the solution of

$$-\Delta\kappa = 1, \quad r < \rho < R; \quad \kappa(r) = \kappa(R) = 0. \tag{2.4}$$

Set $\underline{u}_0 = -\chi - \mu\kappa$, then since $u_0 \in C^1(\overline{B_{r,R}})$, we have $\underline{u}_0 \le u_0$ in $B_{r,R}$ if $\mu > 0$ is suitably large, which implies that $\underline{u} \le u$ in $B_{r,R} \times (0, \infty)$. Moreover, $\Delta\underline{u}_0 + e^{|\nabla \underline{u}_0|} \ge \mu + 1 > 0$. So by the maximum principle, we have $\underline{u}_t \ge 0$ in $B_{r,R}$ for all $t > 0$. As a consequence, there exists a function $S_M \in \overline{B_{r,R}}$ such that for all $x \in B_{r,R}$, $\underline{u}(x, t) \to S_M(x)$ as $t \to \infty$. Similar to the proof of [7, Theorem 3.2] or [10, Theorem 3.1], we have

$$|\nabla \underline{u}| \le C \ln \frac{1}{\delta(x)} \quad \text{in } B_{r,R} \times (0, \infty),$$

where $\delta(x) = \mathrm{dist}(x, \partial B_{r,R})$. Parabolic estimates imply that for any small $\varepsilon > 0$, for some $0 < \alpha < 1$, there holds

$$\|\underline{u}\|_{C^{2+\alpha,1+\alpha/2}(\overline{B_{r+\varepsilon,R-\varepsilon}} \times [t,t+1])} \le C(\varepsilon),\ t > 0.$$

By the diagonal procedure, there exists a sequence $t_n \to \infty$ such that $\underline{u}_n = \underline{u}(x, t_n + t)$ converges in $C^{2,1}_{loc}(\overline{B_{r,R}} \times [0, 1])$ to $S_M(x)$. So $S_M(x) \in C^2(B_{r,R}) \cap C(\overline{B_{r,R}})$ is the unique steady state of Problem (1.1).

(2) Define $w(t) = u(t) - S_M$, $\phi(t) = \|w(t)\|_\infty$. It follows from [7] that $\phi(t)$ is non-increasing for all $t > 0$. Set

$$l = \lim_{t\to\infty} \phi(t) \in [0, \infty).$$

We know that

$$|\nabla u| \le C \ln \frac{1}{\delta(x)}, \quad |u(x,t)| \le \widehat{C}\delta(x)\Big(\ln \frac{1}{\delta(x)} + 1\Big) + \widetilde{C} \quad \text{in } B_{r,R} \times [0, \infty). \quad (2.5)$$

Choose a sequence $t_n \to \infty$ and set $u_n(\cdot, t_n + \cdot)$ and $f_n(\cdot, \cdot) = f(\cdot, t_n + \cdot)$, where $f(x,t) = e^{|\nabla u|}$. Then the functions u_n then satisfy $\partial_t u_n - \Delta u_n = f_n(x,t)$ in $Q := B_{r,R} \times (0, \infty)$, with the sequence $f_n(\cdot, t)$ and $u_n(\cdot, t)$ bounded in $L^\infty_{loc}(Q)$ for $t > 0$. Theorem 1.1 in [7] implies that ∇u_n is bounded in $C^{\beta,\beta/2}_{loc}(Q)$ for some $0 < \beta < 1$. Using local parabolic Schauder estimates, we obtain that u_n is bounded in $C^{2+\gamma,1+\gamma/2}_{loc}(Q)$ for some $0 < \gamma < 1$. Therefore, u_n converges in $C^{2,1}_{loc}(Q)$ to a function $z \in C^{2,1}(Q)$, which solves

$$z_t - \Delta z = e^{|\nabla z|} \quad \text{in } Q.$$

Moreover, (2.5) implies that $\{u(\tau); \tau \ge 0\}$ is relatively compact in $C(\overline{Q})$. For each fixed $t \ge 0$, we may thus find a subsequence n_k such that $u_{n_k}(t)$ converges to $z(t)$ in $C(\overline{Q})$. It follows that

$$z(t) \in C(\overline{Q}) \text{ and } \|z(t) - S_M\|_\infty = \lim_{k\to\infty} \|u(t_{n_k} + t) - S_M\|_\infty = l, \ t \ge 0.$$

Setting $\widetilde{w}(t) := z(t) - S_M$, then $\widetilde{w}(t)$ satisfies

$$\widetilde{w}_t - \Delta\widetilde{w} = \widetilde{b}(x,t) \cdot \nabla\widetilde{w} \quad \text{in} \quad Q,$$

where $\widetilde{b}(x,t) = \int_0^1 e^{|\nabla S_M + s\nabla\widetilde{w}|} \frac{\nabla S_M + s\nabla\widetilde{w}}{|\nabla S_M + s\nabla\widetilde{w}|} ds \in C(Q)$. Assume for contradiction that $l > 0$. Since $\widetilde{w}(\cdot, 2) \in C_0(\overline{B_{r,R}})$, there exists $x_0 \in B_{r,R}$, such that $|\widetilde{w}(x_0, 2)| = \|\widetilde{w}(2)\|_\infty = l = \|\widetilde{w}\|_{L^\infty(B_{r,R})}$. For each $\rho < \delta(x_0)$, since $\widetilde{b} \in L^\infty(B(x_0, \rho) \times (1, 2))$, we may apply the strong maximum principle to deduce that $|\widetilde{w}| = l$ in $B(x_0, \rho) \times [1, 2]$. But by letting $\rho \to \delta(x_0)$, this contradicts $\widetilde{w}(\cdot, 2) \in C_0(\overline{B_{r,R}})$. Therefore, $l = 0$. Since the sequence t_n was arbitrary, we conclude that $\lim_{t\to\infty} \|u(t) - S_M\|_\infty = 0$, and the assertion (2) is proved.

3 Subcritical case $M < M_c$

In this section, we assume that $u_0 \le S_M$ in $B_{r,R}$. By the maximum principle, we have $-\chi - \mu\kappa \le u \le S_M$ for $t < T$, where μ is a suitably large constant. Similar to the proof of [7, Theorem 3.2] or [10, Theorem 3.1], we can get that ∇u blows up only on the boundary. So u exists globally and ∇u is uniformly bounded in $B_{r,R} \times [0,\infty)$. So standard arguments imply that $u(\cdot,t) \to S_M(\cdot)$ as $t \to \infty$.

We consider the eigenvalue problem

$$\begin{cases} -\varphi_{\rho\rho} - \frac{N-1}{\rho}\varphi_\rho - e^{S_{M,\rho}}\varphi_\rho = \lambda\varphi, & r < \rho < R, \\ \varphi(r) = \varphi(R) = 0. \end{cases} \tag{3.1}$$

By (2.1), we get

$$e^{S_{M,\rho}} = -S_{M,\rho\rho} - \frac{N-1}{\rho}S_{M,\rho}.$$

So Equation (3.1) can be written as

$$-\varphi_{\rho\rho} + \Big(S_{M,\rho\rho} + \frac{N-1}{\rho}S_{M,\rho} - \frac{N-1}{\rho}\Big)\varphi_\rho = \lambda\varphi.$$

It is equivalent to

$$-\big(a(\rho)\varphi_\rho\big)_\rho = \lambda a(\rho)\varphi, \quad r < \rho < R; \quad \varphi(r) = \varphi(R) = 0, \tag{3.2}$$

where $a(\rho)$ satisfies

$$\frac{a'(\rho)}{a(\rho)} = -S_{M,\rho\rho} - \frac{N-1}{\rho}S_{M,\rho} + \frac{N-1}{\rho}.$$

Let $\varphi(\rho)$ be the first eigenfunction and λ_1 be the corresponding eigenvalue.

Let $\underline{u}$ be the (global) solution of (1.1) with $-\chi - \mu\kappa$ as the initial data for some $\mu > 0$ such that $-\chi - \mu\kappa \le u_0$. By the comparison principle, we get $\underline{u} \le u$. Therefore $S_M - u \le \underline{v} := S_M - \underline{u}$. Since $\underline{u}$ is radially symmetric, then, by Taylor's expansion up to second order, we obtain

$$\begin{aligned} \underline{v}_t - \underline{v}_{\rho\rho} - \frac{N-1}{\rho}\underline{v}_\rho &= e^{S_{M,\rho}} - e^{\underline{u}_\rho} \\ &= e^{S_{M,\rho}} - e^{S_{M,\rho} - \underline{v}_\rho} \\ &= e^{S_{M,\rho}}\underline{v}_\rho - F(x,\underline{v}_\rho), \end{aligned} \tag{3.3}$$

where $F(x,\underline{v}_\rho) = \frac{1}{2}e^{S_{M,\rho} - \theta(x,\underline{v}_\rho)(S_{M,\rho} - \underline{v}_\rho)}\underline{v}_\rho^2$, $\theta \in (0,1)$. So we have

$$\underline{v}_t - \underline{v}_{\rho\rho} - \frac{N-1}{\rho}\underline{v}_\rho \le e^{S_{M,\rho}}\underline{v}_\rho.$$

Let $\varphi(\rho)$ be the first eigenfunction of (3.2) and choose a constant $C > 0$ such that $u_0 + \chi + \mu\kappa \le C\varphi$. We observe that $Ce^{-\lambda_1 t}\varphi$ is a super-solution of (3.3). Then by the comparison principle, we get $S_M - u \le \underline{v} \le Ce^{-\lambda_1 t}\varphi$. By the strong maximum principle, we get $u(\cdot,t_0) < S_M(\cdot)$ and $-u_\nu(\cdot,t_0) < -S_{M,\nu}(\cdot)$ on the boundary of $B_{r,R}$. Without loss of generality we assume that $t_0 = 0$. So there is a radially symmetric function $\vartheta(\rho)$ such that $u_0 < \vartheta < S_M$. Let $\overline{u}$ be the solution of (1.1) with ϑ as the initial data. Then by comparison principle, we have $u \leq \overline{u} \leq S_M$. Let $\overline{v} = S_M - \overline{u}$, by the Taylor's expansion up to the second order, we also get (3.3) with replaced $\underline{v}$ by $\overline{v}$. Since $|F| \leq C_1|\overline{v}_\rho|^2$ for some constant C_1 independent of $\overline{v}$ due to $\overline{v}_\rho$ is uniformly bounded in $\overline{B_{r,R}} \times [0,\infty)$, we obtain

$$\overline{v}_t - \overline{v}_{\rho\rho} - \frac{N-1}{\rho}\overline{v}_\rho \geq e^{S_{M,\rho}}\overline{v}_\rho - C_1|\overline{v}_\rho|^2.$$

Let $z = 1 - e^{-C_1\overline{v}}$, then

$$z_t - z_{\rho\rho} - \frac{N-1}{\rho}z_\rho \geq e^{S_{M,\rho}}z_\rho.$$

So $S_M - u \geq \overline{v} \geq C_1^{-1}z \geq ce^{-\lambda_1 t}\varphi$ if $c > 0$ is suitably small. Thus we have

$$ce^{-\lambda_1 t}\varphi \leq S_M - u \leq Ce^{-\lambda_1 t}\varphi, \quad x \in B_{r,R},\ t > 0, \tag{3.4}$$

which implies Theorem 2.1 (1).

4 Critical case $M = M_c$

In this section, we assume that $u_0 \leq S_{M_c}$ in $B_{r,R}$. We claimed that u exists globally. Assume for contradiction that $T^* < \infty$. By the maximum principle, we have $u \geq -\chi - \mu\kappa$ for some μ, so ∇u blows up only on the boundary ∂B_r by the similar proof of [7, Theorem 3.2] or [10, Theorem 3.1]. Parabolic estimates imply that u can be extended to a function $u \in C^{2,1}(\overline{B_{r+\varepsilon,R}}) \times (0,T^*]$ for $0 < \varepsilon \ll 1$. Since $u < S_{M_c}$ in $B_{r,R}$ for $t > 0$, by the maximum principle, we have $u_\rho > S_{M_c,\rho}$ on ∂B_R for $0 < t \leq T^*$. Fixing $t_0 \in (0,T^*)$, we can find $M < M_c$ close to M_c and $0 < \varepsilon \ll 1$ such that $u < S_M$ on $\partial B_{R-\varepsilon} \times [t_0,T^*]$ and $u < S_M$ in $\overline{B_{r,R-\varepsilon}}$ at $t = t_0$. So we have $u < S_M$ in $B_{r,R-\varepsilon} \times [t_0,T^*]$, contradicting to the blowup of ∇u at $t = T^*$.

Fixing some $t_0 > 0$, we have $u(x,t_0) < S_{M_c}(x)$ for $x \in B_{r,R}$. So there exists a radial function $h(\rho)$ such that $u(x,t_0) < h(\rho) < S_{M_c}(x)$, therefore $u(x,t) \leq H(\rho,t)$ in $B_{r,R} \times [t_0,\infty)$, where H is the solution of Problem (1.1) with $H(\rho,t_0) = h(\rho)$. Also, since $-\chi(\rho) - \mu\kappa(\rho) \leq u_0(x)$ for some μ, we have $K(\rho,t) \leq u(x,t)$ in $B_{r,R} \times [t_0,\infty)$, where K is the solution of Problem (1.1) with $K(\rho,t_0) = -\chi(\rho) - \mu\kappa(\rho)$. So, similarly to Section 3, it is sufficient to consider the asymptotic behavior of the radial solution of Problem (1.1).

In the following, we use the idea of [6] to study the asymptotic behavior of the radial solution of Problem (1.1).

We consider the degenerate eigenvalue problem

$$-(a(\rho)\varphi_\rho)_\rho = \lambda a(\rho)\varphi, \quad r < \rho < R; \quad \varphi(r) = \varphi(R) = 0, \tag{4.1}$$

and its regularized problem

$$-(a(\rho)\varphi_{\varepsilon,\rho})_\rho = \lambda_\varepsilon a(\rho)\varphi_\varepsilon, \quad r+\varepsilon<\rho<R; \quad \varphi_\varepsilon(r+\varepsilon)=\varphi_\varepsilon(R)=0. \tag{4.2}$$

Denote by λ_ε the first eigenvalue of (4.2) and by φ_ε the corresponding eigenfunction. Let $\lambda_1 = \inf\{\int_r^R a(\rho)(v_\rho)^2 d\rho; v \in J, \int_r^R a(\rho)v^2 d\rho = 1\}$, where $J = \{v \in H^1_{loc}((r,R]); \int_r^R a(\rho)(v_\rho)^2 d\rho < \infty, v(R) = 0\}$. Then from the similar proof of Proposition 5.1 in [6], we know that λ_1 is well defined, $0 < \lambda_1 = \lim_{\varepsilon\to 0}\lambda_\varepsilon < \infty$, and there exists $0 < \varphi \in J \cap C^2((r,R])$ which solves (4.1) with $\lambda = \lambda_1$.

Set $v = S_{M_c} - u$, then

$$\begin{aligned} v_t - \Delta v &= e^{|\nabla S_{M_c}|} - e^{|\nabla u|} \\ &= e^{|\nabla S_{M_c}|}\frac{\nabla S_{M_c}}{|\nabla S_{M_c}|}\cdot\nabla v - F(x,\nabla v), \end{aligned} \tag{4.3}$$

where $F(x,\nabla v) = \frac{1}{2}e^{|\nabla S_{M_c} - \theta(x,\nabla v)\nabla v|}|\nabla v|^2$, $\theta \in (0,1)$. So we have

$$v_t - \Delta v \le e^{|\nabla S_{M_c}|}\frac{\nabla S_{M_c}}{|\nabla S_{M_c}|}\cdot\nabla v \quad \text{in } (r,R)\times(0,\infty).$$

So

$$S_{M_c} - u = v \le Ce^{-\lambda_1 t}\varphi \tag{4.4}$$

if C is suitably large. Since $|F| \le C_\varepsilon|\nabla v|^2$ in $[r+\varepsilon,R]\times(0,\infty)$, we also have

$$v_t - \Delta v \ge e^{|\nabla S_{M_c}|}\frac{\nabla S_{M_c}}{|\nabla S_{M_c}|}\cdot\nabla v - C_\varepsilon|\nabla v|^2 \quad \text{in } [r+\varepsilon,R]\times(0,\infty).$$

Let $z = 1 - e^{-C_\varepsilon v}$, then

$$z_t - \Delta z \ge e^{|\nabla S_{M_c}|}\frac{\nabla S_{M_c}}{|\nabla S_{M_c}|}\cdot\nabla v.$$

So

$$S_{M_c} - u = v \ge C_\varepsilon^{-1}z \ge ce^{-\lambda_\varepsilon t}\varphi_\varepsilon \tag{4.5}$$

in $[r+\varepsilon,R]$, where $c>0$ is suitably small. The first assertion of Theorem 2.1 (2) is proved.

We consider the radial problem

$$\begin{cases} u_t - u_{\rho\rho} - \dfrac{N-1}{\rho}u_\rho = e^{|u_\rho|}, & r<\rho<R, \\ u(r,t)=0,\ u(R,t)=M_c, & t>0. \end{cases} \tag{4.6}$$

Let $v(\rho,t)$ be the solution of (4.3) with $v_0(\rho) = -\chi(\rho) - \mu\kappa(\rho)$ ($\mu > 0$), then $v(\rho,t)$ is nondecreasing in time by the maximum principle. Therefore $v_\rho(r,t)$ is also nondecreasing in time. So we have $\lim_{t\to\infty} v_\rho(r,t) = \infty$. For any radial function $u_0 \in X$ one can find μ suitable large such that $u_0 > v_0$, so we have

$$\lim_{t\to\infty} u_\rho(r,t) = \infty.$$

For $M < M_c$, as in [3], let $N_M(t)$ be the number of intersections of $u(\rho,t)$ and S_M. It is known that $N_M(t)$ is non-increasing. It is obvious that there exists M_0 close enough to M_c such that $N_M(1) = 1$ if $M_0 \le M < M_c$. Denote by $S_{M(t)}$ the solution of (2.1) with $S_{M,\rho}(r) = u_\rho(r,t)$. By $\lim_{t\to\infty} u_\rho(r,t) = \infty$, there exists $t_0 > 1$ such that $M(t) > M_0$ for all $t > t_0$. By Hopf's lemma, if $N_M(t) = 1$, then $u_\rho(r,t) < S_{M,\rho}(r)$. Therefore, $N_{M(t)}(t) = 0$. So $N_{M(t)}(s) = 0$ for $s > t$ since $N_M(t)$ is non-increasing. Thus we have by Hopf's lemma $u_\rho(r,s) > S_{M(t),\rho}(r) = u_\rho(r,t)$ for $s > t$, i.e., $u_\rho(r,t)$ is strictly increasing in time for $t > t_0$.

By (4.4), we have

$$u(\rho,t) \ge S_{M_c}(\rho) - Ce^{-\lambda_1 t},$$

and by (2.2)

$$\frac{u(\rho,t)}{\rho-r} \ge \Big(1 + \ln\frac{1}{c(\rho-r)}\Big) - C(\rho-r)^{-1}e^{-\lambda_1 t}.$$

Using the method in [9] or [1], we can prove that $u_{\rho\rho} < 0$ for $t \gg 1$ and $r < \rho < r+\varepsilon$. Therefore, taking $\rho - r = Ce^{-\lambda_1 t}$, we have

$$u_\rho(r,t) \ge \frac{u(\rho,t)}{\rho-r} \ge Ct \quad \text{for } t \text{ large.} \tag{4.7}$$

On the other hand, for t large, $u(\rho,t) > S_{M(t)}(\rho)$, therefore

$$\begin{aligned}
S_{M_c}(\rho) - u(\rho,t) &\le S_{M_c}(\rho) - S_{M(t)}(\rho)\\
&\le U_{M_c}(\rho) - U_{M(t)}(\rho)\\
&= (\rho-r)\Big(1+\ln\frac{1}{\rho-r}\Big)\\
&\quad +(\rho-r+e^{-\alpha(t)})\ln(\rho-r+e^{-\alpha(t)}) - (\rho-r) + \alpha(t)e^{-\alpha(t)}\\
&\le Ce^{-\alpha(t)},
\end{aligned}$$

where $U_M(\rho)$ is the solution of $U_{\rho\rho}+e^{|U_\rho|} = 0$ in (r,R) and $U(r) = 0, U(R) = M$, and $\alpha(t) = u_\rho(r,t)$. By (4.5), we have

$$e^{-\alpha(t)} \ge \|S_{M_c} - u(t)\|_\infty \ge ce^{-\lambda_\varepsilon t},$$

therefore we get

$$u_\rho(r,t) \le C\lambda_\varepsilon t \quad \text{for } t \text{ large.} \tag{4.8}$$

From (4.7) and (4.8), the second part of Theorem 2.1 (2) follows.

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