Electronic Journal of Qualitative Theory of Differential Equations
2012, No. **90**, 1-12; **http://www.math.u-szeged.hu/ejqtde/**

Minimal Wave Speed of Traveling Wavefronts in Delayed Belousov-Zhabotinskii Model

Shuxia Pan*†, Jie Liu
Department of Applied Mathematics, Lanzhou University of Technology,
Lanzhou, Gansu 730050, People's Republic of China

Abstract

This paper is concerned with the traveling wavefronts of Belousov-Zhabotinskii model with time delay. By constructing upper and lower solutions and applying the theory of asymptotic spreading, the minimal wave speed is obtained under the weaker condition than that in the known results. Moreover, the strict monotonicity of any monotone traveling wavefronts is also established.

Keywords: Comparison principle; asymptotic spreading; minimal wave speed.
AMS Subject Classification (2010): 35C07, 35K57, 37C65.

1 Introduction

For the traveling wavefronts of parabolic equations, one well studied model is the following simplified Belousov-Zhabotinskii system

$$\begin{cases} \frac{\partial U(x,t)}{\partial t} = \Delta U(x,t) + U(x,t)\left[1 - U(x,t) - rV(x,t)\right], \\ \frac{\partial V(x,t)}{\partial t} = \Delta V(x,t) - bU(x,t)V(x,t), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}, t > 0$, $r \in (0,1)$ and $b > 0$ are constants, and $U, V \in \mathbb{R}$ correspond to the concentration of Bromic acid and bromide ion in chemical reaction, respectively. Moreover, (1.1) was also derived from biochemical and biological fields, see Belousov [2], Murray [7–9], Zaikin and Zhabotinskii [13], Zhabotinskii [14]. Recalling the chemical and biological backgrounds of (1.1), much attention has been paid to the following asymptotical

*Corresponding author. E-mail: shxpan@yeah.net.
†Supported by the Development Program for Outstanding Young Teachers in Lanzhou University of Technology (1010ZCX019).

boundary conditions (see Kanel [3], Kapel [4], Troy [10], Ye and Wang [12])

$$\begin{cases} \lim_{x\to-\infty} U(x,t) = 0, \lim_{x\to-\infty} V(x,t) = 1, \\ \lim_{x\to\infty} U(x,t) = 1, \lim_{x\to\infty} V(x,t) = 0, \end{cases} \tag{1.2}$$

which formulates a spatial-temporal transformation process of bromide element with different atomicity in the chemical reaction.

Recently, Lin and Li [5], Ma [6], Wu and Zou [11] further investigated the traveling wavefronts of the following delayed system

$$\begin{cases} \frac{\partial U(x,t)}{\partial t} = \Delta U(x,t) + U(x,t)\left[1 - U(x,t) - rV(x,t-\tau)\right], \\ \frac{\partial V(x,t)}{\partial t} = \Delta V(x,t) - bU(x,t)V(x,t), \end{cases} \tag{1.3}$$

where $\tau \geq 0$ is the time delay and other parameters are the same to those in (1.1). Clearly, if $\tau = 0$, then (1.3) becomes (1.1). So, we only discuss (1.3) in what follows.

Because (1.3) is not a monotone system in the sense of the standard partial ordering of the phase space of (1.3), we first make a change of variables such that we can study it by usual partial ordering when the theory of classical monotone dynamical systems is utilized. Let $u = U$ and $v = 1 - V$, then (1.3) becomes

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + u(x,t)\left[1 - r - u(x,t) + rv(x,t-\tau)\right], \\ \frac{\partial v(x,t)}{\partial t} = \Delta v(x,t) + bu(x,t)\left[1 - v(x,t)\right], \end{cases} \tag{1.4}$$

and (1.2) leads to

$$\lim_{x\to-\infty} u(x,t) = \lim_{x\to-\infty} v(x,t) = 0, \lim_{x\to\infty} u(x,t) = \lim_{x\to\infty} v(x,t) = 1. \tag{1.5}$$

Hereafter, a traveling wavefront of (1.4) is a special solution taking the following form

$$(u(x,t), v(x,t)) = (\phi(\xi), \psi(\xi)), \xi = x + ct,$$

in which $(\phi(\xi), \psi(\xi)) \in C^2(\mathbb{R}, \mathbb{R}^2)$ is the wave profile which is monotone in $\xi \in \mathbb{R}$ while $c > 0$ formulates the wave speed. Clearly, (ϕ, ψ) should satisfy

$$\begin{cases} c\phi'(\xi) = \phi''(\xi) + \phi(\xi)\left[1 - r - \phi(\xi) + r\psi(\xi - c\tau)\right], \\ c\psi'(\xi) = \psi''(\xi) + b\phi(\xi)\left[1 - \psi(\xi)\right], \end{cases} \tag{1.6}$$

and the asymptotical boundary conditions formulated by

$$\lim_{\xi\to-\infty} \phi(\xi) = \lim_{\xi\to-\infty} \psi(\xi) = 0, \lim_{\xi\to\infty} \phi(\xi) = \lim_{\xi\to\infty} \psi(\xi) = 1. \tag{1.7}$$

In literature, the existence of (1.6)-(1.7) has been widely studied by many investigators. For example, when $\tau = 0$, Troy [10] proved that for each $b > 0$, there exist $r^* > 0, c^* \in (0, 2]$ such that (1.6)-(1.7) has a positive solution if we take $c = c^*, r = r^*$. However, the accurate presentation of c^*, r^* is not given and whether such a c^* is the minimal wave speed is also open, here the minimal wave speed c^* implies that (1.6)-(1.7) has no positive solution for $c < c^*$ while has a positive solution for $c \geq c^*$. Moreover, for $\tau = 0$, Murray [8, pp.326] obtained the following bounds on minimal wave speed c in terms of r, b given by

$$\frac{\left(\sqrt{r^2 + 2b/3} - r\right)}{\sqrt{2(b + 2r)}} \leq c \leq 2.$$

Furthermore, some results of existence of (1.6)-(1.7) with $\tau = 0$ were obtained by constructing upper and lower solutions and it was proved that $c_1 =: 2\sqrt{1-r}$ is the minimal wave speed if $b \leq 1 - r$, we refer to Ye and Wang [12].

Recently, Ma [6], Wu and Zou [11] studied the existence of traveling wave solutions of (1.3) for $c > c_1$ and $b \leq 1 - r$. Lin and Li [5] further proved the nonexistence of traveling wave solutions of (1.3) when the wave speed $c < c_1$. At the same time, Lin and Li [5] also investigated the existence of (1.6) when $c = c_1$ holds, but the authors did not confirm (1.7). Therefore, even if $b \leq 1-r$, some problems of the existence/nonexistence of (1.6)-(1.7) remain open. Moreover, the role of time delay has *not* been reflected in these results, and we shall further consider the existence and nonexistence of (1.6)-(1.7).

In this paper, by constructing upper and lower solutions and utilizing the theory of asymptotic spreading, we prove that c_1 is the minimal wave speed of (1.6)-(1.7) if

$$b \leq \frac{(1-r)e^{(1-r)\tau}}{r},$$

which extends/completes some known results. In particular, it also confirms the conjecture of Lin and Li [5, Remark 2.4] by obtaining the asymptotic behavior of traveling wave solutions with $c = c_1$. Moreover, we also consider the strict monotonicity of traveling wavefronts and prove the strict monotonicity of any monotone solutions of (1.6)-(1.7).

2 Preliminaries

In this section, we provide some known results such that we can follow our discussion in the subsequent section. Let X be defined by

$$X = C(\mathbb{R}, \mathbb{R}^2) = \left\{u(x) | u(x) : \mathbb{R} \to \mathbb{R}^2 \text{ is uniformly continuous and bounded}\right\}.$$

If $(\phi, \psi) \in X$, then we denote $(H_1, H_2) : X \to X$ as follows

$$\begin{cases} H_1(\phi, \psi)(\xi) = 2\phi(\xi) + \phi(\xi)\left[1 - r - \phi(\xi) + r\psi(\xi - c\tau)\right], \\ H_2(\phi, \psi)(\xi) = b\psi(\xi) + b\phi(\xi)\left[1 - \psi(\xi)\right]. \end{cases}$$

Using the notation, (1.6) equals to

$$\begin{cases} c\phi'(\xi) = \phi''(\xi) - 2\phi(\xi) + H_1(\phi, \psi)(\xi), \\ c\psi'(\xi) = \psi''(\xi) - b\psi(\xi) + H_2(\phi, \psi)(\xi). \end{cases}$$

Let $c > 0$. Choose constants as follows

$$\gamma_1(c) = \frac{c - \sqrt{c^2 + 8}}{2}, \qquad \gamma_2(c) = \frac{c + \sqrt{c^2 + 8}}{2},$$
$$\gamma_3(c) = \frac{c - \sqrt{c^2 + 4b}}{2}, \qquad \gamma_4(c) = \frac{c + \sqrt{c^2 + 4b}}{2}.$$

Moreover, for $(\phi, \psi) \in X$, define an operator $F = (F_1, F_2) : X \to X$ by

$$\begin{aligned} F_1(\phi, \psi)(\xi) &= \frac{1}{\gamma_2(c) - \gamma_1(c)} \left[\int_{-\infty}^{\xi} e^{\gamma_1(c)(\xi - s)} + \int_{\xi}^{\infty} e^{\gamma_2(c)(\xi - s)}\right] H_1(\phi, \psi)(s) ds, \\ F_2(\phi, \psi)(\xi) &= \frac{1}{\gamma_4(c) - \gamma_3(c)} \left[\int_{-\infty}^{\xi} e^{\gamma_3(c)(\xi - s)} + \int_{\xi}^{\infty} e^{\gamma_4(c)(\xi - s)}\right] H_2(\phi, \psi)(s) ds, \end{aligned}$$

which was earlier used by Wu and Zou [11]. Then, it is evident that a fixed point of F is a solution of (1.6) and it is sufficient to study the fixed point of F. In particular, we present a nice property of F as follows.

Lemma 2.1 *Assume that* $(0, 0) \le (\phi_1(\xi), \psi_1(\xi)) \le (\phi_2(\xi), \psi_2(\xi)) \le (1, 1), \xi \in \mathbb{R}$. *Then*

$$(0, 0) \le F_i(\phi_1, \psi_1)(\xi) \le F_i(\phi_2, \psi_2)(\xi) \le (1, 1), i = 1, 2.$$

Lemma 2.1 is a special form of the comparison principle listed by [6, 11] and we omit its proof here. To proceed our discussion, we need the following definition of upper and lower solutions (see [6, 11]).

Definition 2.2 Assume that $(\overline{\phi}(\xi), \overline{\psi}(\xi)) \in X$ satisfies $(0, 0) \le (\overline{\phi}(\xi), \overline{\psi}(\xi)) \le (1, 1), \xi \in \mathbb{R}$ and is twice differentiable on $\mathbb{R} \backslash \mathbb{T}$, where $\mathbb{T}$ contains finite points of $\mathbb{R}$. If $\overline{\phi}'(\xi), \overline{\psi}'(\xi), \overline{\phi}''(\xi), \overline{\psi}''(\xi)$ are essentially bounded and

$$\begin{cases} c\overline{\phi}'(\xi) \ge (\le) \overline{\phi}''(\xi) + \overline{\phi}(\xi)\left[1 - r - \overline{\phi}(\xi) + r\overline{\psi}(\xi)\right], \\ c\overline{\psi}'(\xi) \ge (\le) \overline{\psi}''(\xi) + b\overline{\phi}(\xi)\left[1 - \overline{\psi}(\xi)\right] \end{cases} \tag{2.1}$$

for $\xi \in \mathbb{R} \setminus \mathbb{T}$, then $(\overline{\phi}(\xi), \overline{\psi}(\xi))$ is called an *upper* (a *lower*) solution of (1.6).

By what we have done, we present the following existence result of (1.6).

Lemma 2.3 *Assume that* $(\overline{\phi}(\xi), \overline{\psi}(\xi))$ *is an upper solution of* (1.6) *while* $(\underline{\phi}(\xi), \underline{\psi}(\xi))$ *is a lower solution of* (1.6). *Also suppose that*

(A1) $(\underline{\phi}(\xi), \underline{\psi}(\xi)) \leq (\overline{\phi}(\xi), \overline{\psi}(\xi)), \xi \in \mathbb{R}$;

(A2) $(\overline{\phi}'(\xi-), \overline{\psi}'(\xi-)) \geq (\overline{\phi}'(\xi+), \overline{\psi}'(\xi+))$, $(\underline{\phi}'(\xi-), \underline{\psi}'(\xi-)) \leq (\underline{\phi}'(\xi+), \underline{\psi}'(\xi+)), \xi \in \mathbb{T}$;

(A3) $\sup_{s<\xi} \underline{\phi}(\xi) \leq \inf_{s>\xi} \overline{\phi}(\xi), \sup_{s<\xi} \underline{\psi}(\xi) \leq \inf_{s>\xi} \overline{\psi}(\xi), \xi \in \mathbb{R}$.

Then (1.6) *has a monotone solution* $(\phi^*(\xi), \psi^*(\xi))$ *such that*

$$(F_1(\underline{\phi}(\xi), \underline{\psi}(\xi)), F_2(\underline{\phi}(\xi), \underline{\psi}(\xi))) \leq (\phi^*(\xi), \psi^*(\xi)) \leq (F_1(\overline{\phi}(\xi), \overline{\psi}(\xi)), F_2(\overline{\phi}(\xi), \overline{\psi}(\xi))) \quad (2.2)$$

for any $\xi \in \mathbb{R}$.

Proof. By Ma [6], Wu and Zou [11], (1.6) has a monotone solution (ϕ^*, ψ^*) satisfying

$$(\underline{\phi}(\xi), \underline{\psi}(\xi)) \leq (\phi^*(\xi), \psi^*(\xi)) \leq (\overline{\phi}(\xi), \overline{\psi}(\xi)), \xi \in \mathbb{R}.$$

By the definition of F, we see that (ϕ^*, ψ^*) is also a fixed point of F. Therefore, Lemma 2.1 further indicates (2.2). The proof is complete. □

We now recall some conclusions of the so-called Fisher equation as follows

$$\begin{cases} \frac{\partial w(x,t)}{\partial t} = \Delta w(x,t) + dw(x,t)\left[1 - w(x,t)\right], \\ w(x,0) = w(x), \end{cases} \quad (2.3)$$

in which $d > 0$ holds and $w(x)$ is bounded and uniformly continuous.

Lemma 2.4 (see [1]) *If* $w(x) > 0$, *then*

$$\liminf_{t\to\infty} \inf_{|x|<ct} w(x,t) = \limsup_{t\to\infty} \sup_{|x|<ct} w(x,t) = 1$$

for any $c \in \left(0, 2\sqrt{d}\right)$.

Moreover, let $w(x,t) = \rho(\xi), \xi = x + ct$ be a traveling wavefront of the following Fisher equation

$$\frac{\partial w(x,t)}{\partial t} = \Delta w(x,t) + dw(x,t)\left[1 - w(x,t)\right]. \quad (2.4)$$

Then the following result is well known and can be found in many textbooks.

Lemma 2.5 **(T1)** *If $c > 2\sqrt{d}$, then* (2.4) *has a unique traveling wavefront $\rho(\xi)$ such that*

$$\lim_{\xi\to-\infty}\rho(\xi)=0,\ \lim_{\xi\to\infty}\rho(\xi)=1,\ \lim_{\xi\to-\infty}\rho(\xi)e^{-\lambda(c)\xi}=1,\ \rho(\xi)<\min\{1,e^{\lambda(c)\xi}\},$$

in which $\lambda(c)=\frac{c-\sqrt{c^2-4d}}{2}$.

(T2) *If $c = 2\sqrt{d}$, then* (2.4) *has a traveling wavefront $\rho(\xi)$ satisfying*

$$\lim_{\xi\to-\infty}\rho(\xi)=0,\ \lim_{\xi\to\infty}\rho(\xi)=1,\ \lim_{\xi\to-\infty}\rho(\xi)e^{-\sqrt{d}\xi}/\xi=-1.$$

We also state the following standard comparison principle.

Lemma 2.6 *Assume that $\overline{w}(x,t)$ is bounded and*

$$\begin{cases}\frac{\partial\overline{w}(x,t)}{\partial t}\geq\Delta\overline{w}(x,t)+d\overline{w}(x,t)\left[1-\overline{w}(x,t)\right],\\ \overline{w}(x,0)\geq w(x).\end{cases}\tag{2.5}$$

Then $\overline{w}(x,t)\geq w(x,t)$.

3 Main Results

We first present our main result as follows.

Theorem 3.1 *Assume that*

$$b\leq\frac{(1-r)e^{(1-r)\tau}}{r}.$$

Then c_1 is the minimal wave speed of (1.6)-(1.7)*. Moreover, if $c\geq c_1$, then a monotone solution of* (1.6)-(1.7) *is strictly monotone.*

We now prove Theorem 3.1 by five lemmas as follows.

Lemma 3.2 *If*

$$b\leq(1-r)e^{(1-r)\tau}/r\ \text{and}\ c>c_1,$$

then (1.6)-(1.7) *has a strict positive solution which is also monotone.*

Proof. Define positive constant $\mu(c)=\frac{c-\sqrt{c^2-4(1-r)}}{2}$, then

$$c\mu(c)=\mu^2(c)+1-r>1-r.$$

Further construct continuous functions as follows

$$\overline{\phi}(\xi)=\min\{e^{\mu(c)\xi},1\},\overline{\psi}(\xi)=\min\{e^{\mu(c)\xi+(1-r)\tau}/r,1\}.$$

Moreover, let $\underline{\phi}(\xi)$ be the unique positive solution of

$$c\phi'(\xi) = \phi''(\xi) + \phi(\xi)\left[1 - r - \phi(\xi)\right], \lim_{\xi\to-\infty} \phi(\xi)e^{-\mu(c)\xi} = 1.$$

Then Lemma 2.5 implies that

$$(\underline{\phi}(\xi), \underline{\psi}(\xi)) \le (\overline{\phi}(\xi), \overline{\psi}(\xi)), \xi \in \mathbb{R}$$

with $\underline{\psi}(\xi) = 0$. Moreover, these functions also satisfy (A1)-(A3) of Lemma 2.3.

Claim 1. $(\overline{\phi}(\xi), \overline{\psi}(\xi))$ is an upper solution while $(\underline{\phi}(\xi), \underline{\psi}(\xi))$ is a lower solution of (1.6).

Applying Lemma 2.3, we see that (1.6) has a positive monotone solution (ϕ^*, ψ^*) satisfying (2.2). Then $\lim_{\xi\to\pm\infty}(\phi^*(\xi), \psi^*(\xi))$ exist and

$$\lim_{\xi\to-\infty}(\phi^*(\xi), \psi^*(\xi)) = (0,0), (1,1) \ge \lim_{\xi\to\infty}(\phi^*(\xi), \psi^*(\xi)) \gg (0,0)$$

by Lemma 2.1. Applying Lebesgue dominated theorem in F, we see that

$$\lim_{\xi\to\infty}(\phi^*(\xi), \psi^*(\xi)) = (1,1)$$

because

$$1 - r - u + rv \le 0, 1 - v \le 0, 0 < u, v \le 1$$

implies $u = v = 1$. Therefore, we only need to verify Claim 1.

Proof of Claim 1. For $\overline{\phi}(\xi)$, it suffices to verify that

$$c\overline{\phi}'(\xi) \ge \overline{\phi}''(\xi) + \overline{\phi}(\xi)\left[1 - r - \overline{\phi}(\xi) + r\overline{\psi}(\xi - c\tau)\right] \tag{3.1}$$

when $\xi \ne 0$. If $\overline{\phi}(\xi) = 1 < e^{\mu(c)\xi}$, then $\overline{\psi}(\xi - c\tau) \le 1$ implies that

$$\begin{aligned} & \overline{\phi}''(\xi) - c\overline{\phi}'(\xi) + \overline{\phi}(\xi)\left[1 - r - \overline{\phi}(\xi) + r\overline{\psi}(\xi - c\tau)\right] \\ \le\ & \overline{\phi}''(\xi) - c\overline{\phi}'(\xi) + \overline{\phi}(\xi)\left[1 - r - \overline{\phi}(\xi) + r\right] \\ =\ & 0. \end{aligned}$$

If $\overline{\phi}(\xi) = e^{\mu(c)\xi} < 1$, then $\overline{\psi}(\xi) \le e^{\mu(c)\xi + (1-r)\tau}/r$ such that

$$\begin{aligned} & -\overline{\phi}(\xi) + r\overline{\psi}(\xi - c\tau) \\ \le\ & -e^{\mu(c)\xi} + e^{\mu(c)(\xi - c\tau) + (1-r)\tau} \\ \le\ & -e^{\mu(c)\xi} + e^{\mu(c)\xi - c\mu(c)\tau + (1-r)\tau} \\ \le\ & 0 \end{aligned}$$

and

$$\overline{\phi}''(\xi) - c\overline{\phi}'(\xi) + \overline{\phi}(\xi)\left[1 - r - \overline{\phi}(\xi) + r\overline{\psi}(\xi - c\tau)\right]$$

$$\begin{aligned}
&\le \overline{\phi}''(\xi) - c\overline{\phi}'(\xi) + \overline{\phi}(\xi)\left[1-r\right] \\
&= e^{\mu(c)\xi}\left[\mu^2(c) - c\mu(c) + 1 - r\right] \\
&= 0.
\end{aligned}$$

For $\overline{\psi}(\xi)$, we need to prove that

$$c\overline{\psi}'(\xi) \ge \overline{\psi}''(\xi) + b\overline{\phi}(\xi)\left[1-\overline{\psi}(\xi)\right] \tag{3.2}$$

if $\xi \in \mathbb{R}$ satisfies $e^{\mu(c)\xi+(1-r)\tau}/r \ne 1$.

When $\overline{\psi}(\xi) = e^{\mu(c)\xi+(1-r)\tau}/r < 1$, then $\overline{\phi}(\xi) \le e^{\mu(c)\xi}$ such that

$$\begin{aligned}
&\overline{\psi}''(\xi) - c\overline{\psi}'(\xi) + b\overline{\phi}(\xi)\left[1-\overline{\psi}(\xi)\right] \\
\le\ &\overline{\psi}''(\xi) - c\overline{\psi}'(\xi) + b\overline{\phi}(\xi) \\
\le\ &\frac{e^{\mu(c)\xi+(1-r)\tau}}{r}[\mu^2(c) - c\mu(c) + bre^{(r-1)\tau}] \\
\le\ &\frac{e^{\mu(c)\xi+(1-r)\tau}}{r}[\mu^2(c) - c\mu(c) + 1 - r] \\
=\ &0.
\end{aligned}$$

If $\overline{\psi}(\xi) = 1 < e^{\mu(c)\xi+(1-r)\tau}/r$, then it is clear that

$$\overline{\psi}''(\xi) - c\overline{\psi}'(\xi) + b\overline{\phi}(\xi)\left[1-\overline{\psi}(\xi)\right] = 0.$$

These complete the proof of (3.2).

Clearly, $(\underline{\phi}(\xi), \underline{\psi}(\xi))$ satisfies

$$\begin{cases} c\underline{\phi}'(\xi) = \underline{\phi}''(\xi) + \underline{\phi}(\xi)\left[1 - r - \underline{\phi}(\xi) + r\underline{\psi}(\xi)\right], \\ c\underline{\psi}'(\xi) \le \underline{\psi}''(\xi) + b\underline{\phi}(\xi)\left[1 - \underline{\psi}(\xi)\right], \end{cases} \tag{3.3}$$

and this indicates Claim 1. By what we have done, the proof is complete. □

Lemma 3.3 *If $b \le (1-r)e^{(1-r)\tau}/r$ and $c = c_1$, then (1.6)-(1.7) has a strict positive solution which is monotone.*

Proof. Let $\mu(c) = \sqrt{1-r}$, then $c\mu(c) = 2(1-r)$. Define continuous functions as follows

$$\overline{\phi}(\xi) = \begin{cases} -K\xi e^{\mu(c)\xi}, \xi \le \xi_1, \\ 1, \xi > \xi_1, \end{cases} \quad \overline{\psi}(\xi) = \begin{cases} -K\xi e^{\mu(c)\xi+(1-r)\tau}/r, \xi \le \xi_2 \\ 1, \xi > \xi_2, \end{cases}$$

in which $K > 1$ such that $\max_{\xi\in\mathbb{R}}\{-K\xi e^{\mu(c)\xi}\} > 1$ and ξ_1 is the smallest ξ such that $-K\xi e^{\mu(c)\xi} = 1$ while ξ_2 is the smallest ξ satisfying $-K\xi e^{\mu(c)\xi}/r = 1$. Clearly, if $K > 1$ is large, then

$$\frac{\xi - c\tau}{\xi} < e^{(1-r)\tau}, \xi < \xi_1 < 0. \tag{3.4}$$

Let $K > 1$ be large enough such that $\overline{\phi}(\xi) > \underline{\phi}(\xi)$, herein $\underline{\phi}(\xi)$ satisfies

$$c_1\phi'(\xi) = \phi''(\xi) + \phi(\xi)\left[1 - r - \phi(\xi)\right],$$

and $\lim_{\xi\to-\infty}\phi(\xi)e^{-\sqrt{d}\xi}/\xi = -1$. By simple analysis, we see that $K > 1$ is well defined.

Claim 2. Let $\underline{\psi}(\xi) = 0$, then $(\overline{\phi}(\xi), \overline{\psi}(\xi))$ is an upper solution while $(\underline{\phi}(\xi), \underline{\psi}(\xi))$ is a lower solution of (1.6). Moreover, $(\overline{\phi}(\xi), \overline{\psi}(\xi))$ and $(\underline{\phi}(\xi), \underline{\psi}(\xi))$ also satisfy (A1)-(A3).

By Claim 2 and Lemma 2.3, we obtain our result. Now, we are in a position of verifying Claim 2.

Proof of Claim 2. For $\overline{\phi}(\xi)$, it suffices to verify that (3.1) holds.

If $\overline{\phi}(\xi) = -K\xi e^{\mu(c)\xi} < 1$, then $\overline{\psi}(\xi) \le -K\xi e^{\mu(c)\xi+(1-r)\tau}/r$ such that

$$\begin{aligned}
-\overline{\phi}(\xi) + r\overline{\psi}(\xi - c\tau) &\le K\xi e^{\mu(c)\xi} - K(\xi - c\tau)e^{\mu(c)(\xi-c\tau)+(1-r)\tau} \\
&= K\xi e^{\mu(c)\xi} - K(\xi - c\tau)e^{\mu(c)\xi+(r-1)\tau} \\
&\le 0
\end{aligned}$$

by (3.4). Therefore, we have

$$\begin{aligned}
&\overline{\phi}''(\xi) - c\overline{\phi}'(\xi) + \overline{\phi}(\xi)\left[1 - r - \overline{\phi}(\xi) + r\overline{\psi}(\xi - c\tau)\right] \\
\le\ &\overline{\phi}''(\xi) - c\overline{\phi}'(\xi) + \overline{\phi}(\xi)\left[1 - r\right] \\
=\ &0.
\end{aligned}$$

Otherwise, $\overline{\phi}(\xi) = 1 < \sup_{\xi\in\mathbb{R}}\{-K\xi e^{\mu(c)\xi}\}$ and $\overline{\psi}(\xi) \le 1$ indicate that

$$\begin{aligned}
&\overline{\phi}''(\xi) - c\overline{\phi}'(\xi) + \overline{\phi}(\xi)\left[1 - r - \overline{\phi}(\xi) + r\overline{\psi}(\xi - c\tau)\right] \\
\le\ &\overline{\phi}''(\xi) - c\overline{\phi}'(\xi) = 0,
\end{aligned}$$

and we finish the verification of (3.1).

For $\overline{\psi}(\xi)$, we need to prove that (3.2) holds. If $\overline{\psi}(\xi) = -K\xi e^{\mu(c)\xi+(1-r)\tau}/r < 1$, then $\overline{\phi}(\xi) = -K\xi e^{\mu(c)\xi}$ such that

$$\begin{aligned}
&\overline{\psi}''(\xi) - c\overline{\psi}'(\xi) + b\overline{\phi}(\xi)\left[1 - \overline{\psi}(\xi)\right] \\
\le\ &\overline{\psi}''(\xi) - c\overline{\psi}'(\xi) + b\overline{\phi}(\xi) \\
\le\ &\overline{\psi}''(\xi) - c\overline{\psi}'(\xi) - bK\xi e^{\mu(c)\xi} \\
=\ &\overline{\psi}''(\xi) - c\overline{\psi}'(\xi) - bre^{(r-1)\tau}Ke^{(1-r)\tau}\xi e^{\mu(c)\xi}/r \\
=\ &\overline{\psi}''(\xi) - c\overline{\psi}'(\xi) + bre^{(r-1)\tau}\overline{\psi}(\xi) \\
\le\ &\overline{\psi}''(\xi) - c\overline{\psi}'(\xi) + (1-r)\overline{\psi}(\xi) \\
=\ &0.
\end{aligned}$$

If $\overline{\psi}(\xi) = 1 < \sup_{\xi \in \mathbb{R}}\{-K\xi e^{\mu(c)\xi+(1-r)\tau}/r\}$, then $\overline{\phi}(\xi) \le 1$ such that

$$\overline{\psi}''(\xi) - c\overline{\psi}'(\xi) + b\overline{\phi}(\xi)\left[1 - \overline{\psi}(\xi)\right] = 0.$$

Therefore, (3.2) is verified.

Moreover, it is evident that (3.3) is true. The proof is complete. □

Lemma 3.4 *Assume that* $c < c_1$. *Then* (1.6)-(1.7) *has no strict positive solution.*

Proof. Were the statement false. Then there exists $c_2 \in (0, c_1)$ such that (1.6)-(1.7) has a positive solution $(\phi_1(\xi), \phi_2(\xi))$ if $c = c_2$. Clearly, $(u(x,t), v(x,t)) = (\phi_1(x+c_2t), \psi_1(x+c_2t))$ also satisfies the following initial value problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + u(x,t)\left[1 - r - u(x,t) + rv(x,t-\tau)\right], \\ \frac{\partial v(x,t)}{\partial t} = \Delta v(x,t) + bu(x,t)\left[1 - v(x,t)\right], \\ u(x,0) = \phi_1(x), v(x,s) = \psi_1(x+cs), s \in [-\tau, 0]. \end{cases} \tag{3.5}$$

Because $\psi_1(x)$ is positive, we know

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} \ge \Delta u(x,t) + u(x,t)\left[1 - r - u(x,t)\right], \\ u(x,0) = \phi_1(x). \end{cases} \tag{3.6}$$

By (3.6) and Lemma 2.6, we see that

$$\phi_1(x + c_2t) \ge \widehat{u}(x,t), \tag{3.7}$$

if $\widehat{u}(x,t)$ satisfies

$$\begin{cases} \frac{\partial \widehat{u}(x,t)}{\partial t} = \Delta \widehat{u}(x,t) + \widehat{u}(x,t)\left[1 - r - \widehat{u}(x,t)\right], \\ \widehat{u}(x,0) = \phi_1(x). \end{cases}$$

Applying Lemma 2.4, we see that

$$\liminf_{t\to\infty} \inf_{-2x=(c_1+c_2)t} \widehat{u}(x,t) > 0,$$

which implies a contradiction to (3.7) since

$$\phi_1(x + c_2t) \to 0 \text{ if } -2x = (c_1 + c_2)t \text{ and } t \to \infty.$$

By what we have done, we complete the proof. □

Summarizing these three lemmas, we obtain the minimal wave speed. Before continuing the paper, we also make a remark as follows.

Remark 3.5 Our results also answer the conjecture of Lin and Li [5, Remark 2.4] by confirming that $\alpha = 0$.

In Lemmas 3.2-3.3, we obtain the existence of monotone solutions of (1.6)-(1.7). We now investigate the strict monotonicity of these traveling wavefronts, which is true for any monotone solutions of (1.6)-(1.7).

Lemma 3.6 *Assume that* $(\phi(\xi), \psi(\xi))$ *is a monotone solution of* (1.6)-(1.7). *Then*

$$0 < \phi(\xi) < 1,\ 0 < \psi(\xi) < 1,\ \xi \in \mathbb{R}.$$

The lemma is clear from the definition of F, and we omit it here.

Lemma 3.7 *Assume that* $(\phi(\xi), \psi(\xi))$ *is a monotone solution of* (1.6)-(1.7). *Then* $\phi(\xi), \psi(\xi)$ *are strict monotone in* $\xi \in \mathbb{R}$.

Proof. If $\psi(\xi)$ is not strictly monotone, then there exists ξ_1 such that $\psi'(\xi_1) = 0$ and $\psi''(\xi_1) \geq 0$. Therefore, we obtain $\phi(\xi_1)(1 - \psi(\xi_1)) \leq 0$, which is impossible by Lemma 3.6. Therefore, we obtain the strict monotone of $\psi(\xi)$.

If $\phi(\xi)$ is not strict monotone, then there exist $\xi_2 < \xi_3$ such that $\phi(\xi_2) = \phi(\xi)$ for $\xi \in [\xi_2, \xi_3]$. Moreover, the smoothness and (1.7) implies that there exists a nonempty interval $I \subset \mathbb{R}$ such that ϕ or ψ is strictly monotone and

$$H_1(\phi, \psi)(s + \xi_3 - \xi_2) > H_1(\phi, \psi)(s), s \in I. \tag{3.8}$$

By Lemma 3.6 and (3.8), we see that

$$\begin{aligned}
F_1(\phi, \psi)(\xi_2) &= \frac{1}{\gamma_2(c) - \gamma_1(c)} \left[\int_{-\infty}^{\xi_2} e^{\gamma_1(c)(\xi_2 - s)} + \int_{\xi_2}^{\infty} e^{\gamma_2(c)(\xi_2 - s)} \right] H_1(\phi, \psi)(s) ds \\
&< \frac{1}{\gamma_2(c) - \gamma_1(c)} \left[\int_{-\infty}^{\xi_2} e^{\gamma_1(c)(\xi_2 - s)} + \int_{\xi_2}^{\infty} e^{\gamma_2(c)(\xi_2 - s)} \right] H_1(\phi, \psi)(s + \xi_3 - \xi_2) ds \\
&= \frac{1}{\gamma_2(c) - \gamma_1(c)} \left[\int_{-\infty}^{\xi_3} e^{\gamma_1(c)(\xi_3 - s)} + \int_{\xi_3}^{\infty} e^{\gamma_2(c)(\xi_3 - s)} \right] H_1(\phi, \psi)(s) ds \\
&= F_1(\phi, \psi)(\xi_3),
\end{aligned}$$

which implies a contradiction. The proof is complete. $\square$

References

[1] D.G. Aronson, H.F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, In *Partial Differential Equations and Related Topics* (Ed. by J.A. Goldstein), Lecture Notes in Mathematics, Vol. 446, pp. 5-49, Springer, Berlin, 1975.

[2] B.P. Belousov, A periodic reaction and its mechanism, In *Reference Handbook on Radiation Medicine after 1958* [in Russian], pp. 145-147, Moscow, 1959.

[3] Ya.I. Kanel, Existence of a travelling wave solution of the Belousov-Zhabotinskii system, *Diiferetsial'npe Uravneniya,* **26** (1990), 652-660.

[4] A.Ya. Kapel, Existence of travelling-wave type solutions for the Belousov-Zhabotinskii system equations, *Sibirskii Mathematicheskii Zhurnal*, **32** (1991), 47-59.

[5] G. Lin, W.T. Li, Travelling wavefronts of Belousov-Zhabotinskii system with diffusion and delay, *Appl. Math. Lett.,* **22** (2009), 341-346.

[6] S. Ma, Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem, *J. Differential Equations,* **171** (2001), 294-314.

[7] J.D. Murray, On a model for temporal oscillations in the Belousov-Zhabotinskii reaction, *J. Chem. Phys.,* **61** (1974), 3611-3613.

[8] J.D. Murray, On traveling wave solutions in a model for Belousov-Zhabotinskii reaction, *J. Theor. Biol.,* **56** (1976), 329-353.

[9] J.D. Murray, *Mathematical Biology* (The Third Edition), Springer-Verlag, New York, 2003.

[10] W.C. Troy, The existence of travelling wavefront solutions of a model of the Belousov-Zhabotinskii reaction, *J. Differential Equations,* **36** (1980), 89-98.

[11] J. Wu, X. Zou, Traveling wave fronts of reaction-diffusion systems with delay, *J. Dynam. Diff. Eqns.,* **13** (2001), 651-687.

[12] Q. Ye, M. Wang, Travelling wavefront solutions of Noyes-field system for Belousov-Zhabotinskii reaction, *Nonlinear Anal. TMA,* **11** (1987), 1289-1302.

[13] A.N. Zaikin, A.M. Zhabotinskii, Concentration wave propagation in two dimensional liquid phase self oscillation system, *Nature*, **225** (1970), 535-537.

[14] A.M. Zhabotinskii, *Concentration Autooscillations,* Nauka, Moscow, 1974.

(Received January 2, 2012)