Spectrum of one dimensional p-Laplacian Operator with indefinite weight

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Abstract

This paper is concerned with the nonlinear boundary eigenvalue problem

$$-(|u'|^{p-2}u')' = \lambda m|u|^{p-2}u \qquad u \in I =]a, b[, \quad u(a) = u(b) = 0,$$

where $p > 1$, λ is a real parameter, m is an indefinite weight, and a, b are real numbers. We prove there exists a unique sequence of eigenvalues for this problem. Each eigenvalue is simple and verifies the strict monotonicity property with respect to the weight m and the domain I, the k-th eigenfunction, corresponding to the k-th eigenvalue, has exactly $k-1$ zeros in (a, b). At the end, we give a simple variational formulation of eigenvalues.

MSC 2000 35P30.
Key words and phrases: p-Laplacian spectrum, simplicity, isolation, strict monotonicity property, zeros of eigenfunctions, variational formulation.

1 Introduction

The spectrum of the p-Laplacian operator with indefinite weight is defined as the set $\sigma_p(-\Delta_p, m)$ of $\lambda := \lambda(m, I)$ for which there exists a nontrivial (weak) solution $u \in W_0^{1,p}(\Omega)$ of problem

$$(\mathcal{V}.\mathcal{P}_{(m,\Omega)}) \qquad \begin{cases} -\Delta_p u &= \lambda m|u|^{p-2}u \quad in \quad \Omega, \\ u &= 0 \qquad\qquad on \quad \partial\Omega, \end{cases}$$

where $p > 1$, Δ_p : is the p-Laplacian operator, defined by $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$, in a bounded domain $\Omega \subset I\!R^N$, and $m \in M(\Omega)$ is an indefinite weight, with

$$M(\Omega) := \{m \in L^\infty(\Omega)/meas\{x \in \Omega,\ m(x) > 0\} \neq 0\}.$$

The values $\lambda(m,\Omega)$ for which there exists a nontrivial solution of $(\mathcal{V}.\mathcal{P}_{(m,\Omega)})$ are called eigenvalues and the corresponding solutions are the eigenfunctions. We will denote $\sigma_p^+(-\Delta_p, m)$ the set of all positive eigenvalues, and by $\sigma_p^-(-\Delta_p, m)$ the set of negative eigenvalues.

For $p = 2$ ($\Delta_p = \Delta$ Laplacian operator) it is well known (*cf* [4, 7, 8]) that,

- $\sigma_2^+(-\Delta, m) = \{\mu_k(m,\Omega), k = 1, 2, \cdots\}$, with $0 < \mu_1(m,\Omega) < \mu_2(m,\Omega) \leq \mu_3(m,\Omega) \leq \cdots \to +\infty$, $\mu_k(m,\Omega)$ repeated according to its multiplicity.
- The k-th eigenfunction corresponding to $\mu_k(m,\Omega)$, has at most k nodal domains.
- The eigenvalues $\mu_k(m,\Omega)$, $k \geq 1$, verify the strict monotonicity property (SMP in brief), i.e if $m,\ m' \in M(\Omega)$, $m(x) \leq m'(x)$ a.e in Ω and $m(x) < m'(x)$ in some subset of nonzero measure, then $\mu_k(m,\Omega) > \mu_k(m',\Omega)$.
- Equivalence between the SMP and the unique continuation one.

For $p \neq 2$ (nonlinear problem), it is well known that the critical point theory of Ljusternik-Schnirelmann (*cf* [15]), provides a sequence of eigenvalues for those problems. Whether or not this sequence, denoted $\lambda_k(m,\Omega)$, constitutes the set of all eigenvalues is an open question when $N \geq 1$, $m \not\equiv 1$, and $p \neq 2$. The principal results for the problem seems to be given in (*cf* [1, 2, 3, 5, 6, 9, 10, 11, 12, 13]), where is shown that there exists a sequence of eigenvalues of $(\mathcal{V}.\mathcal{P}_{(m,\Omega)})$ given by,

$$\lambda_n(m,\Omega) = \inf_{K\in\mathcal{B}_n} \max_{v\in K} \frac{\int_\Omega |\nabla v|^p\, dx}{\int_\Omega m|v|^p\, dx} \tag{1}$$

$\mathcal{B}_n = \{K$, symmetrical compact, $0 \notin K$, and $\gamma(K) \geq n$ $\}$, γ is the genus function, or equivalently,

$$\frac{1}{\lambda_n(m,\Omega)} = \sup_{K\in\mathcal{B}_n} \min_{v\in K} \frac{\int_\Omega m|v|^p\, dx}{\int_\Omega |\nabla v|^p\, dx} \tag{2}$$

which can be written simply,

$$\frac{1}{\lambda_n(m,\Omega)} = \sup_{K\in\mathcal{A}_n} \min_{v\in K} \int_\Omega m|v|^p\, dx \tag{3}$$

$\mathcal{A}_n = \{K \cap S,\ K \in \mathcal{B}_n\}$. S is the unit sphere of $W_0^{1,p}(\Omega)$ endowed with the usual norm ($\|v\|_{1,p}^p = \int_\Omega |\nabla v|^p\, dx$), the equation (2) is the generalized Rayleigh quotient for the problem $(\mathcal{V}.\mathcal{P}_{(m,\Omega)})$. The sequence is ordered as $0 < \lambda_1(m,\Omega) < \lambda_2(m,\Omega) \leq \lambda_3(m,\Omega) \leq \cdots \to +\infty$. The first eigenvalue $\lambda_1(m,\Omega)$ is of special importance. We give some of its properties which will be of interest for us (*cf* [1]). First, $\lambda_1(m,\Omega)$ is given by,

$$\frac{1}{\lambda_1(m,\Omega)} = \sup_{v\in S} \int_\Omega m|v|^p\, dx = \int_\Omega m|\phi_1|^p\, dx \tag{4}$$

$\phi_1 \in S$ is any eigenfunction corresponding to $\lambda_1(m,\Omega)$, for this reason $\lambda_1(m,\Omega)$ is called the principal eigenvalue, also we know that $\lambda_1(m,\Omega) > 0$, simple (i.e if v and u are two eigenfunctions corresponding to $\lambda_1(m,\Omega)$ then $v=\alpha u$ for some $\alpha\in I\!R$), isolate (i.e there is no eigenvalue in $]0,a[$ for some $a>\lambda_1(m,\Omega)$, finally it is the unique eigenvalue which has an eigenfunction with constant sign. We denote $\phi_1(x)$ the positive eigenfunction corresponding to $\lambda_1(m,\Omega)$, $\phi_1(x)$ verifies the strong maximum principle (*cf* [17]), $\frac{\partial\phi_1}{\partial n}(x)<0$, for x in $\partial\Omega$ satisfying the interior ball condition.

In [14] Otani considers the case $N=1$, $m(x)\equiv 1$, and proves that , $\sigma_p(-\Delta_p,1)=\{\mu_k(m,I), k=1,2,\cdots\}$, the sequence can be ordered as $0<\mu_1(m,\Omega)<\mu_2(m,\Omega)<\mu_3(m,\Omega)<\cdots\to+\infty$, the k-th eigenfunction has exactly $k-1$ zeros in $I=(a,b)$. In [10] Elbert proved the same results in the case $N=1$, $m(x)\geq 0$ and continuous, the author gives an asymptotic relation of eigenvalues.

In this paper we consider the general case, $N=1$ and $m(x)$ can change sign and is not necessarily continuous. We prove that $\sigma_p^+(-\Delta_p,m)=\{\lambda_k(m,I), k=1,2,\cdots\}$, the sequence can be ordered as $0<\lambda_1(m,\Omega)<\lambda_2(m,\Omega)<\lambda_3(m,\Omega)<\cdots\lambda_k(m,I)\to+\infty$ as $k\to+\infty$, the k-th eigenfunction has exactly $k-1$ zeros in $I=(a,b)$. The eigenvalues verify the SMP with respect to the weight m and the domain I.

In the next section we denote by: $M(I):=\{m\in L^\infty(I)/meas\{x\in I,\, m(x)>0\}\neq 0\}$, $m_{/J}$ the restriction of m on J for a subset J of I, $Z(u)=\{t\in I/\,u(t)=0\}$, a nodal domain ω of u is a component of $I\backslash Z(u)$, where $(u,\lambda(m,I))$ is a solution of $(\mathcal{V}.\mathcal{P}_{(m,I)})$. $\tilde{u_{/\omega}}$ is the extension, by zero, on I of $u_{/\omega}$

2 Results and technical Lemmas

We first state our main results

Theorem 1 *Assume that N=1 ($\Omega=]a,b[=I$), $m\in M(I)$ such that $m\not\equiv 1$ and $p\neq 2$, we have*

1. *Every eigenfunction corresponding to the k-th eigenvalue $\lambda_k(m,I)$, has exactly k-1 zeros.*
2. *For every k, $\lambda_k(m,I)$ is simple and verifies the strict monotonicity property with respect to the weight m and the domain I.*
3. *$\sigma_p^+(-\Delta_p,m)=\{\lambda_k(m,I), k=1,2,\cdots\}$, for any $m\in M(I)$. The eigenvalues are ordered as $0<\lambda_1(m,\Omega)<\lambda_2(m,\Omega)<\lambda_3(m,\Omega)<\cdots\lambda_k(m,I)\to+\infty$ as $k\to+\infty$.*

Corollary 1 *For any integer n, we have the simple variational formulation,*

$$\frac{1}{\lambda_n(m,I)}=\sup_{F\in\mathcal{F}_n}\min_{F\cap S}\int_a^b m|v|^p\,dx \tag{5}$$

$\mathcal{F}_n$={F / F is a n dimensional subspace of $W_0^{1,p}(I)$}.

For the proof of Theorem 1 we need some technical Lemmas.

Lemma 1 *Let* $m, m^{'} \in M(I)$*,* $m(x) \leq m'(x)$*, then for any* n*,* $\lambda_n(m', I) \leq \lambda_n(m, I)$

Proof Making use of equation (2), we obtain immediately $\lambda_n(m', I) \leq \lambda_n(m, I)$.

Lemma 2 *Let* $(u, \lambda(m, I))$ *be a solution of* $(\mathcal{V}.\mathcal{P}_{(m,I)})$*,* $m \in M(I)$*, then* $m_{/\omega} \in M(\omega)$ *for any nodal domain* ω *of* u*.*

Proof Let ω be a nodal domain of u and multiply $(\mathcal{V}.\mathcal{P}_{(m,I)})$ by $\tilde{u_{/\omega}}$ so that we obtain

$$0 < \int_\omega |u^{'}|^p \, dx = \lambda(m, I) \int_\omega m|u|^p \, dx \,. \tag{6}$$

This completes the proof. ∎

Lemma 3 *The restriction of a solution* $(u, \lambda(m, I))$ *of problem* $(\mathcal{V}.\mathcal{P}_{(m,I)})$*, on a nodal domain* ω*, is an eigenfunction of problem* $(\mathcal{V}.\mathcal{P}_{(m_{/\omega},\omega)})$*, and we have* $\lambda(m, I) = \lambda_1(m/_\omega, \omega)$*.*

Proof Let $v \in W_0^{1,p}(\omega)$ and let $\tilde{v}$ be the extension by zero of v on Ω. It is obvious that $\tilde{v} \in W_0^{1,p}(\Omega)$. Multiply $(\mathcal{V}.\mathcal{P}_{(m,\Omega)})$ by $\tilde{v}$

$$\int_\omega |u'|^{p-2} u'v' \, dx = \lambda(m, I) \int_\omega m|u|^{p-2} uv \, dx \tag{7}$$

for all $v \in W_0^{1,p}(\omega)$. Hence the restriction of u in ω is a solution of problem $(\mathcal{V}.\mathcal{P}_{(m_{/\omega},\omega)})$ with constant sign. We then have $\lambda(m, \Omega) = \lambda_1(m_{/\omega}, \omega), \omega)$, which completes the proof. ∎

Lemma 4 *Each solution* $(u, \lambda(m, I))$ *of the problem* $(\mathcal{V}.\mathcal{P}_{(m,I)})$ *has a finite number of zeros.*

Proof This Lemma plays an essential role in our work. We start by showing that u has a finite number of nodal domains. Assume that there exists a sequence I_k, $k \geq 1$, of nodal domains (intervals), $I_i \cap I_j = \emptyset$ for $i \neq j$. We deduce by Lemmas 3 and 1, respectively, that

$$\lambda(m, I) = \lambda_1(m, I_k) \geq \lambda_1(C, I_k) = \frac{\lambda_1(1, I_k)}{C} = \frac{\lambda_1(1,]0, 1[)}{C(meas(I_k))^p}, \tag{8}$$

where $C = \|m\|_\infty$.
From equation (8) we deduce $(meas(I_k)) \geq (\frac{\lambda_1(1,]0,1[)}{\lambda(m,I)C})^{\frac{1}{p}}$, for all k, so

$$meas(I) = \sum_{k\geq 1} (meas(I_k)) = +\infty.$$

This yields a contradiction.

Let $\{I_1, I_2, \cdots I_k\}$ be the nodal domains of u. Put $I_i =]a_i, b_i[$, where $a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots a_k < b_k \leq b$. It is clear that the restriction of u on $]a, b_1[$ is a nontrivial eigenfunction with constant sign corresponding to $\lambda(m, I)$. The maximum principle (*cf* [17]) yields $u(t) > 0$ for all $t \in]a, b_1[$, so $a = a_1$. By a similar argument we prove that $b_1 = a_2$, $b_2 = a_3, \cdots b_k = b$, which completes the proof. ∎

Lemma 5 *(cf [16]) Let* u *be a solution of problem* $(\mathcal{V}.\mathcal{P}_{(m,\Omega)})$ *and* $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *then* $u \in C^{1,\alpha}(\Omega) \cap C^1(\bar{\Omega})$ *for some* $\alpha \in (0, 1)$*.*

3 Proof of main results

Proof of Theorem 1

For $n = 1$, we know that $\lambda_1(m, I)$ is simple, isolate and the corresponding eigenfunction has constant sign. Hence it has no zero in (a, b) and it remains to prove the SMP.

Proposition 1 *$\lambda_1(m, I)$ verifies the strict monotonicity property with respect to weight m and the domain I. i.e If $m, m^{'} \in M(I)$, $m(x) \leq m'(x)$ and $m(x) < m'(x)$ in some subset of I of nonzero measure then,*

$$\lambda_1(m', I) < \lambda_1(m, I) \tag{9}$$

and, if J is a strict sub interval of I such that $m_{/J} \in M(J)$ then,

$$\lambda_1(m, I) < \lambda_1(m_{/J}, J). \tag{10}$$

Proof Let $m, m^{'} \in M(I)$ as in Proposition 1 and recall that the principal eigenfunction $\phi_1 \in S$ corresponding to $\lambda_1(m, I)$ has no zero in I; i.e $\phi_1(t) \neq 0$ for all $t \in I$. By (3), we get

$$\frac{1}{\lambda_1(m, I)} = \int_I m|\phi_1|^p\,dx < \int_I m^{'}|\phi_1|^p\,dx \leq \sup_{v\in S}\int_I m^{'}|v|^p\,dx = \frac{1}{\lambda_1(m', I)}. \tag{11}$$

Then inequality (9) is proved. To prove inequality (10), let J be a strict sub interval of I and $m_{/J} \in M(J)$. Let $u_1 \in S$ be the (principal) positive eigenfunction of $(\mathcal{V}.\mathcal{P}_{(m,J)})$ corresponding to $\lambda_1(m_{/J}, J)$, and denote by $\tilde{u}_1$ the extension by zero on I. Then

$$\frac{1}{\lambda_1(m_{/J}, J)} = \int_J m|u_1|^p\,dx = \int_I m|\tilde{u}_1|^p\,dx < \sup_{v\in S}\int_I m|v|^p\,dx = \frac{1}{\lambda_1(m, I)}. \tag{12}$$

The last strict inequality holds from the fact that $\tilde{u}_1$ vanishes in I/J so can't be an eigenfunction corresponding to the principal eigenvalue $\lambda_1(m, I)$. ■

For $n = 2$ we start by proving that $\lambda_2(m, I)$ has a unique zero in (a, b).

Proposition 2 *There exists a unique real $c_{2,1} \in I$, for which we have $Z(u) = \{c_{2,1}\}$ for any eigenfunction u corresponding to $\lambda_2(m, I)$. For this reason, we will say that $c_{2,1}$ is the zero of $\lambda_2(m, I)$.*

Proof Let u be an eigenfunction corresponding to $\lambda_2(m, I)$. u changes sign in I. Consider $I_1 =]a, c[$ and $I_2 =]c', b[$ two nodal domains of u, by Lemma 3, $\lambda_1(m_{/I_1}, I_1) = \lambda_2(m, I) = \lambda_1(m_{/I_2}, I_2)$. Assume that $c < c'$, choose $d \in]c, c'[$ and put $J_1 =]a, d[$, $J_2 =]d, b[$; hence $J_1 \cap J_2 = \emptyset$, and for $i = 1, 2$, $I_i \subset J_i$ strictly, and $m_{/J_i} \in M(J_i)$. Making use of Lemma 3, by (10), we get

$$\lambda_1(m_{/J_1}, J_1) < \lambda_1(m_{/I_1}, I_1) = \lambda_2(m, I) \tag{13}$$

and

$$\lambda_1(m_{/J_2}, J_2) < \lambda_1(m_{/I_2}, I_2) = \lambda_2(m, I). \tag{14}$$

Let $\phi_i \in S$ be an eigenfunction corresponding to $\lambda_1(m_{/J_i}, J_i)$, by (4) we have for $i = 1, 2$

$$\frac{1}{\lambda_1(m, J_i)} = \int_{J_i} m|\phi_i|^p\,dx \tag{15}$$

$\tilde{\phi}_i$ is the extension by zero of ϕ_i on I. Consider the two dimensional subspace $F = \langle\tilde{\phi}_1, \tilde{\phi}_2\rangle$ and put $K_2 = F \cap S \subset W_0^{1,p}(I)$. Obviously $\gamma(K_2) = 2$ and we remark that for $v = \alpha\tilde{\phi}_1 + \beta\tilde{\phi}_2$, $\|v\|_{1,p} = 1 \Longleftrightarrow |\alpha|^p + |\beta|^p = 1$. Hence by (3), (13), (14) and (15) we obtain,

$$\begin{aligned}
\tfrac{1}{\lambda_2(m,I)} &\geq \min_{v\in K_2}\int_I m|v|^p\,dx \\
&= \min_{v=\alpha\tilde{\phi}_1+\beta\tilde{\phi}_2\in K_2}(|\alpha|^p\int_{J_1} m|\phi_1|^p + |\beta|^p\int_{J_2} m|\phi_2|^p) \\
&= |\alpha_0|^p \textstyle\int_{J_1} m|\phi_1|^p + |\beta_0|^p \int_{J_2} m|\phi_2|^p) \\
&= \tfrac{|\alpha_0|^p}{\lambda_1(m_{/J_1},J_1)} + \tfrac{|\alpha_0|^p}{\lambda_1(m_{/J_2},J_2)} \\
&> \tfrac{|\alpha_0|^p+|\beta_0|^p}{\lambda_2(m,I)} \\
&= \tfrac{1}{\lambda_2(m,I)},
\end{aligned}$$

a contradiction; hence $c = c'$. On the other hand, let v be another eigenfunction corresponding to $\lambda_2(m,I)$. Denote by d its unique zero in (a,b). Assume, for example, that $c < d$. By Lemma 3 and relation (10), we get

$$\lambda_2(m,I) = \lambda_1(m_{/]a,d[},]a,d[) < \lambda_1(m_{/]a,c[},]a,c[) = \lambda_2(m,I). \tag{16}$$

This is a contradiction so $c = d$. We have proved that every eigenfunction corresponding to $\lambda_2(m,I)$ has one, and only one, zero in (a,b), and that the zero is the same for all eigenfunctions, which completes the proof of the Proposition. ■

Lemma 6 *$\lambda_2(m,I)$ is simple, hence $\lambda_2(m,I) < \lambda_3(m,I)$.*

Proof Let u and v be two eigenfunctions corresponding to $\lambda_2(m,I)$. By Lemma 3 the restrictions of u and v on $]a,c_{2,1}[$ and $]c_{2,1},b[$ are eigenfunctions corresponding to $\lambda_1(m_{/]a,c_{2,1}[},]a,c_{2,1}[)$ and $\lambda_1(m_{/]c_{2,1},b[},]c_{2,1},b[)$, respectively. Making use of the simplicity of the first eigenvalue, we get $u = \alpha v$ in $]a,c_{2,1}[$ and $u = \beta v$ in $]c_{2,1},b[$. But both of u and v are eigenfunctions, so then by Lemma 5, there are $C^1(I)$. The maximum principle (*cf* [17]) tell us that $u'(c_{2,1}) \neq 0$, so $\alpha = \beta$. Finally, by the simplicity of $\lambda_2(m,I)$ and the theorem of multiplicity (*cf* [15]) we conclude that $\lambda_2(m,I) < \lambda_3(m,I)$. ■

Proposition 3 *$\lambda_2(m,I)$ verifies the SMP with respect to the weight m and the domain I.*

Proof Let m, $m' \in M(I)$ such that, $m(x) \leq m'(x)$ a.e in I and $m(x) < m'(x)$ in some subset of nonzero measure. Let $c_{2,1}$ and $c'_{2,1}$ be the zeros of $\lambda_2(m,I)$ and $\lambda_2(m',I)$ respectively. We distinguish three cases :

1. $c_{2,1} = c'_{2,1} = c$, then $meas(\{x \in I/\,m(x) < m'(x) \cap]a,c[\}) \neq 0$, or $meas(\{x \in I/\,m(x) < m'(x) \cap]c,b[\}) \neq 0$, by Lemma 3 and (9) we obtain

$$\lambda_2(m',I) = \lambda_1(m'_{/]a,c[},]a,c[) < \lambda_1(m_{/]a,c[},]a,c[) = \lambda_2(m,I), \tag{17}$$

or

$$\lambda_2(m',I) = \lambda_1(m'_{/]c,b[},]c,b[) < \lambda_1(m_{/]c,b[},]c,b[) = \lambda_2(m,I). \tag{18}$$

2. $c_{2,1} < c'_{2,1}$, by Lemmas 1, 3 and (10), we get

$$\begin{aligned}\lambda_2(m', I) = \lambda_1(m'_{/]a,c'_{2,1}[},]a, c'_{2,1}[) &\leq \lambda_1(m_{/]a,c'_{2,1}[},]a, c'_{2,1}[) \\ &< \lambda_1(m_{/]a,c_{2,1}[},]a, c_{2,1}[) = \lambda_2(m, I).\end{aligned} \tag{19}$$

3. $c'_{2,1} < c_{2,1}$, as before, by Lemmas 1, 3 and (10), we have

$$\begin{aligned}\lambda_2(m', I) = \lambda_1(m'_{/]c'_{2,1},b[},]c'_{2,1}, b[) &\leq \lambda_1(m_{/]c'_{2,1},b[},]c'_{2,1}, b[) \\ &< \lambda_1(m_{/]c_{2,1},b[},]c_{2,1}, b[) = \lambda_2(m, I).\end{aligned} \tag{20}$$

For the SMP with respect to the domain, put $J =]c, d[$ a strict sub interval of I with $m_{/J} \in M(J)$, and denote $c'_{2,1}$ the zero of $\lambda_2(m_{/J}, J)$. As in the SMP with respect to the weight, three cases are distinguished:

1. $c_{2,1} = c'_{2,1} = l$, then $]c, l[$ is a strict sub interval of $]a, l[$ or $]l, d[$ is a strict sub interval of $]l, b[$. By Lemma 3 and (10), we get

$$\lambda_2(m, I) = \lambda_1(m_{/]a,l[},]a, l[) < \lambda_1(m_{/]c,l[},]c, l[) = \lambda_2(m_{/J}, J) \tag{21}$$

or

$$\lambda_2(m, I) = \lambda_1(m_{/]l,b[},]l, b[) < \lambda_1(m_{/]l,d[},]l, d[) = \lambda_2(m_{/J}, J). \tag{22}$$

2. $c_{2,1} < c'_{2,1}$, again by Lemma 3 and (10), we obtain

$$\lambda_2(m, I) = \lambda_1(m_{/]c_{2,1},b[},]c_{2,1}, b[) < \lambda_1(m_{/]c'_{2,1},d[},]c'_{2,1}, b[) = \lambda_2(m_{/J}, J). \tag{23}$$

3. $c'_{2,1} < c_{2,1}$, for the same reason as in the last case, we get

$$\lambda_2(m, I) = \lambda_1(m_{/]a,c_{2,1}[},]a, c_{2,1}[) < \lambda_1(m_{/]c,c'_{2,1}[},]c, c_{2,1}[) = \lambda_2(m_{/J}, J). \tag{24}$$

The proof is complete. ■

Lemma 7 *If any eigenfunction u corresponding to some eigenvalue $\lambda(m, I)$ is such that $Z(u) = \{c\}$ for some real number c, then $\lambda(m, I) = \lambda_2(m, I)$.*

Proof We shall prove that $c = c_{2,1}$. Assume, for example, that $c < c_{2,1}$. By Lemma 1 and (10) we get

$$\lambda(m, I) = \lambda_1(m_{/]c,b[},]c, b[) < \lambda_1(m_{/]c_{2,1},b[},]c_{2,1}, b[) = \lambda_2(m, I). \tag{25}$$

On the other hand,

$$\lambda_2(m, I) = \lambda_1(m_{/]a,c_{2,1}[},]a, c_{2,1}[) < \lambda_1(m_{/]a,c[},]a, c[) = \lambda(m, I), \tag{26}$$

a contradiction. Hence, $c = c_{2,1}$ and $\lambda(m, I) = \lambda_2(m, I)$. The proof is complete. ■

For $n > 2$, we use a recurrence argument. Assume that, for any k, $1 \leq k \leq n$, that the following hypothesis:

1. **H.R.1** For any eigenfunction u corresponding to the k-th eigenvalue $\lambda_k(m, I)$, there exists a unique $c_{k,i}$, $1 \le i \le k-1$, such that $Z(u) = \{c_{k,i},\ 1 \le i \le k-1\}$.

2. **H.R.2** $\lambda_k(m, I)$ is simple.

3. **H.R.3** $\lambda_1(m, I) < \lambda_2(m, I) < \cdots < \lambda_{n+1}(m, I)$.

4. **H.R.4** If $(u, \lambda(m, I))$ is a solution of $(\mathcal{V}.\mathcal{P}_{(m,I)})$ such that $Z(u) = \{c_i,\ 1 \le i \le k-1\}$, then $\lambda(m, I) = \lambda_k(m, I)$.

5. **H.R.5** $\lambda_k(m, I)$ verifies the SMP with respect to the weight m and the domain I.

Holds, and prove them for $n + 1$.

Proposition 4 *There exists a unique family $\{c_{n+1,i},\ 1 \le i \le n\}$ such that $Z(u) = \{c_{n+1,i},\ 1 \le i \le n\}$, for any eigenfunction u corresponding to $\lambda_{n+1}(m, I)$.*

Proof Let u be an eigenfunction corresponding to $\lambda_{n+1}(m, I)$. By H.R.3 and H.R.4, u has at least n zeros. According to Lemma 4, we can consider the $n + 1$ nodal domains of u, $I_1 =]a, c_1[$, $I_2 =]c_1, c_2[$, ... , $I_n =]c_{n-1}, c_n[$, $I_{n+1} =]c, b[$. We shall prove that $c = c_n$. Remark that the restrictions of u on $]a, c_{i+1}[$, $1 \le i \le n-1$, are eigenfunctions with i zeros, by H.R.4 $\lambda_{n+1}(m, I) = \lambda_i(m_{/]a,c_{i+1}[},]a, c_{i+1}[)$. Assume that $c_n < c$, choose d in $]c_n, c[$ and put, $J_1 =]a, d[$, $J_2 =]d, b[$. Remark that $J_1 \cap J_2 = \emptyset$, $]a, c_n[$ is a strict sub interval of $J_1 \subset I$, and $]c, b[$ is a strict sub interval of $J_2 \subset I$. It is clear that $m_{/J_i} \in M(J_i)$ for $i = 1, 2$, by H.R.4 and H.R.5. We have

$$\lambda_n(m_{/J_1}, J_1) < \lambda_n(m_{/]a,c_n[},]a, c_n[) = \lambda_{n+1}(m, I), \tag{27}$$

and

$$\lambda_1(m_{/J_2}, J_2) < \lambda_1(m_{/]c,b[},]c, b[) = \lambda_{n+1}(m, I). \tag{28}$$

Denote by $(\phi_{n+1}, \lambda_1(m_{/J_2}, J_2))$ a solution of $(\mathcal{V}.\mathcal{P}_{(m,J_2)})$, $(v, \lambda_n(m_{/J_1}, J_1))$ a solution of $(\mathcal{V}.\mathcal{P}_{(m,J_1)})$, ϕ_i, $1 \le i \le n$ the restrictions of v on I_i, and $\tilde{\phi}_i$, $1 \le i \le n$, their extensions, by zero, on I. Let $F_{n+1} = \langle \tilde{\phi}_1, \tilde{\phi}_2, \cdots, \tilde{\phi}_{n+1} \rangle$ and $K_{n+1} = F_{n+1} \cap S$, then $\gamma(K_{n+1}) = n + 1$. We obtain by (3) and the same proof as in Proposition 2

$$\frac{1}{\lambda_{n+1}(m, I)} \ge \min_{K_{n+1}} \int_I m|v|^p\, dx > \frac{1}{\lambda_{n+1}(m, I)}, \tag{29}$$

a contradiction, so $c = c_n$. On the other hand, let v be an eigenfunction corresponding to $\lambda_{n+1}(m, I)$. Denote by d_1, d_2, $\cdots, d_n$ the zeros of v. If $d_1 \ne c_1$, then $\lambda_{n+1}(m, I) = \lambda_1(m_{/]a,d_1[},]a, d_1[) \ne \lambda_1(m_{/]a,c_1[},]a, c_1[) = \lambda_{n+1}(m, I)$, so $d_1 = c_1$, by the same argument we conclude that $d_i = c_i$ for all $1 \le i \le n$. ■

Lemma 8 *$\lambda_{n+1}(m, I)$ is simple, hence. $\lambda_{n+1}(m, I) < \lambda_{n+2}(m, I)$.*

Proof Let u and v be two eigenfunctions corresponding to $\lambda_{n+1}(m, I)$. The restrictions of u and v on $]a, c_{n+1,1}[$ and $]c_{n+1,1}, b[$ respectively, are eigenfunctions corresponding to $\lambda_1(m_{/]a,c_{n+1,1}[},]a, c_{n+1,1}[)$ and $\lambda_n(m_{/]c_{n+1,1},b[},]c_{n+1,1}, b[)$. By H.R.2 and H.R.4 we have $u = \alpha v$ in $]a, c_{n+1,1}[$ and $u = \beta v$ in $]c_{n+1,1}, b[$. On the other hand, u and v are $C^1(I)$ and $u'(c_{n+1,1}) \neq 0$, so $\alpha = \beta$. From the simplicity of $\lambda_{n+1}(m, I)$ and theorem of multiplicity we conclude that $\lambda_{n+1}(m, I) < \lambda_{n+2}(m, I)$. ■

Proposition 5 *$\lambda_{n+1}(m, I)$ verifies the SMP with respect to the weight m and the domain I.*

Proof Let m, $m' \in M(I)$, such that $m(x) \leq m'(x)$ with $m(x) < m'(x)$ in some subset of nonzero measure and $(c'_{n+1,i})_{1 \leq i \leq n}$ the zeros of $\lambda_{n+1}(m')$ three cases are distinguished,

1. $c_{n+1,1} = c'_{n+1,1} = c$, one of the subsets is of nonzero measure,
$$\{x \in I / \, m(x) < m'(x)\} \cap]a, c[\quad \textit{and} \quad \{x \in I / \, m(x) < m'(x)\} \cap]c, b[.$$
By Lemma 3 and (9), we have
$$\lambda_{n+1}(m', I) = \lambda_1(m'_{/]a,c[},]a, c[) < \lambda_1(m_{/]a,c[},]a, c[) = \lambda_{n+1}(m, I) \tag{30}$$
or
$$\lambda_{n+1}(m', I) = \lambda_n(m'_{/]c,b[},]c, b[) < \lambda_n(m_{/]c,b[},]c, b[) = \lambda_{n+1}(m, I). \tag{31}$$
2. $c_{n+1,1} < c'_{n+1,1}$, by Lemmas 1, 3 and (10) we have
$$\begin{array}{rcl} \lambda_{n+1}(m', I) & = & \lambda_1(m'_{/]a,c'_{n+1,1}[},]a, c'_{n+1,1}[) \\ & \leq & \lambda_1(m_{/]a,c'_{n+1,1}[},]a, c'_{n+1,1}[) \\ & < & \lambda_1(m_{/]a,c_{n+1,1}[},]a, c_{n+1,1}[) \quad = \quad \lambda_{n+1}(m, I). \end{array} \tag{32}$$
3. $c'_{n,1} < c_{n,1}$, from the same reason as before, we get
$$\begin{array}{rcl} \lambda_{n+1}(m', I) & = & \lambda_n(m'_{/]c'_{n+1,1},b[},]c'_{n+1,1}, b[) \\ & \leq & \lambda_n(m_{/]c'_{n+1,1},b[},]c'_{n+1,1}[, b) \\ & < & \lambda_n(m_{/]c_{n+1,1},b[},]c_{n+1,1}, b[) \quad = \quad \lambda_{n+1}(m, I). \end{array} \tag{33}$$

By similar argument as in proof of Proposition 3, we prove the SMP with respect to the domain I. ■

Lemma 9 *If $(u, \lambda(m, I))$ is a solution of $(\mathcal{V}.\mathcal{P}_{(m,I)})$ such that $Z(u) = \{d_1, d_2, \cdots d_n\}$, then $\lambda(m, I) = \lambda_{n+1}(m, I)$.*

Proof It is sufficient to prove that $d_i = c_{n+1,i}$ for all $1 \leq i \leq n$. If $c_{n+1,1} < d_1$ then, by Lemma 1, (10), H.R.4 and H.R.5,
$$\begin{array}{rcl} \lambda(m, I) = \lambda_1(m_{/]a,d_1[},]a, d_1[) & < & \lambda_1(m_{/]a,c_{n+1,1}[},]a, c_{n+1,1}[) \\ & = & \lambda_{n+1}(m, I) \\ & = & \lambda_n(m_{/]c_{n+1,1},b[},]c_{n+1,1}, b[) \\ & < & \lambda_n(m_{/]d_1,b[},]d_1, b[) \\ & = & \lambda(m, I), \end{array} \tag{34}$$

a contradiction. If $d_1 < c_{n+1,1}$ then, by Lemma 1, (10), H.R.4 and H.R.5,

$$\begin{aligned}\lambda_{n+1}(m,I) = \lambda_1(m_{/]a,c_{n+1}[},]a,c_{n+1}[) &< \lambda_1(m_{/]a,d_1[},]a,d_1[) \\ &= \lambda(m,I) \\ &= \lambda_n(m_{/]d_1,b[},]d_1,b[) \\ &< \lambda_n(m_{/]c_{n+1,1},b[},]c_{n+1},b[) \\ &= \lambda_{n+1}(m,I),\end{aligned} \tag{35}$$

a contradiction. The proof is then complete, which completes the proof of Theorem 1. ■

Proof of Corollary 1. Since for $F \in \mathcal{F}_n$, the compact $F \cap S \in \mathcal{A}_n$, by (3) we have:

$$\sup_{F\in\mathcal{F}_n} \min_{v\in F\cap S} \int_\Omega m|v|^p\,dx \le \frac{1}{\lambda_n(m,\Omega)}. \tag{36}$$

On the other hand, for a n dimensional subspace F of $W_0^{1,p}(I)$, the compact set $K = F \cap S \in \mathcal{A}_n$. Let u be an eigenfunction corresponding to $\lambda_n(m,I)$ and put

$$F = \langle \tilde{\phi}_1(]a,c_{n,1}[), \tilde{\phi}_1(]c_{n,1},c_{n,2}[), \cdots, \tilde{\phi}_1(]c_{n_1,n},b[) \rangle,$$

to conclude $F \cap S \in \mathcal{A}_n$. By an elementary computation as in Proposition 2, one can show that

$$\frac{1}{\lambda_n(m,I)} = \min_{F\cap S} \int_I m|v|^p\,dx. \tag{37}$$

Then combine (36) with (37) to get (5). Which completes the proof. ■

3.1 Remark

The spectrum of p-Laplacian, with indefinite weight, in one dimension, is entirely determined by the sequence $(\lambda_n(m,I))_{n\ge1}$ if $m(x) \ge 0$ a.e in I. Therefore, if $m(x) < 0$ in some subset $J \subset I$ of nonzero measure, replace m by $-m$; by Theorem 1, since $-m \in M(I)$ we conclude that, the negative spectrum $\sigma_p^-(-\Delta_p,m) = -\sigma_p^+(-\Delta_p,-m)$ of this operator is constituted by a sequence of eigenvalues $\lambda_{-n}(m,I) = -\lambda_n(-m,I)$. Thus the spectrum of the operator is,

$$\sigma_p(-\Delta_p,m) = \sigma_p^+(-\Delta_p,m) \cup \sigma_p^-(-\Delta_p,m).$$

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