Exact Controllability of a Second-Order Integro-Differential Equation with a Pressure Term

M. M. Cavalcanti*, V. N. Domingos Cavalcanti, A. Rocha ** and J. A. Soriano

Departamento de Matemática - Universidade Estadual de Maringá

87020-900 Maringá-PR, Brasil.

** Instituto de Matemática - UFRJ - CP.68530,

CEP: 21944 - Rio de Janeiro - RJ - Brasil.

Abstract. This paper is concerned with the boundary exact controllability of the equation

$$u'' - \Delta u - \int_0^t g(t-\sigma)\Delta u(\sigma)d\sigma = -\nabla p$$

where Q is a finite cylinder $\Omega\times]0,T[$, Ω is a bounded domain of $\mathbf{R}^n$, $u = (u_1(x,t),\cdots,u_2(x,t))$, $x=(x_1,\cdots,x_n)$ are $n-$ dimensional vectors and p denotes a pressure term. The result is obtained by applying HUM (Hilbert Uniqueness Method) due to J.L.Lions. The above equation is a simple model of dynamical elasticity equations for incompressible materials with memory.

Key words: HUM, Exact Controllability, Memory.

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1. Introduction

Let Ω be a bounded domain of $\mathbf{R}^n$ with regular boundary Γ. Let $Q = \Omega\times]0,T[$ be a cylinder whose lateral boundary is given by $\Sigma = \Gamma\times]0,T[$.

Let us consider the following system

*corresponding author e-mail: marcelo@gauss.dma.uem.br

$$
(*) \qquad \left| \begin{array}{l} u'' - \Delta u - \displaystyle\int_0^t g(t-\sigma)\Delta u(\sigma) d\sigma = -\nabla p \quad \text{in} \quad Q \\ div\, u = 0 \quad \text{in} \quad Q \\ u = v \quad \text{on} \quad \Sigma \\ u(0) = u^0, \quad u'(0) = u^1 \quad \text{in} \quad \Omega \end{array} \right.
$$

where $\Delta u = (\Delta u_1, \cdots, \Delta u_n), u'' = (u_1'', \cdots, u_n''), div\, u = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i}$ and $p = p(x,t)$ is the pressure term.

The exact controllability problem for system $(*)$ is formulated as follows: given $T > 0$ large enough , for every initial data $\{u^0, u^1\}$ in a suitable space, it is possible to find a control v such that the solution of $(*)$ satisfies

$$
u(T) = u'(T) = 0. \tag{1.1}
$$

System $(*)$ with $g = 0$ was studied by J. L. Lions [9], motivated by dynamical elasticity equations for incompressible materials, as follows.

Let Ω be a three dimensional solid body, made of an elastic, isotropic and incompressible material (like some rubber types) under external forces f. From the Newton's Second Law and considering small deflections of Ω we get

$$
m\frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\left[-p\delta_{ij} + 2\mu\epsilon_{ij}(u)\right] = f(x,t), \quad i = 1,2,3.
$$

Mathematically, the incompressibility condition is represented by

$$
|det\,(I + \nabla \cdot u)| = 1,
$$

where I is the identity matrix.

Noting that under small deflections, the quadratic terms of the determinant are neglectible, we obtain that $div\, u = 0$ in Ω for all t which leads to the model

$$
m\frac{\partial^2 u_i}{\partial t^2} - \mu\Delta u_i = f_i - \frac{\partial p}{\partial x_i}, \quad i = 1,2,3.
$$

The presence of the memory term in equation $(*)$ is related to the viscoelastic properties of the material.

Assuming that Ω is strictly star-shaped with respect to the origin, that is, there exists $\gamma > 0$ such that

$$m \cdot \nu \geq \gamma > 0 \quad \text{on} \quad \Gamma \tag{1.2}$$

(where $m(x) = x = (x_1, \cdots, x_n)$ and ν is the exterior unitary normal) and $g = 0$, J.L.Lions [9] proved that the normal derivative of the solution u of $(*)$ belongs to $(L^2(\Sigma))^n$ while in A. Rocha [1] the exact controllability was establish.

Inspired by the above mentioned works we study, in a natural way, the exact boundary controllability of system $(*)$ when the kernel g is small, which is the goal of this paper. For this end we employ the multiplier technique to obtain the direct and inverse inequalities. However, the convolution term $g * \Delta u$ brought up some technical difficulties which were bypassed by transforming the problem $(*)$ into an equivalent one using the standard Volterra equations theory.

We can find in the Literature several works in connection with memory terms. The reader is referred to the works of J. U. Kim [3], G. Leugering [7], I. Lasiecka [6] and the classical book of J. Lagnese and J. L. Lions [5].

Our paper is organized as follows. In section 2 we give the notations, present standard and auxiliar results and state the main one. In section 3 we obtain the direct and inverse inequalities related with $(*)$ when $g = 0$ while in section 4 we obtain the same inequalities to the general case. In section 5 we study the ultra weak solutions of $(*)$ which is enough to apply HUM in order to obtain the above mentioned exact controllability.

2. Notations and Main Result

In what follows we consider the Hilbert spaces

$$V = \{v \in (H_0^1(\Omega))^n \;\text{ and }\; div\, v = 0 \quad \text{in } \Omega\} \tag{2.1}$$

and

$$H = \{v \in (L^2(\Omega))^n, div\, v = 0 \quad \text{and} \quad v \cdot \nu = 0 \;\text{ on }\; \Gamma\} \tag{2.2}$$

equipped with their respective inner products

$$((u, v)) = \sum_{i=1}^{n} ((u_i, u_i))_{H_0^1(\Omega)} \tag{2.3}$$

and

$$(u,v)=\sum_{i=1}^{n}(u_i,v_i)_{L^2(\Omega)}. \tag{2.4}$$

We also consider

$$\mathcal{V}=\{\varphi\in(D(\Omega))^n,\quad div\,\varphi=0\} \tag{2.5}$$

$$W=V\cap(H^2(\Omega))^n. \tag{2.6}$$

We have that $\mathcal{V}$ is dense in V with the topology induced by V and

$$H=\overline{\mathcal{V}}^{(L^2(\Omega))^n}. \tag{2.7}$$

Also, we observe that if $f\in(D'(\Omega))^n$ and satisfies $f(v)=0$ for all $v\in\mathcal{V}$, then there exists $p\in D'(\Omega)$ such that $f=-\nabla p$. In particular, if $f\in(H^{-1}(\Omega))^n$ and $f(v)=0$ for all $v\in V$, we have the same conclusion with $p\in L^2(\Omega)/\mathbf{R}$.

In addition, let us define

$$Z=\{v\in(L^2(\Sigma))^n;\int_\Sigma v\cdot\nu\,d\Sigma=0\}. \tag{2.8}$$

Let $g:[0,\infty[\rightarrow]0,\infty[$ be a real funtion satisfying the following hypotheses:

$$g\in C^2[0,\infty[, \tag{2.9}$$

$$\alpha=1-\int_0^\infty g(\sigma)d\sigma>0, \tag{2.10}$$

$$-C_1g\le g'\le -C_2g, \tag{2.11}$$

$$0\le g''\le C_3g \tag{2.12}$$

where C_1 , C_2 and C_3 are positive constants and

$$g(0)<\varepsilon,\ \text{where}\ \ \varepsilon>0\ \ \text{is sufficiently small.} \tag{2.13}$$

As an example of a function which satisfies the conditions (2.9)-(2.13) above one can cite

$$g(t)=k_1e^{-k_2t};\quad 0<k_1<k_2\quad\text{and}\ \ k_1<\varepsilon.$$

Let us consider the following problem

$$\left| \begin{array}{l} u'' - \Delta u - \int_0^t g(t-\sigma)\Delta u(\sigma)d\sigma = f - \nabla p \quad \text{in} \quad Q \\ div\, u = 0 \quad \text{in} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(0) = u^0; \;\; u'(0) = u^1 \quad \text{in} \;\; \Omega \end{array} \right. \tag{2.14}$$

We have the auxiliar results

Lemma 2.1. *Let $\{u^0, u^1, f\} \in W \times V \times L^1(0,T;V)$ and assume that $g \in C^2[0,\infty)$. Then, there exists a unique function $u : Q \to \mathbf{R}^n$ such that*

$$u \in L^\infty(0,T;W), \quad u' \in L^\infty(0,T,V) \quad \textit{and} \quad u'' \in L^\infty(0,T;H),$$

$$u'' - \Delta u - \int_0^t g(t-\sigma)\Delta u(\sigma)d\sigma = f - \nabla p \quad \textit{in} \quad (D'(Q))^n,$$

$$div\, u = 0$$

$$u(0) = u^0, \quad u'(0) = u^1 \quad \textit{in} \quad \Omega.$$

Furthermore

$$u'' - \Delta u - \int_0^t g(t-\sigma)\Delta u(\sigma)d\sigma = f \quad \textit{in} \quad L^1(0,T;H).$$

The proof of the above result is obtained by applying Galerkin's approximation with two estimates. The uniqueness is obtained by the usual energy method.

Remark 1. Using Galerkin's method with additional estimates, we can also prove that if

$$\left\{u^0, u^1, f\right\} \in V \cap \left(H^4(\Omega)\right)^n \times W \times L^1(0,T;W)$$

the unique solution u of problem (2.9) satisfies

$$u \in L^\infty(0,T;V \cap \left(H^4(\Omega)\right)^n), \quad u' \in L^\infty(0,T;W), \quad u'' \in L^1(0,T;V).$$

In this case we have $p \in H^2(\Omega)$.

From Lemma 2.1 and using density arguments we can prove the following result:

Lemma 2.2. *Let* $\{u^0, u^1, f\} \in V \times H \times L^1(0,T;H)$. *Then,*

(i) There exists a unique weak solution u *of problem (2.14) such that*

$$u \in C([0,T];V) \cap C^1([0,T];H).$$

(ii) The linear mapping

$$V \times H \times L^1(0,T;H) \to C([0,T];V) \cap C^1([0,T];H)$$

$$\left\{u^0, u^1, f\right\} \mapsto u$$

is continuous, where u *is the solution of problem (2.14) obtained in (i).*

(iii) The solution u *obtained in (i) satisfies*

$$\frac{1}{2}|u'(t)|^2 + \frac{1}{2}||u(t)||^2 + \int_0^t g(t-\sigma)((u(\sigma), u(t)))d\sigma = \frac{1}{2}|u^1|^2 + \frac{1}{2}||u^0||^2$$

$$+g(0)\int_0^t ||u(s)||^2 ds + \int_0^t \int_0^s g'(s-\sigma)((u(\sigma), u(s)))\, d\sigma ds + \int_0^t (f(\sigma), u'(\sigma))\, d\sigma,$$

$\forall t \in [0,T]$.

(iv) The solution u *is such that*

$$u'' - \Delta u - \int_0^t g(t-\sigma)\Delta u(\sigma)d\sigma = f - \nabla p \ \ in \ \ (D'(Q))^n$$

where $p \in D'(Q)$.

From the item (iii) of the above lemma and making use of Gronwall's lemma we obtain the following energy inequality

$$E(t) \le C(T)\left[E(0) + ||f||_{L^1(0,T;H)}\right],$$

where $C(T)$ is a positive constant which depends on $T > 0$ and

$$E(t) = \frac{1}{2}\left|u'(t)\right|^2 + \frac{1}{2}\left|\left|u(t)\right|\right|^2$$

is the energy related with problem (2.14).

Remark 2. As it is stated in J. L. Lions [8], (Chap. I, Lemma 3.7) if $\phi \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right)^n$, then

$$\frac{\partial \phi_i}{\partial x_k} = \nu_k \frac{\partial \phi_i}{\partial \nu} \quad \text{on} \quad \Gamma; \quad \forall i, k \in \{1, \cdots, n\}. \tag{2.15}$$

Moreover, if $div\, \phi = 0$ in Ω, then as in J. L. Lions [8] (Chap. II, section 5) we have

$$\frac{\partial \phi}{\partial \nu} \cdot \nu = 0 \quad \text{on} \quad \Gamma \tag{2.16}$$

and consequently

$$\nu_i \frac{\partial \phi_i}{\partial x_k} = \nu_i \nu_k \frac{\partial \phi_i}{\partial \nu} = 0 \quad \text{on} \quad \Gamma. \tag{2.17}$$

We observe that throughout this paper repeated indices indicate summation from 1 to n.

Let $x^0 \in \mathbf{R}^n$, $m(x) = x - x^0$, $x \in \mathbf{R}^n$ and

$$R = \max\{\, ||m(x)||; x \in \overline{\Omega}\, \}.$$

Let us define

$$\Gamma(x^0) = \{x \in \Gamma;\ m(x) \cdot \nu(x) > 0\},$$
$$\Gamma_*(x^0) = \{x \in \Gamma;\ m(x) \cdot \nu(x) \leq 0\},$$
$$\Sigma(x^0) = \Gamma(x^0) \times [0, T]$$

and

$$\Sigma_*(x^0) = \Gamma_*(x^0) \times [0, T].$$

Now, we are in position to state our main result:

Theorem 2.1. *Provided that the hypotheses (1.2), (2.9)-(2.13) hold there exists for every initial data $\{u^0, u^1\} \in H \times V'$, a control $v \in Z$ such that the ultra weak solution of the system*

$$\left|\begin{array}{l} u'' - \Delta u - \displaystyle\int_0^t g(t-\sigma)\Delta u(\sigma)d\sigma = -\nabla p \quad in \quad Q \\ div\, u = 0 \quad in \quad Q \\ u = v \quad on \quad \Sigma \\ u(0) = u^0; \quad u'(0) = u^1 \quad in \quad \Omega \end{array}\right. \tag{2.18}$$

satisfies the condition:

$$u(T) = 0, \;\; u'(T) = 0$$

for all $T > T_0$ *where* $T_0 = 2R$.

3. Direct and Inverse Inequalities with null kernel

In this section we are going to obtain the direct and inverse inequalities to problem (2.14) when the kernel of the memory $g = 0$. For this end we will employ the multiplier technique. Although the result below plays an essential role in our intent, we will omit its proof since it is exactly as if we were dealing with the wave equation whose proof can be found in J. L. Lions [8], lemma 3.7, pp. 40-43.

Proposition 3.1. *Let* $\{u^0, u^1, f\} \in V \cap (H^4(\Omega))^n \times W \times L^1(0,T;W)$. *Then, for each strong solution* u *of (2.14) with* $g = 0$, *we have the following identity*

$$\frac{1}{2}\int_\Sigma (m\cdot\nu)\left(\frac{\partial u}{\partial \nu}\right)^2 d\Sigma = (u'(t), m\cdot\nabla u)_0^T + \frac{n}{2}\int_Q \left[|u'|^2 - |\nabla u|^2\right] dxdt \tag{3.1}$$

$$+\int_Q |\nabla u|^2\, dxdt - \int_Q f_i\, m_k\, \frac{\partial u_i}{\partial x_k} dxdt + \int_Q \frac{\partial p}{\partial x_i}\, m_k\, \frac{\partial u_i}{\partial x_k} dxdt.$$

where ∇u *means*

$$\begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} \cdots \dfrac{\partial u_n}{\partial x_1} \\ \cdot \quad \cdots \quad \cdot \\ \dfrac{\partial u_1}{\partial x_n} \cdots \dfrac{\partial u_n}{\partial x_n}. \end{pmatrix}.$$

Theorem 3.2. (**Direct Inequality**)

Assume that

$$\left\{u^0, u^1, f\right\} \in V \times H \times L^1(0,T;H)$$

and (1.2) holds. Then, for each weak solution u *of (2.14) there exists a positive constant* $C > 0$ *such that*

$$\int_\Sigma \left|\frac{\partial u}{\partial \nu}\right|^2 d\Sigma \le C(T+1)\left(E(0) + \left(\int_0^T |f(s)|ds\right)^2\right) \tag{3.2}$$

Proof. Assuming the direct inequality proven for regular initial data, one obtains the general result using density arguments and extension by continuity. Let us consider then

$$\{u^0, u^1, f\} \in V \cap \left(H^4(\Omega)\right)^n \times W \times L^1(0,T;W).$$

From the identity (3.1) and taking (1.2) and lemma 2.2(ii) into account it follows that

$$\frac{\gamma}{2}\int_\Sigma \left|\frac{\partial u}{\partial \nu}\right|^2 d\Sigma \tag{3.3}$$

$$\leq C(T+1)\left(E(0) + \left(\int_0^T |f(t)|dt\right)^2\right) + \left|\int_0^T \int_\Omega \frac{\partial p}{\partial x_i} m_k \frac{\partial u_i}{\partial x_k} dx\, dt\right|$$

where C is a positive constant.

Next, we are going to prove that

$$J = \int_Q \frac{\partial p}{\partial x_i} m_k \frac{\partial u_i}{\partial x_k} dx\, dt = 0. \tag{3.4}$$

Indeed, from (2.17), using Gauss's formula and taking into account that $div\, u = 0$ in Q we have

$$J = -\int_Q p \frac{\partial}{\partial x_i}\left(m_k \frac{\partial u_i}{\partial x_k}\right) dxdt + \int_\Sigma p m_k \frac{\partial u_i}{\partial x_k} \nu_i d\Sigma$$

$$= -\int_Q p \frac{\partial u_i}{\partial x_i} dxdt - \int_Q p m_k \frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_i}\right) dxdt + \int_\Sigma p m_k \frac{\partial u_i}{\partial x_k} \nu_i d\Sigma = 0$$

which proves (3.4). □

In order to give a sense to $\frac{\partial u}{\partial \nu}$ when u is a weak solution of problem (2.14) let us consider Y the space of the weak solutions of the same problem when

$$\left\{u^0, u^1, f\right\} \in V \times H \times L^1(0,T;H).$$

Since problem (2.14) is a linear one and has uniqueness of solution the linear map

$$V \times H \times L^1(0,T;H) \to Y$$
$$\left\{u^0, u^1, f\right\} \longmapsto u$$

is injective. Therefore, the vector space Y is a Banach space with the norm

$$||u||_Y = ||u^0|| + |u^1| + ||f||_{L^1(0,T;H)}.$$

Representing by X the space of the strong solutions of (2.14), that is, when

$$\left\{u^0, u^1, f\right\} \in W \times V \times L^1(0,T,V),$$

we have $X \hookrightarrow Y$ with injection continuous and dense.

Considering the linear map

$$\gamma : X \to \left(L^2(\Sigma)\right)^n$$

$$u \mapsto \frac{\partial u}{\partial \nu}$$

from the direct inequality we have

$$\left|\frac{\partial u}{\partial \nu}\right|^2_{(L^2(\Sigma))^n} \leq C\left(||u^0||^2 + |u^1|^2 + ||f||^2_{L^1(0,T;H)}\right),$$

that is,

$$\left|\frac{\partial u}{\partial \nu}\right|^2_{(L^2(\Sigma))^n} \leq C||u||^2_Y.$$

Consequently $\gamma : X \to (L^2(\Sigma))^n$ is continuous in X with the topology induced by Y and since X is dense in Y with respect to the same topology, the map γ has a linear and continuous extension

$$\hat{\gamma} : Y \to \left(L^2(\Sigma)\right)^n$$

defined as follows:

Considering $u \in Y$, there exists $u_\mu \in X$ such that

$$u_\mu \to u \quad \text{in} \quad Y$$

and therefore

$$\hat{\gamma} u = \lim_{\mu\to\infty} \gamma u_\mu = \lim_{\mu\to\infty} \frac{\partial u_\mu}{\partial \nu}.$$

Motivated by the above definition, the normal derivative $\frac{\partial u}{\partial \nu}$ of $u \in Y$ is given by

$$\frac{\partial u}{\partial \nu} = \lim_{\mu \to \infty} \frac{\partial u_\mu}{\partial \nu}.$$

Theorem 3.3 (**Inverse Inequality**) For all $T \geq T_0$ and for all weak solution u of (2.14) with $f = 0$ we have the following inequality

$$(T - T_0)E(0) \leq \frac{R}{2} \int_{\Sigma(x^0)} \left| \frac{\partial u}{\partial \nu} \right|^2 d\Sigma. \tag{3.5}$$

Proof. We proceed as in Theorem 3.2, that is, we consider initially regular initial data and then we obtain the desired result by a density argument. From (3.1) and (3.4) we can write

$$(u'(t),\, m \cdot \nabla u)_0^T + \frac{n-1}{2} \int_Q \left[|u'|^2 - |\nabla u|^2 \right] dxdt - \frac{1}{2} \int_\Sigma (m \cdot \nu) \left| \frac{\partial u}{\partial \nu} \right|^2 d\Sigma \tag{3.6}$$

$$+ \int_0^T E(t)\, dt = 0$$

where

$$E(t) = \frac{1}{2} \left[|u'(t)|^2 + |\nabla u(t)|^2 \right].$$

In what follows we are going to proceed as in J. L. Lions [8] using arguments due to L. F. Ho [2] and V. Komornik [4].

Since

$$E(t) = E(0) \tag{3.7}$$

and taking into account that

$$(u'(t), u(t))_0^T - \int_0^T |u'(t)|^2 dt + \int_0^T |\nabla u(t)|^2 dt = 0 \tag{3.8}$$

the expression (3.6) becomes

$$\left(u',\, m \cdot \nabla u + \frac{n-1}{2} u \right)_0^T + TE(0) = \frac{1}{2} \int_\Sigma (m \cdot \nu) \left(\frac{\partial u}{\partial \nu} \right)^2 d\Sigma. \tag{3.9}$$

But

$$\left| m \cdot \nabla u + \frac{n-1}{2} u \right|^2 = |m \cdot \nabla u|^2 - \frac{n^2-1}{4} |u|^2 \leq |m \cdot \nabla u|^2 \leq R^2 |\nabla u|^2$$

which implies

$$\left|\left(u',\, m\cdot\nabla u + \frac{n-1}{2}u\right)_0^T\right| \leq R\left[|u'(T)|\,|\nabla u(T)| + |u'(0)|\,|\nabla u(0)|\right] \leq 2RE(0). \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$(T-T_0)E(0) \leq \frac{R}{2}\int_{\Sigma(x^0)} \left|\frac{\partial u}{\partial \nu}\right|^2 d\Sigma.$$

This concludes the proof. □

4. Direct and Inverse Inequalities in the general case

In this section we will establish the direct and inverse inequalities related with the weak solutions of the homogeneous problem

$$\left|\begin{array}{l} \varphi'' - \Delta\varphi - \displaystyle\int_0^t g(t-\sigma)\Delta\varphi(\sigma)d\sigma = -\nabla p \quad \text{in} \quad Q \\ div\,\varphi = 0 \quad \text{in} \quad Q \\ \varphi = 0 \quad \text{on} \quad \Sigma \\ \varphi(0) = \varphi^0; \varphi'(0) = \varphi^1 \quad \text{in} \quad \Omega \end{array}\right. \tag{4.1}$$

using the inequalities obtained in section 3 when $g = 0$. We begin making some considerations. In what follows X will represent the cylinder Q or its lateral boundary Σ. Let

$$K : L^2(X) \to L^2(X)$$

be the linear operator defined by

$$(K\varphi)(x,t) = \int_0^t g(t-\tau)\varphi(x,\tau)d\tau,\ \forall\varphi\in L^2(X).$$

We note that from Cauchy-Schwarz inequality and Fubini's theorem we have

$$||K\varphi||_{L^2(X)} \leq ||g||_{L^1(0,\infty)}||\varphi||_{L^2(X)}.$$

Then, K is well defined and since $||g||_{L^1(0,\infty)} < 1$ we conclude that $||K||_{\mathcal{L}(L^2(X))} < 1$ and consequently the operator $(I-K)^{-1}$ exists and belongs to $\mathcal{L}(L^2(X)\,)$. Moreover, in this case we have

$$\sum_{n=0}^{\infty} K^n = (I-K)^{-1}$$

where the above equality is understood in the space $\mathcal{L}(L^2(X)\,)$.

By standard Volterra equations theory for any $\theta \in L^2(X)$ there exists a unique solution of the Volterra equation

$$\varphi + K\varphi = \theta$$

Furthermore, φ and θ are related by the equations

$$\theta(x,t) = \varphi(x,t) + \int_0^t g(t-\sigma)\varphi(x,\sigma)d\sigma \tag{4.2}$$

$$\varphi(x,t) = \theta(x,t) + \int_0^t h(t-\sigma)\theta(x,\sigma)d\sigma \tag{4.3}$$

where

$$h(t-\sigma) = \sum_{n=1}^{\infty}(-1)^n g_n(t-\sigma)$$

and

$$g_1(t) = g(t) \quad \text{and} \quad g_n(t-\tau) = \int_\tau^t g_1(t-\sigma)g_{n-1}(\sigma-\tau)d\sigma \quad \text{if} \quad n \geq 2.$$

Also, g and h are related by the formula

$$g(t) = -h(t) - \int_0^t g(t-\sigma)h(\sigma)d\sigma. \tag{4.4}$$

From (4.1) and (4.3) we obtain the equivalent problem for θ

$$\left|\begin{array}{l} \theta'' - \Delta\theta + h(0)\theta' + h'(0)\theta + \int_0^t h''(t-\sigma)\theta(\sigma)d\sigma = -\nabla p \quad \text{in} \quad Q \\ div\,\theta = 0 \quad \text{in} \quad Q \\ \theta = 0 \quad \text{on} \quad \Sigma \\ \theta(0) = \varphi^0, \quad \theta'(0) = \varphi^1 + g(0)\varphi^0 \quad \text{in} \quad \Omega. \end{array}\right. \tag{4.5}$$

We note that problem (4.5) can be written as

$$\theta = \hat{\theta} + \zeta$$

where $\hat{\theta}$ and ζ are the unique solutions of the following problems

$$\left| \begin{array}{l} \hat{\theta}'' - \Delta\hat{\theta} = -\nabla\hat{p} \quad \text{in} \;\; Q \\ div\,\hat{\theta} = 0 \quad \text{in} \quad Q \\ \hat{\theta} = 0 \;\; \text{on} \;\; \Sigma \\ \hat{\theta}(0) = \varphi^0, \;\; \hat{\theta}'(0) = \varphi^1 + g(0)\varphi^0 \quad \text{in} \quad \Omega \end{array} \right.$$

and

$$\left| \begin{array}{l} \zeta'' - \Delta\zeta = f - \nabla q \quad \text{in} \;\; Q \\ div\,\zeta = 0 \quad \text{in} \quad Q \\ \zeta = 0 \;\; \text{on} \;\; \Sigma \\ \zeta(0) = \zeta'(0) = 0 \quad \text{in} \quad \Omega. \end{array} \right.$$

with

$$p = \hat{p} + q \quad \text{and} \quad f(t) = -h(0)\theta'(t) - h'(0)\theta(t) - \int_0^t h''(t-\sigma)\theta(\sigma)d\sigma. \quad (4.6)$$

From the direct and inverse inequalities obtained in section 3 there exists L_1, L_2 and L_3 positive constants such that

$$L_1\left(||\hat{\theta}(0)|| + |\hat{\theta}'(0)|\right) \leq \left|\left|\frac{\partial\hat{\theta}}{\partial\nu}\right|\right|_{(L^2(\Sigma(x^0)))^n} \leq L_2\left(||\hat{\theta}(0)|| + |\hat{\theta}'(0)|\right) \quad (4.7)$$

and

$$\left|\left|\frac{\partial\zeta}{\partial\nu}\right|\right|_{(L^2(\Sigma(x^0)))^n} \leq L_3||f||_{L^1(0,T;H)}. \quad (4.8)$$

On the other hand, from (4.6) we get

$$|f(t)| \leq |h(0)|\,|\theta'(t)| + |h'(0)|\,|\theta(t)| + \int_0^t |h''(t-\sigma)|\,|\theta(\sigma)|\,d\sigma. \quad (4.9)$$

But, from (4.4) it follows that

$$h(0) = -g(0), \quad (4.10)$$

$$h'(0) = -g'(0) + g^2(0), \quad (4.11)$$

$$h''(t) = -g''(t) - g(t)\left[-g'(0) + g^2(0)\right] + g'(t)g(0) - \int_0^t g(\sigma)h''(t-\sigma)d\sigma. \quad (4.12)$$

Using assumption (2.13) and from (4.10) we obtain

$$|h(0)| < \varepsilon. \quad (4.13)$$

Considering the assumptions (2.11) and (2.13), from (4.11) we get

$$|h'(0)| < -g'(0) + \varepsilon^2 \leq C_1 g(0) + \varepsilon^2 < C_1\varepsilon + \varepsilon^2. \quad (4.14)$$

Provided that the asumptions (2.11)-(2.13) hold and from (4.12) we deduce

$$|h''(t)| \leq \left[C_3 + C_1\varepsilon + \varepsilon^2 + \varepsilon C_1\right] g(t) + \int_0^t g(\sigma)\,|h''(t-\sigma)|\,d\sigma. \quad (4.15)$$

Integrating (4.15) over [0,T] it follows that

$$\int_0^T |h''(t)|\,dt \leq C(\varepsilon)\int_0^\infty g(t)\,dt + \left(\int_0^\infty g(\sigma)d\sigma\right)\int_0^T |h''(t)|\,dt$$

and from (2.10) and (2.13) we obtain

$$\int_0^T |h''(t)| \leq \frac{C(\varepsilon)}{\alpha}\int_0^\infty g(\sigma)\,d\sigma \leq \varepsilon C_2^{-1}C(\varepsilon). \quad (4.16)$$

Now, integrating (4.9) over [0,T] and combining (4.13), (4.14) and (4.16) we conclude

$$\int_0^T |f(t)|\,dt \leq T\varepsilon||\theta'||_{C^0([0,T];H)} + T\left(C_1\varepsilon + \varepsilon^2\right)||\theta||_{C^0([0,T];H)}$$
$$+\varepsilon C_2^{-1}C(\varepsilon)||\theta||_{C^0([0,T];H)}$$

and therefore, using lemma (2.2) (ii), we infer

$$\int_0^T |f(t)|\,dt \leq \varepsilon C(T)\left(|\varphi^1 + g(0)\varphi^0| + ||\varphi^0||\right). \quad (4.17)$$

From (4.8) and (4.17) we can write

$$\left|\left|\frac{\partial\zeta}{\partial\nu}\right|\right| \leq \varepsilon C(T)\left(|\varphi^1 + g(0)\varphi^0| + ||\varphi^0||\right) \quad (4.18)$$

and since $\theta = \hat{\theta} + \zeta$ from (4.7) and (4.18) it holds that

$$\left\| \frac{\partial \theta}{\partial \nu} \right\|_{(L^2(\Sigma(x^0)))^n} \geq \left\| \frac{\partial \hat{\theta}}{\partial \nu} \right\| - \left\| \frac{\partial \zeta}{\partial \nu} \right\| \tag{4.19}$$

$$\geq L_1 \left(||\hat{\theta}(0)|| + |\hat{\theta}'(0)| \right) - \varepsilon C(T) \left(|\varphi^1 + g(0)\varphi^0| + ||\varphi^0|| \right)$$
$$= (L_1 - \varepsilon C(T)) \left(|\varphi^1 + g(0)\varphi^0| + ||\varphi^0|| \right)$$
$$\geq (L_1 - \varepsilon C(T)) \left(|\varphi^1| - g(0)|\varphi^0| + ||\varphi^0|| \right).$$

Finally, from assumption (2.13) and since $|v| \leq \lambda ||v||$, $\forall v \in V$ from (4.19) we have

$$\left\| \frac{\partial \theta}{\partial \nu} \right\|_{(L^2(\Sigma(x^0)))^n} \geq (L_1 - \varepsilon C(T))(1 - \varepsilon\lambda)||\varphi^0|| + (L_1 - \varepsilon C(T))|\varphi^1|.$$

Choosing ε small enough, we conclude from the last inequality that there exists a positive constant C such that

$$\left\| \frac{\partial}{\partial \nu} \left(\varphi + \int_0^t g(t-\sigma)\varphi d\sigma \right) \right\|_{(L^2(\Sigma(x^0))^n} \geq C \left(||\varphi^0|| + |\varphi^1| \right). \tag{4.20}$$

The direct inequality follows imediately from (4.7) and (4.8). This concludes the desired result. □

5. Regularity of the Ultra Weak Solution

We begin this section considering the nonhomogeneous boundary value problem:

$$\left| \begin{array}{l} u'' - \Delta u + \int_0^t g(t-\sigma)\Delta u(\sigma) d\sigma = -\nabla p \quad \text{in } Q \\ div\, u = 0 \quad \text{in} \quad Q \\ u = v \quad \text{in} \quad \Sigma \\ u(0) = u^0, \quad u'(0) = u^1 \quad \text{in} \quad \Omega, \end{array} \right. \tag{5.1}$$

with

$$\left\{u^0, u^1, v\right\} \in H \times V' \times Z. \tag{5.2}$$

The solution u of (5.1) is defined by the transposition method c.f. J. L. Lions [8] Chap. I, section 4. More precisely $u \in L^\infty(0,T;H)$ is a ultra weak solution of (5.1) if its verifies

$$\int_0^T (u(t), f(t))\, dt = -\left(u^0, \theta'(0)\right) + \langle u^1, \theta(0)\rangle - \int_\Sigma \frac{\partial\theta}{\partial\nu} v d\Sigma, \quad \forall f \in L^1(0,T,H)$$

where

$$\theta(x,t) = \varphi(x,t) + \int_0^t g(t-\sigma)\varphi(x,\sigma)d\sigma$$

and φ is the solution of the following problem

$$\left|\begin{array}{l} \varphi'' - \Delta\varphi + \displaystyle\int_t^T g(\sigma - t)\Delta\varphi(\sigma)d\sigma = f - \nabla q \quad \text{in} \quad Q \\ div\ \varphi = 0 \quad \text{in} \quad Q \\ \varphi = 0 \quad \text{in} \quad \Sigma \\ \varphi(T) = \varphi'(T) = 0 \quad \text{in} \quad \Omega. \end{array}\right. \tag{5.3}$$

Considering analogous arguments to those ones used in J. L. Lions [8] we also have

$$u \in C^0\left([0,T], H\right) \cap C^1\left([0,T], V'\right). \tag{5.4}$$

However, the case we are interested in is the one which $f = 0$ and the control v, like in the wave equation, is given by

$$v = \frac{\partial\theta}{\partial\nu} = \frac{\partial}{\partial\nu}\left(\varphi + \int_t^T g(\sigma - t)\varphi(\sigma)d\sigma\right). \tag{5.5}$$

From the direct inequality and taking into account the reversibility of the problem (5.3) we have just proved that

$$\frac{\partial\theta}{\partial\nu} \in L^2(\Sigma). \tag{5.6}$$

Furthermore, from (2.16) we have

$$\frac{\partial\theta}{\partial\nu}\cdot\nu = 0 \quad \text{on} \quad \Sigma \tag{4.7}$$

which proves that $\frac{\partial\theta}{\partial\nu} \in Z$. $\square$

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