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On the Lyapunov Functional of Leslie-Gower Predator-Prey Models with Time-Delay and Holling's Functional Responses

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Abstract

The global stability on the dynamical behavior of the Leslie-Gower predator-prey system with delayed prey specific growth is analyzed by constructing the corresponding Lyapunov functional. Three different types of famous Holling's functional responses are considered in the present study. The sufficient conditions for the global stability analysis of the unique positive equilibrium point are derived accordingly. A numerical example is presented to illustrate the effect of different Holling-Type functional responses on the global stability of the Leislie-Gower predator-prey model.

Keywords: Leslie-Gower predator-prey models, global stability, time delay, Lyapunov functional, Holling's functional response

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1 Introduction

Predator-prey models have been studied by many authors for a long time. Most of studies are interested in the global stability of the unique positive equilibrium point of the predator-prey systems with or without delay. Popular methods in the global stability of predator-prey

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system without time delay can be categorized into the four different types: to construct a Lyapunov function [1, 2, 6, 7, 8] to employ the Dulac Criterion plus the Poincaré-Bendixson Theorem [10] the limit cycle stability analysis [10, 11, 13] and the comparison method [11, 13]. But a more realistic model should include some of the past states of the population system; that is, a real system should be modeled with time delay. As discussed in the references [14, 15, 17], the global stability analysis for the system with time delay relied mainly on constructing a corresponding Lyapunov functional.

The global stability for the Lotka-Volterra model has been extensively addressed, e.g. see [24]. As one of the famous model in describing the dynamic behavior of predator-prey system, the carrying capacity of the predator population in the Lotka-Volterra model is independent of the prey population. Actually, the carrying capacity of the predator population should depend on the prey population which results into the so-called Leslie-Gower model which is a Kolmogorov-Type model and is of the form:

$$\dot{x}(t) = x(t)\left[r\left(1-\frac{x(t)}{K}\right)\right] - p(x)y(t), \tag{1.1}$$

$$\dot{y}(t) = y(t)\left[\delta - \beta\frac{y(t)}{x(t)}\right] \tag{1.2}$$

where $x(t)$ and $y(t)$ denote the density of prey and predator, respectively; r, β and δ are positive constants; and K is the environment carrying capacity. Also, $p(x)$ denotes the functional response of the predator. This system has an unique positive equilibrium point. Various modifications of Leslie-Gower models and associated global stability problem can be referred to [12] and the references cited therein.

There are many different kinds of the predator-prey models with time delay in the literature, for more details we can refer to [3], [5] and [20]. The discussion on the effect of time delay to the dynamic behavior of the system (1.1) and (1.2) are mainly focus on the Leslie-Gower terms in (1.2). Alternatively, we are concerned with the effect of the single time delay τ on the logistic term of the prey, $x(t-\tau)/K$, in (1.1) which was first proposed and discussed by [18]. The time delay appearing in the intra-specific interaction term of the prey equation represents a delayed prey growth effect. The stability, bifurcation, and periodic solutions about similar predator-prey systems are extensively studied in literature, e.g., [4, 9, 16, 21, 22, 23, 25, 26].

In present study, we establish global stability analysis of Leslie-Gower predator-prey models with time delay. Three different functional responses of the predator, i.e., $p(x)$ term in (1.1), are considered by constructing their corresponding Lyapunov functionals according to the Korobeinikov approach for the non-delay model[13]. This paper is organized as follows. In section 2, we introduce some useful definitions and theorems and the bound of the dynamical behavior of the Leslie-Gower predator-prey system with a single delay. In section 3, we analyze the global stability by constructing the Lyapunov functional to Holling's type I, II and III functional responses . Finally, we illustrate our results by some numerical examples.

2 The Model with Time Delay

2.1 Preliminaries

Define $\mathcal{C} \equiv C([-\tau, 0], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence; i.e., for $\phi \in \mathcal{C}$, the norm of ϕ is defined as $\|\phi\| = \sup_{\theta \in [-\tau,0]} |\phi(\theta)|$, where $|\cdot|$ is any norm in $\mathbb{R}^n$. Define $\mathbf{x}_t \in \mathcal{C}$ as $\mathbf{x}_t(\theta) = \mathbf{x}(t+\theta)$, $\theta \in [-\tau, 0]$. Consider the following general nonlinear autonomous system of delay differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}_t), \tag{2.1}$$

where $\mathbf{f} : \Omega \to \mathbb{R}^n$ and Ω is a subset of $\mathcal{C}$. In this paper, we need the following definitions, theorems and lemmas.

Definition 2.1. [15]

1. The solution $\mathbf{x} = \mathbf{0}$ of the system (2.1) is said to be stable if, for any $\sigma \in \mathbb{R}$, $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, \sigma)$ such that $\phi \in B(0, \delta)$ implies $\mathbf{x}_t(\sigma, \phi) \in B(\mathbf{0}, \varepsilon)$ for $t \geq \sigma$. Otherwise, we say $\mathbf{x} = \mathbf{0}$ is unstable.

2. The solution $\mathbf{x} = \mathbf{0}$ of the system (2.1) is said to be asymptotically stable if it is stable and there is a $b_0 = b(\sigma) > 0$ such that $\phi \in B(0, b_0)$ implies $\mathbf{x}(\sigma, \phi)(t) \to \mathbf{0}$ as $t \to \infty$.

3. The solution $\mathbf{x} = \mathbf{0}$ of the system (2.1) is said to be uniformly stable if the number δ in the definition of stable is independent of σ.

4. The solution $\mathbf{x} = \mathbf{0}$ of the system (2.1) is said to be uniformly asymptotically stable if it is uniformly stable and there is a $b_0 > 0$ such that, for every $\eta > 0$, there is a $t_0(\eta)$ such that $\phi \in B(0, b_0)$ implies $\mathbf{x}_t(\sigma, \phi) \in B(\mathbf{0}, \eta)$ for $t \geq \sigma + t_0(\eta)$, for every $\sigma \in \mathbb{R}$.

Definition 2.2. [23] System (2.1) is said to be uniformly persistent if there exists a compact region $D \subset \operatorname{int} \mathbb{R}^2_+$ such that every solution of the system (2.1) eventually enters and remains in the region D.

Lemma 2.3. [15] *Let $u(\cdot)$ and $w(\cdot)$ be nonnegative continuous scalar functions such that $u(0) = w(0) = 0$; $w(s) > 0$ for $s > 0$, $\lim_{s\to\infty} u(s) = +\infty$ and that $V : C \to \mathbb{R}$ is continuous and satisfies*

$$V(\phi) \geq u(|\phi(0)|), \quad \dot{V}(\phi) \leq -w(|\phi(0)|).$$

Then $\mathbf{x} = \mathbf{0}$ is globally asymptotically stable. That is, every solution of the system (2.1) *approaches $\mathbf{x} = \mathbf{0}$ as $t \to +\infty$.*

2.2 The Leslie-Gower System

Consider the Leslie-Gower predator-prey system with time delay τ modeled by

$$\begin{aligned} \dot{x}(t) &= x(t)\left[r\left(1 - \frac{x(t-\tau)}{K}\right) - \frac{p(x)}{x(t)}y(t)\right], \\ \dot{y}(t) &= y(t)\left[\delta - \beta\frac{y(t)}{x(t)}\right] \end{aligned} \tag{2.2}$$

with the initial conditions

$$\begin{aligned} &x(\theta) = \phi(\theta) \geq 0,\ \theta \in [-\tau, 0],\ \phi \in C([-\tau, 0], \mathbb{R}), \\ &x(0) > 0,\ y(0) > 0, \end{aligned} \tag{2.3}$$

where r, K, c, β, δ and τ are positive constants, x and y denote the densities of prey and predator population, respectively. The biological population is to be discussed and we need only to consider the first quadrant in xy-plane. The following consistent condition with (2.2) is assumed:

(A) $p \in C^1([0,\infty), [0,\infty))$; $p(0) = 0$ and $p'(x) \geq 0$ for all $x \geq 0$.

The popular functional responses of the predator, $p(x)$, in the literature are $p(x) = cx$ $p(x) = c\frac{x}{1+x}$, and $p(x) = c\frac{x^2}{1+x^2}$ of the Holling-type I, II, and III, respectively, for some positive constant c. And it is evident that $p(x) \leq c\max\{x, x^2\}$ for these three responses.

Lemma 2.4. *Every solution of the system* (2.2) *with the initial conditions* (2.3) *exists in the interval* $[0,\infty)$ *and remains positive for all* $t \geq 0$.

proof. It is true because

$$\begin{aligned} x(t) &= x(0)\exp\left\{\int_0^t \left[r\left(1 - \frac{x(s-\tau)}{K}\right) - \frac{p(x(s))}{x(s)}y(s)\right]\, ds\right\}, \\ y(t) &= y(0)\exp\left\{\int_0^t \left[\delta - \beta\frac{y(s)}{x(s)}\right]\, ds\right\} \end{aligned}$$

and $x(0) > 0$ and $y(0) > 0$. □

Lemma 2.5. *Let* $(x(t), y(t))$ *denote the solution of* (2.2) *with the initial conditions* (2.3), *then*

$$0 < x(t) \le M, \; 0 < y(t) \le L \tag{2.4}$$

eventually for all large t, *where*

$$M = Ke^{r\tau}, \tag{2.5}$$

$$L = \frac{\delta}{\beta} M. \tag{2.6}$$

proof. Now, we want to show that there exists a $T > 0$ such that $x(t) \le M$ for $t > T$. By Lemma 2.4, we know that solutions of the system (2.2) with the initial conditions (2.3) are positive, hence by assumption (A), and (2.2) becomes

$$\begin{aligned} \dot{x}(t) &= rx(t)\left(1 - \frac{x(t-\tau)}{K}\right) - p(x(t))y(t) \\ &\le x(t)\left[r\left(1 - \frac{x(t-\tau)}{K}\right)\right]. \end{aligned} \tag{2.7}$$

Taking $M^* = K(1 + k_1)$ for $0 < k_1 < e^{r\tau} - 1$. The situation of $x(t)$ with respect to M^* is categorized into two possible cases.

Case 1: Suppose $x(t)$ is not oscillatory about M^*. That is, there exists a $T_0 > 0$ such that either

$$x(t) \le M^* \text{ for } t > T_0 \tag{2.8}$$

or

$$x(t) > M^* \text{ for } t > T_0. \tag{2.9}$$

If (2.8) holds, then for $t > T = T_0$,

$$x(t) \le M^* = K(1 + k_1) < Ke^{r\tau} = M.$$

Suppose (2.9) holds, (2.7) implies that for $t > T_0 + \tau$

$$\begin{aligned} \dot{x}(t) &\le rx(t)\left[1 - \frac{x(t-\tau)}{K}\right] \\ &< -k_1 r x(t). \end{aligned}$$

It follows that

$$\int_{T_0+\tau}^{t} \frac{\dot{x}(s)}{x(s)}\, ds < \int_{T_0+\tau}^{t} (-k_1 r)\, ds = -k_1 r(t - T_0 - \tau),$$

then $0 < x(t) < x(T_0+\tau)\ e^{-k_1 r(t-T_0-\tau)} \to 0$ as $t \to \infty$. That is, $\lim_{t\to\infty} x(t) = 0$ by the Squeeze Theorem. It contradicts to (2.9). Therefore, there exist a $T_1 > T_0$ such that $x(T_1) \le M$. If $x(t) \le M$ for all $t \ge T_1$, and then there exist a $T > 0$ such that $x(t) \le M$ for all $t \ge T$.

Case 2: Suppose $x(t)$ is oscillatory about M^*, then there must exist a $T_2 > T_1$ such that T_2 be the first time which $x(T_2) > M^*$. Therefore, there exists a $T_3 > T_2$ such that T_3 be the first time which $x(T_3) < M^*$ by above discussion. By above, we know that $x(T_1) \le M^*$, $x(T_2) > M^*$ and $x(T_3) \le M^*$ where $T_1 < T_2 < T_3$. Then, by the Intermediate Value Theorem, there exists T_4 and T_5 such that

$$x(T_4) = M^*,\ T_1 \le T_4 < T_2,$$
$$x(T_5) = M^*,\ T_2 < T_5 \le T_3$$

and $x(t) > M^*$ for $T_4 < t < T_5$. Hence there is a $T_6 \in (T_4, T_5)$ such that $x(T_6)$ is a local maximum and (2.7), we have

$$0 = \dot{x}(T_6) \le x(T_6)\left[r\left(1 - \frac{x(T_6-\tau)}{K}\right)\right]$$

and

$$x(T_6-\tau) \le K.$$

Integrating both sides of (2.7) on the interval $[T_6-\tau, T_6]$, we have

$$\ln\left[\frac{x(T_6)}{x(T_6-\tau)}\right] = \int_{T_6-\tau}^{T_6} \frac{\dot{x}(s)}{x(s)}\ ds \le \int_{T_6-\tau}^{T_6} \left[r\left(1-\frac{x(s-\tau)}{K}\right)\right]\ ds \le r\tau$$

and

$$x(T_6) \le x(T_6-\tau)\ e^{r\tau} \le Ke^{r\tau} = M.$$

Applying the same operation on the amplitude of the trajectory $x(t)$, we can find a sequences of T_6 such that every $x(T_6)$ is a local maximum of $x(t)$, and its amplitude is less then M. Hence we can conclude that there exists a $T > 0$ such that

$$x(t) \le M \quad \text{for } t \ge T. \tag{2.10}$$

Now, we want to show that $y(t)$ is bounded above by L eventually for all large t. By (2.10), it follows that for $t > T$

$$\begin{aligned}\dot{y}(t) &= y(t)\left[\delta - \beta\frac{y(t)}{x(t)}\right] \\ &\le y(t)\left[\delta - \frac{\beta}{M}y(t)\right] \\ &= \delta y(t)\left[1 - \frac{y(t)}{\frac{\delta M}{\beta}}\right].\end{aligned}$$

Therefore, $y(t) \le \delta M/\beta = L$ for $t > T$. The proof is complete. □

Lemma 2.6. *Suppose that the system* (2.2) *satisfies*

$$r - c\ \max\{1, M\}\ L > 0, \tag{2.11}$$

where L defined by (2.6)*. Then the system* (2.2) *is uniformly persistent. That is, there exists m, l and $T^* > 0$ such that $m \le x(t) \le M$ and $l \le y(t) \le L$ for $t \ge T^*$, $i = 1, 2$.*

proof. By Lemma 2.5, equation (2.2) follows that for $t \ge T + \tau$

$$\dot{x}(t) \ge x(t)\left[r\left(1 - \frac{M}{K}\right) - \frac{p(x(t))}{x(t)}L\right] \ge x(t)\left[r\left(1 - \frac{M}{K}\right) - c\ \max\{1, M\}\ L\right]. \tag{2.12}$$

Integrating both sides of (2.12) on $[t - \tau, t]$, where $t \ge T + \tau$, then we have

$$x(t) \ge x(t - \tau)\ e^{(r(1-\frac{M}{K}) - c\ \max\{1,M\}\ L)\tau}.$$

That is,

$$x(t - \tau) \le x(t)\ e^{-(r(1-\frac{M}{K}) - c\ \max\{1,M\}\ L)\tau}. \tag{2.13}$$

From (2.2) that for $t \ge T + \tau$

$$\begin{aligned}\dot{x}(t) &= x(t)r\left(1 - \frac{x(t-\tau)}{K}\right) - p(x(t))y(t) \\ &\ge x(t)\left[r - c\ \max\{1, M\}\ L - \frac{r}{K}e^{-(r(1-M/K) - c\ \max\{1,M\}\ L)\tau}\ x(t)\right] \\ &= (r - c\ \max\{1, M\}\ L)\ x(t)\ \left[1 - \frac{x(t)}{\frac{K(r - c\ \max\{1,M\}\ L)}{r}\ e^{(r(1-\frac{M}{K}) - c\ \max\{1,M\}\ L)\tau}}\right].\end{aligned}$$

It follows that

$$\liminf_{t\to\infty} x(t) \ge K\frac{r - c\max\{1, M\}L}{r}\ e^{(r - r\frac{M}{K} - c\max\{1,M\}L)\tau} \equiv \bar{m}$$

and $\bar{m} > 0$. Hence $x(t) > \bar{m} - \varepsilon_1 \equiv m > 0$ with a positive number ε_1, for large t. Next,

$$\begin{aligned}\dot{y}(t) &\geq y(t)\left[\delta - \frac{\beta}{m}y(t)\right] \\ &= \delta y(t)\left[1 - \frac{y(t)}{\frac{\delta m}{\beta}}\right],\end{aligned}$$

it implies

$$\liminf_{t\to\infty} y(t) \geq \frac{\delta m}{\beta} \equiv \bar{l}.$$

Therefore, $y(t) > \bar{l} - \varepsilon_2 \equiv l > 0$ with a positive number ε_2, for large t. Let

$$D = \{(x, y) \mid m \leq x \leq M,\ l \leq y \leq L\}$$

be a bounded compact region in $\mathbb{R}^2_+$ that has positive distance from coordinate hyperplanes. Hence we obtain that there exists a $T^* > 0$ such that if $t \geq T^*$, then every positive solution of the system (2.2) with the initial conditions (2.3) eventually enters and remains in the region D; that is, the system (2.2) is uniformly persistent. □

3 The Lyapunov Functional

The equilibrium point $E^* = (x^*, y^*)$ of the Leslie-Gower system

$$\begin{aligned}\dot{x}(t) &= x(t)\left[r\left(1 - \frac{x(t-\tau)}{K}\right) - \frac{p(x)}{x(t)}y(t)\right], \\ \dot{y}(t) &= y(t)\left[\delta - \beta\frac{y(t)}{x(t)}\right]\end{aligned}$$

satisfies

$$\begin{gathered} r\left(1 - \frac{x^*}{K}\right) = \frac{p(x^*)}{x^*}y^* = \frac{\delta}{\beta}p(x^*) \\ \delta x^* = \beta y^* \end{gathered}$$

or equivalently, is the solution of the systems

$$\begin{gathered} \frac{x^*}{K} + \frac{\delta}{r\beta}p(x^*) = 1, \\ \beta y^* - \delta x^* = 0. \end{gathered}$$

Once the equilibrium point E^* is found, we can obtain an perturbed system to construct the Lyapunov functional. Now three different types of Holling's functional responses are considered:

Type I: $p(x) = c\,x$,

Type II: $p(x) = c\frac{x}{1+x}$,

Type III: $p(x) = c\frac{x^2}{1+x^2}$.

The Lyapunov functionals for each Holling's functional responses are derived based upon the formula proposed by Korobeinikov [13] for non-delay model. Although the Tsai's paper [22] presents a similar result, ours provides larger delay bound for asymptotically stability which will be demonstrated by various examples in next section.

3.1 Holling-Type I Functional Response

When $p(x) = cx$, the equilibrium point $E^*(x^*, y^*)$ is then given by

$$x^* = \frac{K\beta r}{\beta r + \delta K c},$$
$$y^* = \frac{\delta K r}{\beta r + \delta K c}.$$

Theorem 3.1. *Let $p(x) = c\,x$ be the functional response of Holling-Type I and the time delay τ satisfies*

$$r - c\,\max\{1, M\}\,L > 0, \tag{3.1}$$
$$M^2\tau < 2\frac{\beta K}{rc}, \tag{3.2}$$
$$\left(\frac{r}{\delta K} + \frac{c}{2\beta}\right) M\tau < \frac{1}{\delta}, \tag{3.3}$$
$$B^2 - 4AC < 0, \tag{3.4}$$

where

$$A = \frac{r}{\delta K} - \frac{rMc\tau}{2\beta K} - \frac{r^2 M\tau}{\delta K^2}, \tag{3.5}$$
$$B = \max\left\{\left|-\frac{c}{\beta} + \frac{1}{M}\right|, \left|-\frac{c}{\beta} + \frac{1}{m}\right|\right\}, \tag{3.6}$$
$$C = \frac{1}{M} - \frac{rMc\tau}{2\beta K}, \tag{3.7}$$

with M and m defined in Lemmas 2.5 and 2.6, respectively, then the unique positive equilibrium E^* of the system (2.2) is globally asymptotically stable.

proof. Define $z(t) = (z_1(t), z_2(t))$ by

$$z_1(t) = \frac{x(t) - x^*}{x^*}, \quad z_2(t) = \frac{y(t) - y^*}{y^*}.$$

From (2.2), the perturbed system is given by

$$\dot{z}_1(t) = [1 + z_1(t)]\left[-cy^* z_2(t) - \frac{rx^* z_1(t-\tau)}{K}\right], \tag{3.8}$$

$$\dot{z}_2(t) = [1 + z_2(t)]\left[\frac{\delta x^* z_1(t) - \beta y^* z_2(t)}{x^*[1 + z_1(t)]}\right]. \tag{3.9}$$

Let

$$V_1(z(t)) = \frac{\{z_1(t) - \ln[1 + z_1(t)]\} + \{z_2(t) - \ln[1 + z_2(t)]\}}{\beta y^*}, \tag{3.10}$$

then from (3.8) and (3.9), we have

$$\dot{V}(z(t)) = \left[-\frac{c}{\beta} + \frac{1}{x^*[1 + z_1(t)]}\right] z_1(t) z_2(t) - \frac{1}{x^*[1 + z_1(t)]} z_2^2(t) - \frac{r z_1(t) z_1(t-\tau)}{\delta K}. \tag{3.11}$$

Suppose the inequality $r - c \ \max\{1, M\} \ L > 0$ hold, then by Lemma 2.6, there exists a $T^* > 0$ such that $m \leq x^*[1 + z_1(t)] \leq M$ for $t > T^*$. The equation (3.11) implies that

$$\dot{V}_1(z(t)) \leq \left(-\frac{c}{\beta} + \frac{1}{x^*[1 + z_1(t)]}\right) z_1(t) z_2(t) - \frac{1}{M} z_2^2(t) - \frac{r z_1(t) z_1(t-\tau)}{\delta K} \tag{3.12}$$

and since

$$\begin{aligned}
& -\frac{r z_1(t) z_1(t-\tau)}{\delta K} \\
= \ & -\frac{r z_1(t)}{\delta K}\left[z_1(t) - \int_{t-\tau}^{t} \dot{z}_1(s) ds\right] \\
= \ & -\frac{r}{\delta K} z_1^2(t) + \frac{r}{\delta K}\int_{t-\tau}^{t} [1 + z_1(s)]\left[-cy^* z_1(t) z_2(s) - \frac{rx^* z_1(t) z_1(s-\tau)}{K}\right] ds \\
\leq \ & -\frac{r}{\delta K} z_1^2(t) \ + \frac{r}{\delta K}\int_{t-\tau}^{t} [1 + z_1(s)]\left[\frac{c\delta x^*}{2\beta}(z_1^2(t) + z_2^2(s)) + \frac{rx^*}{2K}(z_1^2(t) + z_1^2(s-\tau))\right] ds \\
\leq \ & \left(-\frac{r}{\delta K} + \frac{rMc\tau}{2\beta K} + \frac{r^2 M\tau}{2\delta K^2}\right) z_1^2(t) \ + \frac{rMc}{2\beta K}\int_{t-\tau}^{t} z_2^2(s) ds + \frac{r^2 M}{2\delta K^2}\int_{t-\tau}^{t} z_1^2(s-\tau) ds
\end{aligned}$$

then we have

$$\begin{aligned}\dot{V}_1(z(t)) \;\le\; & \left(-\frac{c}{\beta}+\frac{1}{x^*[1+z_1(t)]}\right)z_1(t)z_2(t)+\left(-\frac{r}{\delta K}+\frac{rMc\tau}{2\beta K}+\frac{r^2M\tau}{2\delta K^2}\right)z_1^2(t)\\ & +\frac{r^2M}{2\delta K^2}\int_{t-\tau}^{t} z_1^2(s-\tau)ds-\frac{1}{M}z_2^2(t)+\frac{rMc}{2\beta K}\int_{t-\tau}^{t} z_2^2(s)ds. \qquad (3.13)\end{aligned}$$

Let

$$V_2(z(t))=\frac{rMc}{2\beta K}\int_{t-\tau}^{t}\int_{s}^{t} z_2^2(\gamma)d\gamma ds+\frac{r^2M}{2\delta K^2}\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma-\tau)d\gamma ds \qquad (3.14)$$

and

$$V_3(z(t))=\frac{r^2M\tau}{2\delta K^2}\int_{t-\tau}^{t} z_1^2(s)ds, \qquad (3.15)$$

then

$$\begin{aligned}\dot{V}_2(z(t)) \;=\; & \frac{r^2M\tau}{2\delta K^2}z_1^2(t-\tau)-\frac{r^2M}{2\delta K^2}\int_{t-\tau}^{t} z_1^2(\gamma-\tau)d\gamma\\ & +\frac{rMc\tau}{2\beta K}z_2^2(t)-\frac{rMc}{2\beta K}\int_{t-\tau}^{t} z_2^2(\gamma)d\gamma \qquad (3.16)\end{aligned}$$

and

$$\dot{V}_3(z(t))=\frac{r^2M\tau}{2\delta K^2}z_1^2(t)-\frac{r^2M\tau}{2\delta K^2}z_1^2(t-\tau). \qquad (3.17)$$

Now we define a Lyapunov functional $V(z(t))$ as

$$V(z(t))=V_1(z(t))+V_2(z(t))+V_3(z(t)), \qquad (3.18)$$

then from (3.13), (3.16) and (3.17) it follows that for $t\ge T^*$

$$\begin{aligned}\dot{V}(z(t)) \;\le\; & -\left(\frac{r}{\delta K}-\frac{rMc\tau}{2\beta K}-\frac{r^2M\tau}{\delta K^2}\right)z_1^2(t)+\left(-\frac{c}{\beta}+\frac{1}{x^*[1+z_1(t)]}\right)z_1(t)z_2(t)\\ & -\left(\frac{1}{M}-\frac{rMc\tau}{2\beta K}\right)z_2^2(t). \qquad (3.19)\end{aligned}$$

By (3.19), there is $\varepsilon>0$ such that

$$\dot{V}(z(t))\le-\varepsilon(z_1^2(t)+z_2^2(t)) \qquad (3.20)$$

if and only if $A>0$, $C>0$ and $B_1^2-4AC<0$ where A and C are defined by (3.5) and (3.7), and

$$B_1=-\frac{c}{\beta}+\frac{1}{x^*[1+z_1(t)]}$$

for all possible trajectory $(x^*[1+z_1(t)], y^*[1+z_2(t)])$. Since $m \le x^*[1+z_1(t)] \le M$ for $t > T^*$, i.e.,

$$-\frac{c}{\beta} + \frac{1}{M} \le -\frac{c}{\beta} + \frac{1}{x^*[1+z_1(t)]} \le -\frac{c}{\beta} + \frac{1}{m}$$

and by define

$$B = \max\left\{\left|-\frac{c}{\beta} + \frac{1}{M}\right|, \left|-\frac{c}{\beta} + \frac{1}{m}\right|\right\}$$

Then the condition for $\dot{V}(z(t)) \le 0$ becomes

$$\begin{aligned} M^2\tau &< 2\frac{\beta K}{rc}, \\ M\tau &< \frac{2\beta K}{\delta K c + 2\beta r}, \\ B^2 - 4AC &< 0, \end{aligned}$$

where A, B and C are given by equations (3.5)-(3.7).

Define $w(s) = \varepsilon s^2$, then w is nonnegative continuous on $[0, \infty)$, $w(0) = 0$ and $w(s) > 0$ for $s > 0$. It follows that for $t \ge T^*$

$$\dot{V}(z(t)) \le -\varepsilon[z_1^2(t) + z_2^2(t)] = -w(|z(t)|).$$

Now, we want to find a function u such that $V(z(t)) \ge u(|z(t)|)$. From (3.10), (3.14) and (3.15) that

$$V(z(t)) \ge V_1(z(t)) = \frac{1}{\beta y^*}\{z_1(t) - \ln[1+z_1(t)] + z_2(t) - \ln[1+z_2(t)]\}. \tag{3.21}$$

By the Taylor Theorem, we have

$$z_i(t) - \ln[1+z_i(t)] = \frac{z_i^2(t)}{2[1+\theta_i(t)]^2}, \tag{3.22}$$

where $\theta_i(t) \in (0, z_i(t))$ or $(z_i(t), 0)$ for $i = 1, 2$. Consider the all the possible cases for θ_i:

1. $0 < \theta_i(t) < z_i(t)$, for $i = 1, 2$,
2. $-1 < z_i(t) < \theta_i(t) < 0$, for $i = 1, 2$,
3. $0 < \theta_1(t) < z_1(t), \quad -1 < z_2(t) < \theta_2(t) < 0$,
4. $-1 < z_1(t) < \theta_1(t) < 0, \quad 0 < \theta_2(t) < z_2(t)$,

we can find a parameter $\widetilde{N}$ defined by

$$\widetilde{N} = \min\left\{ \frac{1}{2\beta y^*}\left(\frac{x^*}{M}\right)^2, \frac{1}{2\beta y^*}\left(\frac{y^*}{L}\right)^2 \right\}$$

such that

$$V(z(t)) \geq \widetilde{N}|z(t)|^2.$$

Define $u(s) = \widetilde{N}s^2$, then u is nonnegative continuous on $[0,\infty)$ with $u(0) = 0$, $u(s) > 0$ for $s > 0$ and $\lim_{s\to\infty} u(s) = +\infty$. Thus we have

$$V(z(t)) \geq u(|z(t)|) \text{ for } t \geq T^*.$$

Hence the equilibrium point E^* of the system (2.2) is globally asymptotically stable by Lemma 2.3. □

Remark 1. In the proof of Theorem 3.1, the corresponding Lyapunov functional (3.18) for $\tau = 0$ becomes

$$V(x,y) \;=\; \frac{1}{\beta y^*}\left[\frac{x}{x^*} - 1 + \ln\frac{x^*}{x} + \frac{y}{y^*} - 1 + \ln\frac{y^*}{y}\right] \tag{3.23}$$

which the same as the Lyapunov functional by Korobeinikov [13] with an extra constant -2 and multiplicative constant $1/\beta y^*$ such that $V(x^*,y^*) = 0$.

3.2 Holling-Type II Functional Response

When $p(x) = c\frac{x}{1+x}$, the equilibrium point $E^*(x^*,y^*)$ is obtained by solving

$$x^{*2} + K\left(\frac{1}{K} + \frac{\delta c}{r\beta} - 1\right)x^* - 1 = 0,$$

$$y^* = \frac{\delta}{\beta}x^*.$$

Theorem 3.2. *Let $p(x) = c\,\frac{x}{1+x}$ be the functional response of Holling-Type II and the time delay τ satisfies*

$$r - c\,\max\{1,M\}\,L \;>\; 0, \tag{3.24}$$

$$\frac{M^2}{1+m}\tau \;<\; 2\frac{K\beta}{rc}, \tag{3.25}$$

$$\left(\frac{r}{\delta K} + \frac{c}{2\beta}\frac{3x^*+5}{(1+x^*)(1+m)}\right)M\tau + \frac{Kc}{\beta r}\left(\frac{1}{1+m} - \frac{1}{(1+x^*)(1+M)}\right) \;<\; \frac{1}{\delta}, \tag{3.26}$$

$$B^2 - 4AC \;<\; 0, \tag{3.27}$$

where

$$A = \frac{r(K - rM\tau)}{\delta K^2} + \frac{c}{\beta(1+x^*)(1+M)} - \frac{c}{\beta(1+m)} - \frac{rMc\tau(3x^*+5)}{2\beta K(1+m)(1+x^*)} \tag{3.28}$$

$$B = \max\left\{\left|-\frac{c}{\beta(1+m)} + \frac{1}{M}\right|, \left|-\frac{c}{\beta(1+M)} + \frac{1}{m}\right|\right\}, \tag{3.29}$$

$$C = \frac{1}{M} - \frac{rMc\tau}{2\beta K(1+m)} \tag{3.30}$$

with M and m *defined in Lemmas 2.5 and 2.6, respectively, then the unique positive equilibrium* E^* *of the system* (2.2) *is globally asymptotically stable.*

proof. Define $z(t) = (z_1(t), z_2(t))$ by

$$z_1(t) = \frac{x(t) - x^*}{x^*}, \quad z_2(t) = \frac{y(t) - y^*}{y^*}.$$

From (2.2), the perturbed system is given by

$$\begin{aligned} \dot{z}_1(t) &= [1 + z_1(t)]\left[\frac{cy^* z_1(t)}{1 + x^*[1 + z_1(t)]} - \frac{cy^* z_1(t)}{(1+x^*)\left(1 + x^*[1+z_1(t)]\right)} - \frac{cy^* z_2(t)}{1 + x^*[1+z_1(t)]}\right. \\ &\qquad \left. - \frac{rx^* z_1(t-\tau)}{K}\right], \qquad (3.31) \\ \dot{z}_2(t) &= [1 + z_2(t)]\left[\frac{\delta x^* z_1(t) - \beta y^* z_2(t)}{x^*[1+z_1(t)]}\right]. \qquad (3.32) \end{aligned}$$

Let

$$V_1(z(t)) = \frac{\{z_1(t) - \ln[1+z_1(t)]\} + \{z_2(t) - \ln[1+z_2(t)]\}}{\beta y^*}, \tag{3.33}$$

then from (3.31) and (3.32), we have

$$\begin{aligned} \dot{V}_1(z(t)) &= \frac{1}{\beta y^*}\left(\frac{z_1(t)\dot{z}_1(t)}{1+z_1(t)} + \frac{z_2(t)\dot{z}_2(t)}{1+z_2(t)}\right) \\ &= \left(-\frac{c}{\beta\left(1 + x^*[1+z_1(t)]\right)} + \frac{1}{x^*[1+z_1(t)]}\right) z_1(t) z_2(t) + \frac{c z_1^2(t)}{\beta\left(1 + x^*[1+z_1(t)]\right)} \\ &\quad - \frac{c z_1^2(t)}{\beta(1+x^*)\left(1 + x^*[1+z_1(t)]\right)} - \frac{z_2^2(t)}{x^*[1+z_1(t)]} - \frac{r z_1(t) z_1(t-\tau)}{\delta K} \\ &\le \left(-\frac{c}{\beta\left(1 + x^*[1+z_1(t)]\right)} + \frac{1}{x^*[1+z_1(t)]}\right) z_1(t) z_2(t) \\ &\quad + \left(\frac{c}{\beta(1+m)} - \frac{c}{\beta(1+x^*)(1+M)}\right) z_1(t)^2 - \frac{1}{M} z_2^2(t) - \frac{r\, z_1(t)\, z_1(t-\tau)}{\delta K}. \end{aligned}$$

Suppose the inequality $r - c\ \max\{1, M\}\ L > 0$ hold, then by Lemma 2.6, there exists a $T^* > 0$ such that $m \le x^*[1 + z_1(t)] \le M$ for $t > T^*$. Since

$$
\begin{aligned}
-\frac{rz_1(t)z_1(t-\tau)}{\delta K} &= -\frac{rz_1(t)}{\delta K}\left[z_1(t) - \int_{t-\tau}^{t} \dot{z}_1(s)ds\right] \\
&= -\frac{r}{\delta K}z_1^2(t) + \frac{r}{\delta K}\int_{t-\tau}^{t} [1 + z_1(s)] \\
&\quad \left[\frac{cy^*z_1(t)z_1(s)}{1 + x^*[1 + z_1(s)]} - \frac{cy^*z_1(t)z_1(s)}{(1 + x^*)\left(1 + x^*[1 + z_1(s)]\right)} - \frac{cy^*z_1(t)z_2(s)}{1 + x^*[1 + z_1(s)]}\right. \\
&\quad \left. -\frac{rx^*z_1(t)z_1(s-\tau)}{K}\right] ds \\
&\le -\frac{r}{\delta K}z_1^2(t) + \frac{r}{\delta K}\int_{t-\tau}^{t} x^*[1 + z_1(s)] \\
&\quad \left[\left(\frac{c\delta}{\beta\left(1 + x^*[1 + z_1(s)]\right)} + \frac{c\delta}{\beta(1 + x^*)\left(1 + x^*[1 + z_1(s)]\right)}\right)\frac{z_1^2(t) + z_1^2(s)}{2}\right. \\
&\quad \left. +\frac{c\delta}{\beta\left(1 + x^*[1 + z_1(s)]\right)}\frac{z_1^2(t) + z_2^2(s)}{2} + \frac{r}{K}\frac{z_1^2(t) + z_1^2(s-\tau)}{2}\right] ds \\
&\le \left(-\frac{r}{\delta K} + \frac{rMc\tau}{\beta K(1 + m)} + \frac{rMc\tau}{2\beta K(1 + x^*)(1 + m)} + \frac{r^2M\tau}{2\delta K^2}\right) z_1^2(t) \\
&\quad +\frac{rM}{2K}\left(\frac{c}{\beta(1 + m)} + \frac{c}{\beta(1 + x^*)(1 + m)}\right)\int_{t-\tau}^{t} z_1^2(s)ds \\
&\quad +\frac{rMc}{2K\beta(1 + m)}\int_{t-\tau}^{t} z_2^2(s)ds + \frac{r^2M}{2\delta K^2}\int_{t-\tau}^{t} z_1^2(s-\tau)ds
\end{aligned}
$$

then we have

$$
\begin{aligned}
\dot{V}_1(z(t)) &\le \left(-\frac{c}{\beta\left(1 + x^*[1 + z_1(t)]\right)} + \frac{1}{x^*(1 + z_1(t))}\right) z_1(t)z_2(t) \\
&\quad +\left(\frac{c}{\beta(1 + m)} - \frac{c}{\beta(1 + x^*)(1 + M)} - \frac{r}{\delta K} + \frac{rMc\tau}{K\beta(1 + m)}\right. \\
&\qquad \left. +\frac{rMc\tau}{2K\beta(1 + x^*)(1 + m)} + \frac{r^2M\tau}{2\delta K^2}\right) z_1^2(t) \\
&\quad -\frac{1}{M}z_2^2(t) + \frac{rMc}{2K\beta}\left(\frac{1}{1 + m} + \frac{1}{(1 + x^*)(1 + m)}\right)\int_{t-\tau}^{t} z_1^2(s)ds \\
&\quad +\frac{rMc}{2K\beta(1 + m)}\int_{t-\tau}^{t} z_2^2(s)ds + \frac{r^2M}{2\delta K^2}\int_{t-\tau}^{t} z_1^2(s-\tau)ds. \qquad (3.34)
\end{aligned}
$$

Let

$$V_2(z(t)) = \frac{rMc}{2K\beta}\left(\frac{1}{1+m}+\frac{1}{(1+x^*)(1+m)}\right)\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma)d\gamma ds + \frac{rMc}{2K\beta(1+m)}\int_{t-\tau}^{t}\int_{s}^{t} z_2^2(\gamma)d\gamma ds + \frac{r^2M}{2\delta K^2}\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma-\tau)d\gamma ds \quad (3.35)$$

and

$$V_3(z(t)) = \frac{r^2M\tau}{2\delta K^2}\int_{t-\tau}^{t} z_1^2(s)ds, \quad (3.36)$$

then

$$\begin{aligned}\dot{V}_2(z(t)) &= \frac{rMc}{2K\beta}\left(\frac{1}{1+m}+\frac{1}{(1+x^*)(1+m)}\right)z_1^2(t) - \frac{rMc}{2K\beta}\left(\frac{1}{1+m}+\frac{1}{(1+x^*)(1+m)}\right)\int_{t-\tau}^{t} z_1^2(\gamma)d\gamma \\ &+\frac{rMc\tau}{2K\beta(1+m)}z_2^2(t) - \frac{rMc}{2K\beta(1+m)}\int_{t-\tau}^{t} z_2^2(\gamma)d\gamma \\ &+\frac{r^2M\tau}{2\delta K^2}z_1^2(t-\tau) - \frac{r^2M}{2\delta K^2}\int_{t-\tau}^{t} z_1^2(\gamma-\tau)d\gamma. \end{aligned} \quad (3.37)$$

and

$$\dot{V}_3(z(t)) = \frac{r^2M\tau}{2\delta K^2}z_1^2(t) - \frac{r^2M\tau}{2\delta K^2}z_1^2(t-\tau). \quad (3.38)$$

Now we define a Lyapunov functional $V(z(t))$ as

$$V(z(t)) = V_1(z(t)) + V_2(z(t)) + V_3(z(t)), \quad (3.39)$$

then from (3.34), (3.37) and (3.38) it follows that for $t \geq T^*$

$$\begin{aligned}\dot{V}(z(t)) &= \dot{V}_1(z(t)) + \dot{V}_2(z(t)) + \dot{V}_3(z(t)) \\ &\leq -\left(\frac{r(K-rM\tau)}{\delta K^2} + \frac{c}{\beta(1+x^*)(1+M)} - \frac{c}{\beta(1+m)} - \frac{rMc\tau(3x^*+5)}{2K\beta(1+x^*)(1+m)}\right)z_1^2(t) \\ &+\left(-\frac{c}{\beta\left(1+x^*[1+z_1(t)]\right)} + \frac{1}{x^*[1+z_1(t)]}\right)z_1(t)z_2(t) \\ &-\left(\frac{1}{M} - \frac{rMc\tau}{2K\beta(1+m)}\right)z_2^2(t) \end{aligned} \quad (3.40)$$

By (3.40), there is $\varepsilon > 0$ such that

$$\dot{V}(z(t)) \leq -\varepsilon(z_1^2(t) + z_2^2(t)) \quad (3.41)$$

if and only if $A > 0$, $C > 0$ and $B_2^2 - 4AC < 0$ where A and C are defined by (3.28) and (3.30), and

$$B_2 = -\frac{c}{\beta\left(1 + x^*[1 + z_1(t)]\right)} + \frac{1}{x^*[1 + z_1(t)]}$$

for all possible trajectory $(x^*[1+z_1(t)], y^*[1+z_2(t)])$. Since $m \leq x^*[1+z_1(t)] \leq M$ for $t > T^*$, i.e.,

$$-\frac{c}{\beta(1+m)} + \frac{1}{M} \leq -\frac{c}{\beta\left(1 + x^*[1 + z_1(t)]\right)} + \frac{1}{x^*[1 + z_1(t)]} \leq -\frac{c}{\beta(1+M)} + \frac{1}{m}$$

and by define

$$B = \max\left\{\left|-\frac{c}{\beta(1+m)} + \frac{1}{M}\right|, \left|-\frac{c}{\beta(1+M)} + \frac{1}{m}\right|\right\}$$

Then the condition for $\dot{V}(z(t)) \leq 0$ becomes

$$\begin{aligned} \frac{M^2}{1+m}\tau &< 2\frac{K\beta x}{rc}, \\ \left(\frac{r}{\delta K} + \frac{c}{2\beta}\frac{3x^* + 5}{(1+x^*)(1+m)}\right)M\tau + \frac{Kc}{\beta r}\left(\frac{1}{1+m} - \frac{1}{(1+x^*)(1+M)}\right) &< \frac{1}{\delta}, \\ B^2 - 4AC &< 0, \end{aligned}$$

where A, B and C are given by (3.28)-(3.30).

Define $w(s) = \varepsilon s^2$, then w is nonnegative continuous on $[0, \infty)$, $w(0) = 0$ and $w(s) > 0$ for $s > 0$. It follows that for $t \geq T^*$

$$\dot{V}(z(t)) \leq -\varepsilon[z_1^2(t) + z_2^2(t)] = -w(|z(t)|).$$

To find a function u such that $V(z(t)) \geq u(|z(t)|)$, since the following relationship is still hold for Holling's Type II

$$V(z(t)) \geq V_1(z(t)) = \frac{1}{\beta y^*}\left\{z_1(t) - \ln[1 + z_1(t)] + z_2(t) - \ln[1 + z_2(t)]\right\}.$$

and then by the same argument proposed in the proof of Theorem 3.1, we can establish that this is a nonnegative continuous function u defined on $[0, \infty)$ with $u(0) = 0$, $u(s) > 0$ for $s > 0$ and $\lim_{s\to\infty} u(s) = +\infty$ and

$$V(z(t)) \geq u(|z(t)|) \text{ for } t \geq T^*.$$

Hence the equilibrium point E^* of the system (2.2) is globally asymptotically stable by Lemma 2.3. □

3.3 Holling-Type III Functional Response

When $p(x) = c\frac{x^2}{1+x^2}$, the equilibrium point $E^*(x^*, y^*)$ is obtained by solving

$$x^{*3} + K\left(\frac{\delta c}{r\beta} - 1\right)x^{*2} + x^* - K = 0,$$
$$y^* = \frac{\delta}{\beta}x^*.$$

Theorem 3.3. *Let $p(x) = c\,\frac{x^2}{1+x^2}$ be the functional response of Holling-Type III and the time delay τ satisfies*

$$r - c\ \max\{1, M\}\ L \ > \ 0, \qquad (3.42)$$
$$\frac{M^2(M + 2x^*)}{1 + m^2}\tau \ < \ 2\frac{K\beta}{rc}, \qquad (3.43)$$
$$\left(\frac{r}{\delta K} + \frac{c}{2\beta}\frac{(5 + 3x^{*2})(M + 2x^*) + 2x^*}{\beta(1 + x^{*2})(1 + m^2)}\right)M\tau + \frac{KC}{\beta r}\left(\frac{M + 2x^*}{1 + m^2} + \frac{M + x^*}{(1 + x^*)(1 + m^2)} - \frac{2x^*}{(1 + x^{*2})(1 + M^2)}\right) \ < \ \frac{1}{\delta}, \qquad (3.44)$$
$$B^2 - 4AC \ < \ 0, \qquad (3.45)$$

where

$$A = \frac{r}{\delta K} + \frac{2cx^*}{\beta(1 + x^{*2})(1 + M^2)} - \frac{c(M + 2x^*)}{\beta(1 + m^2)} - \frac{c(M + x^*)}{\beta(1 + x^*)(1 + m^2)} - \frac{3rMc(M + 2x^*)\tau}{2K\beta(1 + m^2)} - \frac{rMc(M + 3x^*)\tau}{K\beta(1 + x^{*2})(1 + m^2)} - \frac{r^2M\tau}{\delta K^2} \qquad (3.46)$$
$$B = \max\left\{\left|-\frac{cM}{\beta(1 + m^2)} + \frac{1}{M}\right|, \left|-\frac{cm}{\beta(1 + M^2)} + \frac{1}{m}\right|\right\} \qquad (3.47)$$
$$C = \frac{1}{M} - \frac{rMc(M + 2x^*)\tau}{2K\beta(1 + m^2)} \qquad (3.48)$$

where M and *m defined in Lemmas 2.5 and 2.6, then the unique positive equilibrium E^* of the system* (2.2) *is globally asymptotically stable.*

proof. Define $z(t) = (z_1(t), z_2(t))$ by

$$z_1(t) = \frac{x(t) - x^*}{x^*}, \quad z_2(t) = \frac{y(t) - y^*}{y^*}.$$

From (2.2), the perturbed system is given by

$$\begin{aligned}
\dot{z}_1(t) &= [1+z_1(t)]\left[\frac{cx^*y^*z_1(t)}{1+x^{*2}[1+z_1(t)]^2} - \frac{2cx^*y^*z_1(t)}{(1+x^{*2})\,(1+x^{*2}[1+z_1(t)]^2)} + \frac{cx^*y^*z_1^2(t)}{1+x^{*2}[1+z_1(t)]^2}\right. \\
&\quad - \frac{cx^*y^*z_1^2(t)}{(1+x^{*2})\,(1+x^{*2}[1+z_1(t)]^2)} - \frac{cx^*y^*z_2(t)}{1+x^{*2}[1+z_1(t)]^2} - \frac{cx^*y^*z_1(t)z_2(t)}{1+x^{*2}[1+z_1(t)]^2} \\
&\quad \left. - \frac{rx^*z_1(t-\tau)}{K}\right], \qquad (3.49) \\
\dot{z}_2(t) &= [1+z_2(t)]\left[\frac{\delta x^*z_1(t)-\beta y^*z_2(t)}{x^*[1+z_1(t)]}\right]. \qquad (3.50)
\end{aligned}$$

Let

$$V_1(z(t)) = \frac{\{z_1(t)-\ln[1+z_1(t)]\}+\{z_2(t)-\ln[1+z_2(t)]\}}{\beta y^*}, \qquad (3.51)$$

then from (3.49) and (3.50), we have

$$\dot{V}_1(z(t)) = \frac{1}{\beta y^*}\left(\frac{z_1(t)\dot{z}_1(t)}{1+z_1(t)} + \frac{z_2(t)\dot{z}_2(t)}{1+z_2(t)}\right)$$

and after some algebraic operation similar to the proof of Theorem 3.1, it follows that

$$\begin{aligned}
\dot{V}_1(z(t)) &= -\frac{cx^*z_1(t)z_2(t)}{\beta\,(1+x^{*2}[1+z_1(t)]^2)} + \frac{z_1(t)z_2(t)}{x^*[1+z_1(t)]} + \frac{cx^*z_1^2(t)}{\beta\,(1+x^{*2}[1+z_1(t)]^2)} \\
&\quad - \frac{2cx^*z_1^2(t)}{\beta(1+x^{*2})\,(1+x^{*2}[1+z_1(t)]^2)} + \frac{cx^*z_1^3(t)}{\beta\,(1+x^{*2}[1+z_1(t)]^2)} \\
&\quad - \frac{cx^*z_1^3(t)}{\beta(1+x^{*2})\,(1+x^{*2}[1+z_1(t)]^2)} - \frac{cx^*z_1^2(t)z_2(t)}{\beta\,(1+x^{*2}[1+z_1(t)]^2)} \\
&\quad - \frac{z_2^2(t)}{x^*[1+z_1(t)]} - \frac{rz_1(t)z_1(t-\tau)}{\delta K} \\
&\le \left(-\frac{cx^*[1+z_1(t)]}{\beta\,(1+x^{*2}[1+z_1(t)]^2)} + \frac{1}{x^*[1+z_1(t)]}\right) z_1(t)z_2(t) \\
&\quad + \frac{c}{\beta}\left(\frac{M+2x^*}{1+m^2} - \frac{2x^*}{(1+x^{*2})(1+M^2)} + \frac{M+x^*}{(1+x^{*2})(1+m^2)}\right) z_1^2(t) \\
&\quad - \frac{z_2^2(t)}{M} - \frac{rz_1(t)z_1(t-\tau)}{\delta K}
\end{aligned}$$

Suppose the following inequality $r - c\ \max\{1, M\}\ L > 0$ hold, then by Lemma 2.6, there exists a $T^*>0$ such that $m \le x^*[1+z_1(t)] \le M$ for $t>T^*$. Since

$$\begin{aligned}
-\frac{rz_1(t)z_1(t-\tau)}{\delta K} &= -\frac{r}{\delta K}z_1^2(t)+\frac{r}{\delta K}\int_{t-\tau}^{t} z_1(t)\dot{z}_1(s)ds \\
&\le \left(-\frac{r}{\delta K}+\frac{rMc\tau(M+3x^*)}{2\beta K(1+m^2)}+\frac{rMcx^*\tau}{K\beta(1+x^{*2})(1+m^2)}+\frac{r^2M\tau}{2\delta K^2}\right)z_1^2(t) \\
&\quad +\frac{rMc}{2K\beta}\left(\frac{2M+3x^*}{1+m^2}+\frac{2(M+2x^*)}{(1+x^{*2})(1+m^2)}\right)\int_{t-\tau}^{t} z_1^2(s)ds \\
&\quad +\frac{rMc(M+2x^*)}{2K\beta(1+m^2)}\int_{t-\tau}^{t} z_2^2(s)ds+\frac{r^2M}{2\delta K^2}\int_{t-\tau}^{t} z_1^2(s-\tau)ds
\end{aligned}$$

then we have

$$\begin{aligned}
\dot{V}_1(z(t)) &\le \left(-\frac{cx^*[1+z_1(t)]}{\beta\left(1+x^{*2}[1+z_1(t)]^2\right)}+\frac{1}{x^*[1+z_1(t)]}\right)z_1(t)z_2(t) \\
&\quad +\left(-\frac{r}{\delta K}-\frac{2cx^*}{\beta(1+x^{*2})(1+M^2)}+\frac{c(M+2x^*)}{\beta(1+m^2)}+\frac{rMc\tau(M+3x^*)}{2K\beta(1+m^2)}\right. \\
&\quad \left.\frac{rMc\tau x^*}{\beta K(1+x^{*2})(1+m^2)}+\frac{r^2M\tau}{2\delta K^2}\right)z_1^2(t) \\
&\quad -\frac{1}{M}z_2^2(t)+\frac{rMc}{2K\beta}\left(\frac{2M+3x^*}{1+m^2}+\frac{2(M+2x^*)}{(1+x^{*2})(1+m^2)}\right)\int_{t-\tau}^{t} z_1^2(s)ds \\
&\quad +\frac{rMc(M+2x^*)}{2K\beta(1+m^2)}\int_{t-\tau}^{t} z_2^2(s)ds+\frac{r^2M}{2\delta K^2}\int_{t-\tau}^{t} z_1^2(s-\tau)ds \qquad (3.52)
\end{aligned}$$

Let

$$\begin{aligned}
V_2(z(t)) &= \frac{rMc}{2K\beta}\left(\frac{2M+3x^*}{1+m^2}+\frac{2(M+2x^*)}{(1+x^{*2})(1+m^2)}\right)\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma)d\gamma ds \\
&\quad +\frac{rMc(M+2x^*)}{2K\beta(1+m^2)}\int_{t-\tau}^{t}\int_{s}^{t} z_2^2(\gamma)d\gamma ds+\frac{r^2M}{2\delta K^2}\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma-\tau)d\gamma ds \qquad (3.53)
\end{aligned}$$

and

$$V_3(z(t))=\frac{r^2M\tau}{2\delta K^2}\int_{t-\tau}^{t} z_1^2(s)ds, \qquad (3.54)$$

then

$$\begin{aligned}
\dot{V}_2(z(t)) &= \frac{rMc\tau}{2K\beta}\left(\frac{2M+3x^*}{1+m^2}+\frac{2(M+2x^*)}{(1+x^{*2})(1+m^2)}\right)z_1^2(t) \\
&\quad -\frac{rM}{2K\beta}\left(\frac{2M+3x^*}{1+m^2}+\frac{2(M+2x^*)}{(1+x^{*2})(1+m^2)}\right)\int_{t-\tau}^{t} z_1^2(\gamma)d\gamma \\
&\quad +\frac{rMc(M+2x^*)\tau}{2K\beta(1+m^2)}z_2^2(t)-\frac{rMc(M+2x^*)}{2K\beta(1+m^2)}\int_{t-\tau}^{t} z_2^2(\gamma)d\gamma \\
&\quad +\frac{r^2M\tau}{2\delta K^2}z_1^2(t-\tau)-\frac{r^2M}{2\delta K^2}\int_{t-\tau}^{t} z_1^2(\gamma-\tau)d\gamma. \qquad (3.55)
\end{aligned}$$

and

$$\dot{V}_3(z(t))=\frac{r^2M\tau}{2\delta K^2}z_1^2(t)-\frac{r^2M\tau}{2\delta K^2}z_1^2(t-\tau). \qquad (3.56)$$

Now we define a Lyapunov functional $V(z(t))$ as

$$V(z(t))=V_1(z(t))+V_2(z(t))+V_3(z(t)), \qquad (3.57)$$

then from (3.52), (3.55) and (3.56) it follows that for $t\geq T^*$

$$\begin{aligned}
\dot{V}(z(t)) &= \dot{V}_1(z(t))+\dot{V}_2(z(t))+\dot{V}_3(z(t)), \\
&\leq -\left(\frac{r}{\delta K}+\frac{2cx^*}{\beta(1+x^{*2})(1+M^2)}-\frac{c(M+2x^*)}{\beta(1+m^2)}-\frac{c(M+x^*)}{\beta(1+x^*)(1+m^2)}\right. \\
&\quad \left.-\frac{3rMc(M+2x^*)\tau}{2K\beta(1+m^2)}-\frac{rMc(M+3x^*)\tau}{K\beta(1+x^{*2})(1+m^2)}-\frac{r^2M\tau}{\delta K^2}\right)z_1^2(t) \\
&\quad +\left(-\frac{cx^*(1+z_1(t))}{\beta\left(1+x^{*2}[1+z_1(t)]^2\right)}+\frac{1}{x^*[1+z_1(t)]}\right)z_1(t)z_2(t) \\
&\quad -\left(\frac{1}{M}-\frac{rMc(M+2x^*)\tau}{2K\beta(1+m^2)}\right)z_2^2(t) \qquad (3.58)
\end{aligned}$$

By (3.58), there is $\varepsilon>0$ such that

$$\dot{V}(z(t))\leq-\varepsilon(z_1^2(t)+z_2^2(t)) \qquad (3.59)$$

if and only if $A>0$, $C>0$ and $B_3^2-4AC<0$ where A and C are defined by (3.46) and (3.48), and

$$B_3=-\frac{cx^*(1+z_1(t))}{\beta\left(1+x^{*2}[1+z_1(t)]^2\right)}+\frac{1}{x^*[1+z_1(t)]}$$

for all possible trajectory $(x^*[1+z_1(t)],y^*[1+z_2(t)])$. Since $m\leq x^*[1+z_1(t)]\leq M$ for $t>T^*$, i.e.,

$$-\frac{cM}{\beta(1+m^2)}+\frac{1}{M}\leq-\frac{cx^*(1+z_1(t))}{\beta\left(1+x^{*2}[1+z_1(t)]^2\right)}+\frac{1}{x^*[1+z_1(t)]}\leq-\frac{cm}{\beta(1+M^2)}+\frac{1}{m}$$

and by define

$$B = \max\left\{\left|-\frac{cM}{\beta(1+m^2)} + \frac{1}{M}\right|, \left|-\frac{cm}{\beta(1+M^2)} + \frac{1}{m}\right|\right\}.$$

Then the condition for $\dot{V}(z(t)) \leq 0$ becomes

$$\begin{aligned}
\frac{M^2(M+2x^*)}{1+m^2}\tau &< 2\frac{K\beta}{rc}, \\
\left(\frac{r}{\delta K} + \frac{c}{2\beta}\frac{(5+3x^{*2})(M+2x^*)+2x^*}{\beta(1+x^{*2})(1+m^2)}\right)M\tau & \\
+\frac{KC}{\beta r}\left(\frac{M+2x^*}{1+m^2} + \frac{M+x^*}{(1+x^*)(1+m^2)} - \frac{2x^*}{(1+x^{*2})(1+M^2)}\right) &< \frac{1}{\delta}, \\
B^2 - 4AC &< 0,
\end{aligned}$$

where A, B and C are given by (3.46)-(3.48).

Define $w(s) = \varepsilon s^2$, then w is nonnegative continuous on $[0, \infty)$, $w(0) = 0$ and $w(s) > 0$ for $s > 0$. It follows that for $t \geq T^*$

$$\dot{V}(z(t)) \leq -\varepsilon[z_1^2(t) + z_2^2(t)] = -w(|z(t)|).$$

To find a function u such that $V(z(t)) \geq u(|z(t)|)$, since the following relationship is still hold for Holling's Type III

$$V(z(t)) \geq V_1(z(t)) = \frac{1}{\beta y^*}\left\{z_1(t) - \ln[1+z_1(t)] + z_2(t) - \ln[1+z_2(t)]\right\}.$$

and then by the same argument proposed in the proof of Theorem 3.1, we can establish that this is a nonnegative continuous function u defined on $[0, \infty)$ with $u(0) = 0$, $u(s) > 0$ for $s > 0$ and $\lim_{s\to\infty} u(s) = +\infty$ and

$$V(z(t)) \geq u(|z(t)|) \text{ for } t \geq T^*.$$

Hence the equilibrium point E^* of the system (2.2) is globally asymptotically stable by Lemma 2.3 □

4 Numerical Example

The following numerical example is used to illustrate the procedures of applying our results to Leslie-Gower model without and with time delay. Consider the system

$$\begin{aligned}
\dot{x}(t) &= x(t)[3 - 10x(t-\tau)] - p(x)y(t), \\
\dot{y}(t) &= y(t)\left[1 - 6\frac{y(t)}{x(t)}\right],
\end{aligned} \tag{4.1}$$

with the initial conditions

$$\begin{aligned} &x_1(\theta) = x_1(0),\ \theta \in [-\tau, 0], \\ &x_1(0) > 0,\ x_2(0) > 0. \end{aligned} \tag{4.2}$$

The corresponding parameter values are

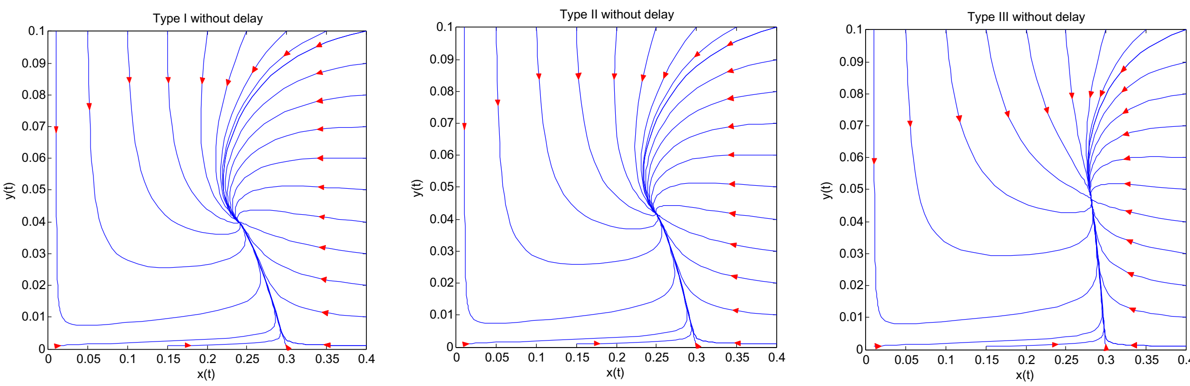
$$r = 3, \quad K = 0.3, \quad c = 15, \quad \delta = 1, \quad \beta = 6.$$

The unique positive equilibrium point $E^* = (x^*, y^*)$ for three-different types of Holling's functional response are listed below.

Table 1: The unique positive equilibrium point E^* of the Leslie-Gower system (4.1) for three types of Holling's functional responses

Holling's functional response	$p(x)$	$E^* = (x^*, y^*)$
Type I	$15x$	$\left(\frac{6}{25}, \frac{1}{25}\right)$
TypeII	$15\frac{x}{1+x}$	$\left(\frac{1}{4}, \frac{1}{24}\right)$
Type III	$15\frac{x^2}{1+x^2}$	$\approx (0.2816, 0.0469)$

When $\tau = 0$, i.e., the model without delay, the unique positive equilibrium point E^* of the system (4.1) is globally asymptotically stable by using the Lyapunov functional (3.23). The corresponding trajectories of the system are depicted in Figure 1.



(a) Holling-Type I (b) Holling-Type II (c) Holling-Type III

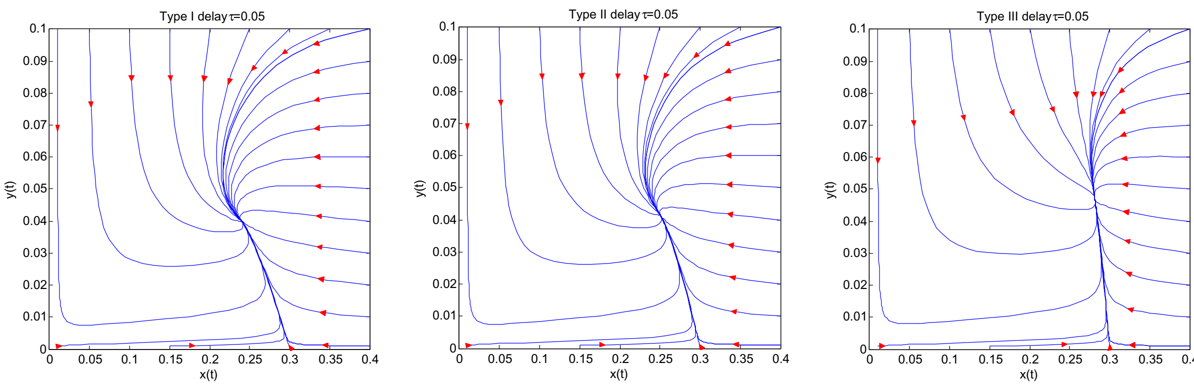
Figure 1: The trajectories of the system (4.1) for three types of Holling's functional responses without delay

Whenever $\tau = 0.05$, we obtain $M = 0.34855$, $L = 0.05809$, $r - c\max\{1, M\}L = 2.12862$, and choose $m = 0.19691$. The corresponding sufficient conditions of Theorems 3.1-3.3 for three different types of Holling's functional responses are verified by the following table

Table 2: The parameters in the sufficient conditions of Theorems 3.1-3.3.

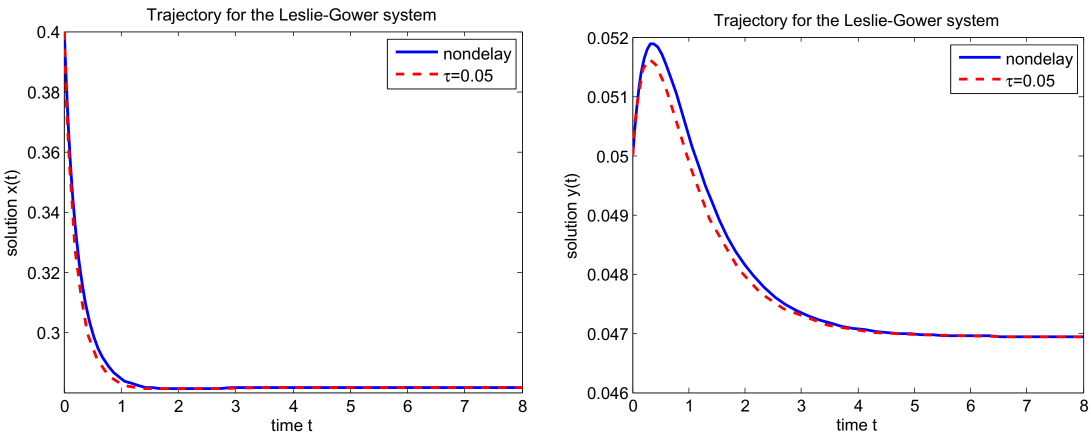
Holling's functional response	A	B	C	$B^2 - AC$
Type I	8.0394	2.5784	2.6512	−78.6074
Type II	6.8076	3.2246	2.6870	−62.7711
Type III	5.0054	4.6395	2.6778	−32.0887

Consequently, by Theorems 3.1-3.3, we conclude that the unique positive equilibrium point E^* of the system (4.1) with initial conditions (4.2) is globally asymptotically stable. The trajectories of the delayed system are depicted in Figure 2. But it is indistinguishable in the phase portraits given by Figures 1 and 2 which correspond to non-delay and delay systems. To observe the effect of time delay on dynamical behavior, we choose the system with Holling-Type III functional response under initial conditions $x(\theta) = 0.4$ for $\theta \in [-\tau, 0]$, $x(0) = 0.4$, and $y(0) = 0.05$. Figure 3 shows the time history of the system trajectories for both cases and the trajectories for the delay system is moving a very little higher and faster than those of the non-delay one.



(a) Holling-Type I (b) Holling-Type II (c) Holling-Type III

Figure 2: The trajectories of the system (4.1) for three types of Holling's functional responses with delay time $\tau = 0.05$



(a) $x(t)$ vs. t (b) $y(t)$ vs. t

Figure 3: Comparison of time history of the trajectory of the system (4.1) for Holling-Type III functional response with delay time $\tau = 0.05$ under initial conditions $x(\theta) = 0.4$ for $\theta \in [-\tau, 0]$, $x(0) = 0.4$, and $y(0) = 0.05$

Based on the sufficient conditions of Theorems 3.1-3.3, it can be numerically verified that the unique positive equilibrium point E^* is globally asymptotically stable whenever the delay time is less than the upper bound defined in Table 3. It is evident that the upper bound on delay time of Holling-Type I from Theorem 3.1 is much larger than those provided by Tsai's paper [22] and Tsai's paper can't provide the information on the time-delay bounds of Holling-Type II & III.

Table 3: The upper bound on delay time of the Leslie-Gower system (4.1) for three types of Holling's functional responses

Methods	Holling-Type I	Holling-Type II	Holling-Type III
Present study	0.127607	0.105255	0.0767035
Tsai's paper [22]	0.006333	–	–

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