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BIFURCATION OF A NONLINEAR ELLIPTIC SYSTEM FROM THE FIRST EIGENVALUE

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Abstract. We study the following bifurcation problem in a bounded domain Ω in $\mathbb{R}^N$:

$$\begin{cases} -\Delta_p u = & \lambda|u|^\alpha|v|^\beta v + f(x,u,v,\lambda) \quad \text{in } \Omega \\ -\Delta_q v = & \lambda|u|^\alpha|v|^\beta u + g(x,u,v,\lambda) \quad \text{in } \Omega \\ (u,v) \in & W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega). \end{cases}$$

We prove that the principal eigenvalue λ_1 of the following eigenvalue problem

$$\begin{cases} -\Delta_p u = & \lambda|u|^\alpha|v|^\beta v \quad \text{in } \Omega \\ -\Delta_q v = & \lambda|u|^\alpha|v|^\beta u \quad \text{in } \Omega \\ (u,v) \in & W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \end{cases}$$

is simple and isolated and we prove that $(\lambda_1, 0, 0)$ is a bifurcation point of the system mentioned above.

1. Introduction

The purpose of this paper is to illustrate a global bifurcation phenomenon for the nonlinear elliptic system

$$(BS) \qquad \begin{cases} -\Delta_p u = & \lambda|u|^\alpha|v|^\beta v + f(x,u,v,\lambda) \quad \text{in } \Omega \\ -\Delta_q v = & \lambda|u|^\alpha|v|^\beta u + g(x,u,v,\lambda) \quad \text{in } \Omega \\ (u,v) \in & W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega), \end{cases}$$

where Ω is a bounded domain not necessary regular in $\mathbb{R}^N$, $N \geq 1$, α, β, p and q are real numbers satisfying suitable conditions which ensure the results. The system (BS) is weakly coupled in the sense that the interaction is present only in the " source terms", while the differential terms have only one dependent variable each. The differential operator involved is the well-known p-Laplacian $\Delta_p u = \nabla.(|\nabla u|^{p-2}\nabla u)$ which reduces to the ordinary Laplacian Δu, when $p = 2$. The nonlinearities f and g satisfy some hypotheses to be specified later. To

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system (BS) we associate the eigenvalue problem system

$$(ES) \qquad \begin{cases} -\Delta_p u = & \lambda|u|^\alpha|v|^\beta v \qquad \text{in } \Omega \\ -\Delta_q v = & \lambda|u|^\alpha|v|^\beta u \qquad \text{in } \Omega \\ (u,v) \in & W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega). \end{cases}$$

We say that λ is an eigenvalue of (ES) if there exists a nontrivial pair $(u,v) \in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ that satisfies (ES) in the following sense

$$(S) \qquad \begin{cases} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi\, dx &= \int_\Omega [\lambda|u|^\alpha|v|^\beta v\phi\, dx \\ \int_\Omega |\nabla v|^{q-2}\nabla v \nabla\psi\, dx &= \int_\Omega [\lambda|u|^\alpha|v|^\beta u\psi\, dx, \end{cases}$$

for any $(\phi,\psi) \in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

The goal of this work is the study of the main properties (simplicity, isolation) of the least positive eigenvalue denote λ_1 of (ES) in theorem 3.1. These properties are well-known in the scalar case of one equation, we refer the reader to the bibliography contained in [14] where several results are cited.

We prove that the eigenvectors (u,v) associated to λ_1 have definite sign in Ω. So, we show that this property implies that λ_1 is simple and isolated in the spectrum. Note that our result of isolation follows by adaptation a technique used by Anane [1] for scalar p-Laplacian in a smooth bounded domain. Concerning the bifurcation problem, we prove the existence of bifurcation branch of nontrivial solutions of (BS) from λ_1, (see Theorem 3.2.). Here we use abstract methods of nonlinear functional analysis based on the generalized topological degree in order to get a new result.

System (BS) has been studied by Chaïb [6] in the case $\Omega = I\!\!R^N$. The author extended Diaz-Saaś inequality in $I\!\!R^N$ and claimed that λ_1 is simple. In [12], the authors obtained similar result by considering for a quite different system. In the case of a bounded domain sufficiently regular the following eigenvalue system

$$\begin{cases} -\Delta_p u = & \lambda|u|^{\alpha-1}|v|^{\beta+1}u \qquad \text{in } \Omega \\ -\Delta_q v = & \lambda|u|^{\alpha+1}|v|^{\beta-1}v \qquad \text{in } \Omega \\ (u,v) \in & W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega) \end{cases}$$

was studied in [5], where the author proved only the simplicity result of the first eigenvalue.

Here, we address to (ES) and (BS) without any regularity assumption on the boundary $\partial\Omega$.

Bifurcation problems in the case of one equation in a smooth bounded domain involving p-Laplacian operator were studied by [2],[9] and [10] under some restrictive conditions on the nonlinearity sources. In any bounded domain, we cite recent results of [7] and [8].

A global bifurcation result from the first eigenvalue of a particular system is considered in [13] under some restrictive hypotheses on the nonlinearities f and g to be specified at the end of this paper.

In this work, we consider a bifurcation system for which we investigate the system improving the conditions on the nonlinearities f and g in any bounded domain with some condition of homogeneity type connecting p, q, α and β. For this end we use a generalized degree of Leray-Shauder.

This paper is organized as follows: in Section 2 we introduce some assumptions, definitions and we prove some auxiliary results that are the key point for proving our results; Section 3 contains a results and proofs of simplicity and isolation of λ_1, finally we verify that the topological degree has a jump when λ crosses λ_1, which implies the bifurcation results.

2. Assumptions and Preliminaries

Through this paper Ω will be a bounded domain of $I\!R^N$. $W_0^{1,p}(\Omega)$ will denote the usual Sobolev space endowed with norm $\|\nabla u\|_p = (\int_\Omega |\nabla u|^p dx)^{\frac{1}{p}}$. We will write $\|.\|_p$ for the L^p-norm. We will also denote for $p > 1$, $p' = \frac{p}{p-1}$ the Hölder conjugate exponent of p and p^* the critical exponent, that is $p^* = \infty$ if $p \geq N$ and $p^* = \frac{Np}{N-p}$ if $1 < p < N$.

Here we use the following norm in product space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$

$$\|(u,v)\| = \|\nabla u\|_p + \|\nabla v\|_q.$$

and $\|.\|_*$ is the dual norm.

2.1. **Assumptions.** We assume that $p, q > 1, \quad \alpha > 0, \ \beta > 0$ and

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1 \tag{2.1}$$

(f) $\quad f : \Omega \times I\!R \times I\!R \times I\!R \longrightarrow I\!R$ satisfies Carathéodory's conditions in the first three variables and

$$\lim_{|(r,s)| \to +\infty|} \frac{f(x,r,s,\lambda)}{|r|^{\gamma-1} r |s|^{\delta+1}} = 0; \tag{2.2}$$

$$f(x,r,s,\lambda) = o(|r|^{p-1}) \ \text{ as } \ |(r,s)| \to 0 \tag{2.3}$$

uniformly a.e. with respect to x and uniformly with respect to λ in bounded intervals of $I\!R$.

(g) $\quad g : \Omega \times I\!R \times I\!R \times I\!R \longrightarrow I\!R$ satisfies Carathéodory's conditions in the first three variables and

$$\lim_{|(r,s)| \to +\infty|} \frac{g(x,r,s,\lambda)}{|r|^{\gamma+1}|s|^{\delta-1}s} = 0; \tag{2.4}$$

$$g(x,r,s,\lambda) = o(|s|^{q-1}) \text{ as } |(r,s)| \to 0 \tag{2.5}$$

uniformly a.e. with respect to x and uniformly with respect to λ in bounded intervals of $I\!R$; with

$$\gamma + 1 > p \text{ or } \delta + 1 > q$$

and

$$\delta + \gamma + 2 < min(p^*, q^*),$$

for some nonnegative reals γ and δ.

Remark 2.1. *The homogeneity assumption* (2.1) *justifies the notion of eigenvalues.*

2.2. **Definitions.** 1.We say that $(\lambda, (u,v))$ in $I\!R \times W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is a solution of (BS) if for any $(\phi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$,

$$\begin{cases} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi \, dx &= \int_\Omega [\lambda |u|^\alpha |v|^\beta v + \quad f(x,u,v,\lambda)] \phi \, dx \\ \int_\Omega |\nabla v|^{q-2} \nabla v \nabla \psi \, dx &= \int_\Omega [\lambda |u|^\alpha |v|^\beta u + \quad g(x,u,v,\lambda)] \psi \, dx, \end{cases} \tag{2.6}$$

for all $(\phi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. We note that the pair $(\lambda, (0,0))$ is a solution of (BS) for every $\lambda \in I\!R$. The pairs of this form will be called the trivial solutions of (BS). We say that $P = (\bar{\lambda}, (0,0))$ is a bifurcation point of (BS) if in any neighbourhood of P in $I\!R \times W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ there exists a nontrivial solution of (BS).

2. Let X be a real reflexive Banach space and let X^* stand for its dual with respect to the pairing $\langle .,. \rangle$. We shall deal with mapping T acting from X into X^*. T is demicontinuous at u in X, if $u_n \to u$ strongly in X, implies that $Tu_n \rightharpoonup Tu$ weakly in X^*. T is said to belong to the class (S_+), if for any sequence u_n weakly convergent to u in X and $\limsup_{n \to +\infty} \langle Tu_n, u_n - u \rangle \leq 0$, it follows that $u_n \to u$ strongly in$X$.

2.3. **Degree theory.** If $T \in (S_+)$ and T is demicontinuous, then it is possible to define the degree

$$Deg[T; D; 0]$$

where $D \subset X$ is a bounded open set such that $Tu \neq 0$ for any $u \in \partial D$. Its properties are analogous to the ones of the Leray-Schauder degree (cf.[3]).

A point $u_0 \in X$ will be called a critical point of T if $Tu_0 = 0$. We say that u_0 is an isolated critical point of T if there exists $\epsilon > 0$ such that for any $u \in B_\epsilon(u_0)$ (open ball in X centered to u_0 and the radius ϵ) , $Tu \neq 0$ if $u \neq u_0$. Then the limit

$$Ind(T, u_0) = \lim_{\epsilon \to 0^+} Deg[T; B_\epsilon(u_0); 0]$$

exists and is called the index of the isolated critical point u_0.

Assume, furthermore, that T is a potential operator, i.e., for some continuously differentiable functional $\Phi : X \to I\!R$, $\Phi'(u) = Tu$, $u \in X$. Then we have the following two lemmas which we can find in [9] or [10]. These lemmas are crucial results in our bifurcation argument.

Lemma 2.1. *Let u_0 be a local minimum of Φ and an isolated critical point of T. Then*

$$Ind(T, u_0) = 1.$$

Lemma 2.2. *Assume that $\langle Tu, u\rangle > 0$ for all $u \in X$, $\|u\|_X = \rho$. Then*

$$Deg[T; B_\rho(0); 0] = 1.$$

2.4. **Preliminaries.** This subsection establishes an abstract framework used to prove our main results.
Let define, for $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and $(\phi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, the operators

$$\langle A_u(u, v), (\phi, \psi)\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi \, dx,$$

$$\langle A_v(u, v), (\phi, \psi)\rangle = \int_\Omega |\nabla v|^{q-2}\nabla v \nabla\psi \, dx,$$

$$\langle B_u(u, v), (\phi, \psi)\rangle = \int_\Omega |u|^\alpha |v|^\beta v\phi \, dx,$$

$$\langle B_v(u, v), (\phi, \psi)\rangle = \int_\Omega |u|^\alpha |v|^\beta u\psi \, dx,$$

with

$$A(u,v)=\frac{\alpha+1}{p}\int_\Omega |\nabla u|^p\,dx+\frac{\beta+1}{q}\int_\Omega |\nabla v|^q\,dx \text{ and } B(u,v)=\int_\Omega |u|^\alpha|v|^\beta uv\,dx$$

are functionals of class C^1 and

$$A_u=\frac{1}{\alpha+1}\frac{\partial A}{\partial u},\;\; A_v=\frac{1}{\beta+1}\frac{\partial A}{\partial v},\;\; B_u=\frac{1}{\alpha+1}\frac{\partial B}{\partial u},\;\; B_v=\frac{1}{\beta+1}\frac{\partial B}{\partial v}.$$

Define also

$$\langle F^\lambda(u,v),(\phi,\psi)\rangle=\int_\Omega f(x,u,v,\lambda)\phi\,dx,$$
$$\langle G^\lambda(u,v),(\phi,\psi)\rangle=\int_\Omega g(x,u,v,\lambda)\psi\,dx.$$

Remarks 2.1. *(i) Due to* (2.6) *a pair* (u,v) *is a weak solution of* (BS) *if and only if*

$$A_u(u,v)-\lambda B_u(u,v)=F^\lambda(u,v) \text{ in } (W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega))'$$

and

$$A_v(u,v)-\lambda B_v(u,v)=G^\lambda(u,v) \text{ in } (W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega))'$$

(ii) The operators A_u *and* A_v *are odd and satisfy* (S_+).

Lemma 2.3. B_u *and* B_v *are well defined, completely continuous and odd functionals.*

Proof.

Step 1. Definition of B_u and B_v.
By Hölder's inequality, we have

$$|\langle B_u(u,v),(\phi,\psi)\rangle|=\left|\int_\Omega |u|^\alpha|v|^\beta v\phi\,dx\right|\leq \|u\|_p^\alpha\|v\|_q^{\beta+1}\|\phi\|_p<+\infty$$

for any $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ and $(\phi,\psi)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$. Thus B_u is well defined. Similarly, we prove that B_v is also well defined.

Step 2. Compactness of B_u and B_v.
Let $\{(u_n,v_n)\}_{n\geq 0}\subset W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ be a sequence such that $(u_n,v_n)\rightharpoonup(u,v)$ weakly in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$. We claim that $B_u(u_n,v_n)\to B_u(u,v)$ strongly in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$, i.e. for all $(\phi,\psi)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$,

$$\sup_{\|\nabla\phi\|_p+\|\nabla\psi\|_q\leq 1}|\langle B_u(u_n,v_n)-B_u(u,v),(\phi,\psi)\langle|=o(1) \text{ as } n\to+\infty.$$

Indeed, Hölder's inequality implies

$$\begin{aligned}
|\langle B_u(u_n,v_n)-B_u(u,v),(\phi,\psi)\rangle| &= \left|\int_\Omega (|u_n|^\alpha |v_n|^\beta v_n - |u|^\alpha |v|^\beta v)\phi\,dx\right| \\
&\le \int_\Omega |v_n|^{\beta+1}\,|(|u_n|^\alpha - |u|^\alpha)|\,|\phi|\,dx \\
&+ \int_\Omega |u|^\alpha \left||v_n|^\beta v_n - |v|^\beta v\right|\,|\phi|\,dx \\
&\le \||u_n|^\alpha - |u|^\alpha\|_{\frac{p}{\alpha}}\,\|\phi\|_p \|v_n\|_q^{\beta+1} \\
&\quad + \|u\|_p^\alpha \|\phi\|_p \left\||v_n|^\beta v_n - |v|^\beta v\right\|_{\frac{q}{\beta+1}}.
\end{aligned}$$

Due to the continuity of Nemytskii operator $u \longrightarrow |u|^\alpha$ (resp. $v \longrightarrow |v|^\beta v$)
from $L^p(\Omega)$ into $L^{\frac{p}{\alpha}}(\Omega)$ (resp. from$L^q(\Omega)$ into $L^{\frac{q}{\beta+1}}(\Omega)$) and Rellich's Theorem, there exists $n_0 \ge 0$ such that for all $n \ge n_0$ we have

$$\||u_n|^\alpha - |u|^\alpha\|_{\frac{p}{\alpha}} = o(1), \tag{2.7}$$

$$\left\||v_n|^\beta v_n - |v|^\beta v\right\|_{\frac{q}{\beta+1}} = o(1). \tag{2.8}$$

Thus (v_n) is bounded in $L^p(\Omega)$. Finally from (2.7) and (2.8) we have the claim.
The compactness of B_v can be proved by the same argument.
The oddness of B_u and B_v is obvious. The proof is completed. ■

Lemma 2.4. *$F^\lambda(.,.)$ and $G^\lambda(.,.)$ are well defined, completely continuous and $F^\lambda(0,0) = G^\lambda(0,0) = 0$ and we have*

$$\lim_{\substack{\|\nabla u\|_p \to 0 \\ \|\nabla v\|_q \to 0}} \frac{F^\lambda(u,v)}{\|\nabla u\|_p^{p-1} + \|\nabla v\|_q^{q-1}} = 0 \;\; in \;\; (W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega))', \tag{2.9}$$

and

$$\lim_{\substack{\|\nabla u\|_p \to 0 \\ \|\nabla v\|_q \to 0}} \frac{G^\lambda(u,v)}{\|\nabla u\|p^{p-1} + \|\nabla v\|_q^{q-1}} = 0 \;\; in \;\; (W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega))', \tag{2.10}$$

uniformly with respect to λ in bounded subsets of $I\!R$.

Proof.
(2.2) and (2.3) imply that for any $\epsilon > 0$ there are two real $\xi = \xi(\epsilon)$ and $M = M(\xi) > 0$ such that for a.e, $x \in \Omega$

$$|f(x, u, v, \lambda)| \leq \epsilon |u|^{p-1} + M|u|^{\gamma}|v|^{\delta+1}.$$

• Compactness:
Here, we show the compactness of F^{λ}.
If $\delta + 1 > p$, let $\theta \in I\!R^{+}$ such that:

$$\begin{cases} \delta + \gamma + 2 < \theta < min(p^*, q^*) & \text{if} \quad \gamma + 1 > p \\ \\ \max(\delta + \gamma + 2, p) < \theta < min(p^*, q^*) & \text{if} \quad \gamma + 1 \leq p. \end{cases}$$

By using Hölder's and Young's inequalities we have

$$\begin{aligned} \int_{\Omega} |f(x, \lambda, u, v)|^{\theta'} dx &\leq \epsilon \int_{|u(x)|+|v(x)| \leq \xi} |u|^{\theta'(p-1)} dx + M \int_{|u(x)|+|v(x)| \geq \xi} |u|^{\theta' \delta} |v|^{\theta'(\gamma+1)} dx \\ &\leq \epsilon \int_{\Omega} |u|^{\theta'(p-1)} dx + M \int_{\Omega} |u|^{\theta' \gamma} |v|^{\theta'(\delta+1)} dx \\ &\leq M_1 \int_{\Omega} |u|^{\theta} dx + M_2 \left(\int_{\Omega} |u|^{\theta} dx \right)^{\frac{\gamma}{\theta-1}} \left(\int_{\Omega} |v|^{\theta} dx \right)^{\frac{\delta+1}{\theta-1}} \end{aligned}$$

Thus F^{λ} is the Nemytskii continuous operator from $L^{\theta}(\Omega) \times L^{\theta}(\Omega)$ into $L^{\theta'}(\Omega)$.
On the other hand, let $\{(u_n, v_n)\}_n \subset W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that $(u_n, v_n) \rightharpoonup (u, v)$ in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and strongly in $L^{\theta}(\Omega) \times L^{\theta}(\Omega)$.
So,

$$\begin{aligned} \|F^{\lambda}(u_n, v_n) - F^{\lambda}(u, v)\|_* &= \sup_{|(\phi,\psi)\| \leq 1} \left| \int_{\Omega} (f(x, \lambda, u_n, v_n) - f(x, \lambda, u, v)) \phi \, dx \right| \\ &\leq \|f(x, \lambda, u_n, v_n) - f(x, \lambda, u, v\|_{\theta'} \to 0, \text{ as } n \to \infty. \end{aligned}$$

Hence F^{λ} is compact.
By a similar argument, we can show that G^{λ} is compact.

• For the limit (2.9) and (2.10), we argue as follows:

Hölder's inequality yields for all $(\phi,\psi)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$,

$$\begin{aligned}
\|F^\lambda(u,v)\|_* &\le \sup_{\|(\phi,\psi)\|\le 1}\left|(\epsilon\int_\Omega |u|^{p-1}\phi\,dx + M\int_\Omega |u|^\gamma |v|^{\delta+1}\phi\,dx)\right| \\
&\le \sup_{\|(\phi,\psi)\|\le 1}(\epsilon\|u\|_p^{p-1}\|\phi\|_p + M|\Omega|^{\omega_0}\|u\|_{p*}^{\gamma}\|v\|_{q*}^{\delta+1}\|\phi\|_{p*}) \\
&\le c\sup_{\|(\phi,\psi)\|\le 1}(\epsilon\|\nabla u\|_p^{p-1} + M\|\nabla u\|_p^{\gamma}\|\nabla v\|_q^{\delta+1})\|\nabla\phi\|_p
\end{aligned}$$

where $\omega_0=\frac{p^*q^*}{p^*q^*-(\gamma+1)p^*-(\delta+1)q^*}$ and $c\in I\!R^+$ is a constant.

So, we distinguish two cases:
If $\gamma+1>p$ then,

$$\frac{\|F^\lambda(u,v)\|_*}{\|\nabla u\|_p^{p-1}+\|\nabla v\|_q^{q-1}}\le c(\epsilon+M\|\nabla u\|_p^{\gamma-(p-1)}\|\nabla v\|_q^{\delta+1}). \tag{2.11}$$

If $\delta+1>q$ then,

$$\frac{\|F^\lambda(u,v)\|_*}{\|\nabla u\|_p^{p-1}+\|\nabla v\|_q^{q-1}}\le c\epsilon+cM\|\nabla u\|_p^{\gamma}\|\nabla v\|_q^{(\delta+1)-(q-1)}. \tag{2.12}$$

Therefore, by passing to the limit in (2.11) or (2.12), (2.9) follows.
There is the same proof of (2.10). ■

Remark 2.2. *Note that every continuous map $T:X\longrightarrow X^*$ is also demicontinuous. Note also, that if $T\in(S_+)$ then $(T+K)\in(S_+)$ for any compact operator $K:X\longrightarrow X^*$.*

Remark 2.3. *λ is an eigenvalue of (ES) if and only if the system*

$$\begin{cases} A_u(u,v)-\lambda B_u(u,v) &= 0\\ A_v(u,v)-\lambda B_v(u,v) &= 0\end{cases}$$

has a nontrivial solution $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

Proposition 2.1. *Under the assumption (2.1), (ES) has a principal eigenvalue λ_1 characterized variationally as follows*

$$\lambda_1=\inf\left\{A(u,v);\ (u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega), B(u,v)=1\right\}.$$

The proof of this proposition is more or less the same as F. De Thélin [5] for the system case (ES) modulo a suitable modification.

Now, let

$$\begin{aligned}\Gamma_p(u,\phi) &= \int_\Omega |\nabla u|^p\,dx + (p-1)\int_\Omega |\nabla\phi|^p\left(\frac{|u|}{\phi}\right)^p dx \\ &- p\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\nabla u\left(\frac{|u|^{p-2}u}{\phi^{p-1}}\right)dx \\ &= \int_\Omega |\nabla u|^p\,dx + \int_\Omega \frac{\Delta_p\phi}{\phi^{p-1}}|u|^p\,dx\end{aligned}$$

for all $(u,\phi)\in(W_0^{1,p}(\Omega)\cap C^1(\Omega))^2$ with $\phi>0$ in Ω.
The following lemma is the heart on the proof of the simplicity.

Lemma 2.5. *For all $(u,\phi)\in(W_0^{1,p}(\Omega)\cap C^{1,\nu}(\Omega))^2$ with $\phi>0$ in Ω and $\nu\in(0,\ 1)$, we have $\Gamma_p(u,\phi)\geq 0$ i.e*

$$\int_\Omega |\nabla u|^p\,dx \geq \int_\Omega \frac{-\Delta_p\phi}{\phi^{p-1}}|u|^p\,dx, \tag{2.13}$$

and if $\Gamma_p(u,\phi)=0$ there is $c\in I\!R$ such that $u=c\phi$.

Proof.
By Young's inequality we have for $\epsilon>0$,

$$\begin{aligned}\int_\Omega \nabla u|\nabla\phi|^{p-2}\nabla\phi\frac{u|u|^{p-2}}{\phi^{p-1}}\,dx &\leq \int_\Omega |\nabla u||\nabla\phi|^{p-1}(\frac{|u|}{\phi})^{p-1}\,dx \\ &\leq \int_\Omega(\frac{\epsilon^p}{p}|\nabla u|^p + \frac{p-1}{p\epsilon^p}|\frac{u}{\phi}|^p|\nabla\phi|^p)\,dx.\end{aligned} \tag{2.14}$$

For $\epsilon=1$ we have by integration over Ω,

$$p\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\nabla u\left(\frac{|u|^{p-2}u}{\phi^{p-1}}\right)dx \leq \int_\Omega |\nabla u|^p\,dx + (p-1)\int_\Omega |\frac{u}{\phi}|^p|\nabla\phi|^p\,dx.$$

Thus

$$\Gamma_p(u,\phi)\geq 0.$$

On the other hand, if $\Gamma_p(u,\phi)=0$ by (2.14), we obtain for $\epsilon=1$,

$$p\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\nabla u\left(\frac{|u|^{p-2}u}{\phi^{p-1}}\right)dx - \int_\Omega |\nabla u|^p\,dx - (p-1)\int_\Omega |\frac{u}{\phi}|^p|\nabla\phi|^p\,dx = 0 \tag{2.15}$$

and

$$\int_\Omega\left\{\nabla u\nabla\phi|\nabla\phi|^{p-2}\frac{u|u|^{p-2}}{\phi^{p-1}} - |\nabla u||\nabla\phi|^{p-1}\left(\frac{|u|}{\phi}\right)^{p-1}\right\}dx = 0. \tag{2.16}$$

Thanks to (2.15) we find $|\nabla u| = |\frac{u}{\phi}\nabla\phi|$. Thus from (2.16), it follows that $\nabla u = \epsilon\frac{u}{\phi}\nabla\phi$, where $|\epsilon| = 1$. Consequently, since $\Gamma_p(u,\phi) = 0$, we obtain $\epsilon = 1$ and $\nabla(\frac{u}{\phi}) = 0$. Therefore, there is $c \in I\!R$ such that $u = c\phi$. ∎

Remarks 2.2. *(i) By adapting the argument of* [5] *we can show that the eigenvectors associated to λ_1 without any additional smoothness assumption on $\partial\Omega$, are in $L^\infty \times L^\infty$.*

(ii) Thanks to an advanced result in regularity theory of [4]*, we deduce with (i) that the positive solution of (ES) associated to λ_1 is in $C^1_{loc}(\Omega) \times C^1_{loc}(\Omega)$.*

(iii) According to the Maximum principle of [17] *applied to each equation we conclude that (ES) has a priori a positive eigenvector associated to λ_1.*

3. Main Results

3.1. Simplicity and isolation results.

Theorem 3.1. *(i) λ_1 is simple.*

(ii) λ_1 is the unique eigenvalue of (ES) having an eigenvector (u,v) not changing its sign, i.e., $uv > 0$ in Ω.

(iii) There is $c > 0$ such that

$$|\Omega^-| \geq (|\lambda|c)^\omega ,$$

where $(\lambda,(u,v))$ is an eigenpair of (ES), $\Omega^- = \{x \in \Omega,\ u(x) < 0 \text{ and } v(x) < 0\}$ and $\omega = \frac{p^*q^*}{p^*q^* - (\alpha+1)q^* - (\beta+1)p^*}$.

(iv) λ_1 is isolated.

Proof.

(i) Let (u,v) and (ϕ,ψ) be two eigenvectors of (ES) associated to λ_1 with (u,v) is positive ($u \geq 0,\ \ v \geq 0$). Thanks to definition of λ_1 and

Hölder's inequality we have in view of remark 2.4 that

$$\begin{aligned}
A(\phi,\psi) &= \lambda_1 B(\phi,\psi)\\
&\le \lambda_1 \int_\Omega u^{\alpha+1}v^{\beta+1}\frac{|\phi|^{\alpha+1}|\psi|^{\beta+1}}{u^{\alpha+1}v^{\beta+1}}\,dx\\
&\le \lambda_1 \int_\Omega u^{\alpha+1}v^{\beta+1}\left[\frac{\alpha+1}{p}\frac{|\phi|^p}{u^p}+\frac{\beta+1}{q}\frac{|\psi|^q}{v^q}\right]dx\\
&\le \lambda_1 \int_\Omega \left[\frac{\alpha+1}{p}\frac{u^\alpha v^{\beta+1}}{u^{p-1}}|\phi|^p+\frac{\beta+1}{q}\frac{u^{\alpha+1}v^\beta}{v^{q-1}}|\psi|^q\right]dx\\
&\le \frac{\alpha+1}{p}\int_\Omega \frac{-\Delta_p u}{u^{p-1}}|\phi|^p\,dx+\frac{\beta+1}{q}\int_\Omega \frac{-\Delta_q v}{v^{q-1}}|\psi|^q\,dx.
\end{aligned}$$

Thanks to lemma 2.5 we have

$$A(\phi,\psi)=\frac{\alpha+1}{p}\int_\Omega \frac{-\Delta_p u}{u^{p-1}}|\phi|^p\,dx+\frac{\beta+1}{q}\int_\Omega \frac{-\Delta_q v}{v^{q-1}}|\psi|^q\,dx.$$

Thus

$$\int_\Omega |\nabla\phi|^p\,dx=\int_\Omega \frac{-\Delta_p u}{u^{p-1}}|\phi|^p\,dx \text{ and } \int_\Omega |\nabla\phi|^p\,dx=\int_\Omega \frac{-\Delta_q v}{v^{q-1}}|\phi|^q\,dx.$$

Hence by Lemma 3.2 again, there exist k_1 and k_2 in $I\!R$ such that $u=k_1\phi$ and $v=k_2\psi$.

(ii) Let (u,v) be a positive eigenvector of (ES) associated to λ and (ϕ,ψ) a positive solution of (ES) associated to λ_1. It is clear that $\lambda_1\le\lambda$ and by Hölder's inequality we have

$$\begin{aligned}
A(\phi,\psi) &= \lambda_1 B(\phi,\psi)\\
&\le \lambda \int_\Omega u^{\alpha+1}v^{\beta+1}\frac{\phi^{\alpha+1}\psi^{\beta+1}}{u^{\alpha+1}v^{\beta+1}}\,dx\\
&\le \lambda \int_\Omega u^{\alpha+1}v^{\beta+1}\left[\frac{\alpha+1}{p}\frac{\phi^p}{u^p}+\frac{\beta+1}{q}\frac{\psi^q}{v^q}\right]dx\\
&\le \lambda \int_\Omega \left[\frac{\alpha+1}{p}\frac{u^\alpha v^{\beta+1}}{u^{p-1}}\phi^p+\frac{\beta+1}{q}\frac{u^{\alpha+1}v^\beta}{v^{q-1}}\psi^q\right]dx\\
&\le \frac{\alpha+1}{p}\int_\Omega \frac{-\Delta_p u}{u^{p-1}}\phi^p\,dx+\frac{\beta+1}{q}\int_\Omega \frac{-\Delta_q v}{v^{q-1}}\psi^q\,dx.
\end{aligned}$$

Therefore, we deduce from Lemma 2.5 that $A(\phi,\psi)=A(u,v)$ i.e

$$\lambda_1\int_\Omega |\phi|^{\alpha+1}|\psi|^{\beta+1}\,dx=\lambda\int_\Omega |u|^{\alpha+1}|v|^{\beta+1}\,dx.$$

So, by normalization, we conclude that $\lambda_1 = \lambda$.
(iii) Let $(\lambda, (u, v))$ be an eigenpair of (ES), $u^-(x) = \min(u(x), 0)$ and $v^-(x) = min(v(x), 0)$.
Thus, by multiplying the first equation of (ES) by u^- we have

$$\begin{aligned}
\int_\Omega |\nabla u^-|^p \, dx &= \lambda \int_\Omega |u|^\alpha |v|^\beta v u^- \, dx \\
&= \lambda \left[\int_{\Omega^-} |u^-|^{\alpha+1} |v^-|^{\beta+1} + \int_\Omega |u|^\alpha |v|^\beta u^- v^+ \right] dx \\
&\leq |\lambda| \int_{\Omega^-} |u^-|^{\alpha+1} |v^-|^{\beta+1} \, dx \\
&\leq |\lambda| |\Omega^-|^\omega \|u^-\|_{p^*}^{\alpha+1} \|v^-\|_{q^*}^{\beta+1} \\
&\leq c|\lambda| |\Omega^-|^\omega \|u^-\|_{1,p}^{\alpha+1} \|v^-\|_{1,q}^{\beta+1} \\
&\leq c|\lambda| |\Omega^-|^\omega \left(\frac{\alpha+1}{p} \int_\Omega |\nabla u^-|^p \, dx + \frac{\beta+1}{q} \int_\Omega |\nabla v^-|^q \, dx \right). \qquad (3.1)
\end{aligned}$$

Similarly, multiplying the second equation of (ES) by v^-, we obtain

$$\int_\Omega |\nabla v^-|^q \, dx \leq c|\lambda| |\Omega^-|^\omega \left(\frac{\alpha+1}{p} \int_\Omega |\nabla u^-|^p \, dx + \frac{\beta+1}{q} \int_\Omega |\nabla v^-|^q \, dx \right). \qquad (3.2)$$

Hence by (3.1) and (3.2) we complete the proof of (iii).
(iv) The proof is a rather a simple adaptation of argument of [1] modulo suitable modification. ∎

3.2. **Bifurcation Result.** Once we have proved in the previous subsection that λ_1 is simple and isolated, we can study the bifurcation when λ is near λ_1.

Proposition 3.1. *If* $(\bar{\lambda}; (0, 0))$ *is a bifurcation point of* (BS)*, then* $\bar{\lambda}$ *is an eigenvalue of* (ES).

Proof.
$(\bar{\lambda}, (0, 0))$ is a bifurcation point of (BS) then there is a sequence $\{\lambda_n, (u_n, v_n)\}_n \subset I\!R \times W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ of nontrivial solutions of (ES) such that

$$\begin{aligned}
\lambda_n &\longrightarrow \bar{\lambda} \text{ in } I\!R \\
(u_n, v_n) &\longrightarrow (0, 0) \text{ in } W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega).
\end{aligned}$$

Thus

$$\begin{aligned} A_u(u_j,v_j)-\lambda_j B_u(u_j,v_j) &= F^{\lambda_j}(u_j,v_j) \\ A_v(u_j,v_j)-\lambda_j B_v(u_j,v_j) &= G^{\lambda_j}(u_j,v_j). \end{aligned} \tag{3.3}$$

Set $\bar{u}_j = \frac{u_j}{\|\nabla u_j\|_p^{p-1}+\|\nabla v_j\|_q^{q-1}}$ and $\bar{v}_j = \frac{v_j}{\|\nabla u_j\|_p^{p-1}+\|\nabla v_j\|_q^{q-1}}$. Then $\bar{u}_j$ and $\bar{v}_j$ are bounded, it follows that there exist a pair $(\bar{u},\bar{v}) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that $(\bar{u}_j,\bar{v}_j) \rightharpoonup (\bar{u},\bar{v})$ weakly in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and a.e in Ω. Thanks to Lemma 2.5 and Lemma 2.4 we deduce the following convergence

$$\lambda_j B_u(\bar{u}_j,\bar{v}_j) + \frac{F^{\lambda_j}(u_j,v_j)}{\|\nabla u_j\|_p^{p-1}+\|\nabla v_j\|_q^{q-1}} \longrightarrow \bar{\lambda} B_u(\bar{u},\bar{v})$$

and

$$\lambda_j B_v(u_j,v_j) + \frac{G^{\lambda_j}(u_j,v_j)}{\|\nabla u_j\|_p^{p-1}+\|\nabla v_j\|_q^{q-1}} \longrightarrow \bar{\lambda} B_v(\bar{u},\bar{v}).$$

According to (3.3), we have

$$A_u(\bar{u}_j,\bar{v}_j) \longrightarrow \bar{\lambda} B_u(\bar{u},\bar{v})$$

and

$$A_v(\bar{u}_j,\bar{v}_j) \longrightarrow \bar{\lambda} B_v(\bar{u},\bar{v}),$$

Since $-\Delta_p$ and $-\Delta_q$ are homeomorphisms, it is clear that

$$(\bar{u}_j,\bar{v}_j) \longrightarrow (A_u)^{-1}(\bar{\lambda} B_u(\bar{u},\bar{v}))$$

and

$$(\bar{u}_j,\bar{v}_j) \longrightarrow (A_v)^{-1}(\bar{\lambda} B_v(\bar{u},\bar{v})).$$

Consequently by the convergence a.e in Ω we conclude that

$$(\bar{u},\bar{v}) = (A_u)^{-1}(\bar{\lambda} B_u(\bar{u},\bar{v}))$$

and

$$(\bar{u},\bar{v}) = (A_v)^{-1}(\bar{\lambda} B_v(\bar{u},\bar{v})).$$

Finally, since $\|\nabla \bar{u}_j\|_p^{p-1} + \|\nabla \bar{v}_j\|_q^{q-1} = 1$ we must have $\bar{u} \not\equiv 0$ and $\bar{v} \not\equiv 0$. It follows that $(\bar{u},\bar{v})$ solves (ES). This complete the proof in view of Remark 2.4.

∎

Let $X = I\!R \times W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ be equipped with the norm

$$\|(\lambda,u,v)\| = (|\lambda|^2 + \|\nabla u\|_p^2 + \|\nabla v\|_q^2)^{\frac{1}{2}}.$$

Definition 3.1. *We say that*

$$C = \{(\lambda, (u, v)) : \ (\lambda, (u, v))\ solves (BS), u \neq 0, v \neq 0\}$$

is a continuum of nontrivial solutions of (BS), if it is a connected subset in E.

Now let, $L = \frac{\alpha+1}{p} A_u + \frac{\beta+1}{q} A_v, \quad G = \frac{1}{2}(B_u + B_v)$ and $F = F^\lambda + G^\lambda$. In virtue of the preceding results $Deg[L - \lambda G - F, B_\epsilon(0, 0), (0, 0)]$ is well defined. By homotopic we have

$$Deg[L - \lambda G - F, B_\epsilon(0, 0), (0, 0)] = Deg[L - \lambda G, B_\epsilon(0, 0), (0, 0)].$$

The main bifurcation result in this paper is the following

Theorem 3.2. *Under the assumptions (f) and (g), the pair $(\lambda_1, (0, 0))$ is a bifurcation point of (BS). Moreover, there is a continuum of nontrivial solutions C of (BS) such that $(\lambda_1, (0, 0)) \in \bar{C}$ and C is either unbounded in E or there is $\bar{\lambda} \neq \lambda_1$, an eigenvalue of (ES), with $(\bar{\lambda}, (0, 0)) \in \bar{C}$.*

Proof.
We will give only a sketch of the proof since it follows the lines of the proof of Theorem 14.18 in [9]. The key point in the proof is the fact that the value of

$$Deg[L - \lambda G; B_\epsilon(0), 0] \tag{3.4}$$

changes when λ crosses λ_1. If this fact is proved then the result follows exactly as in the classical bifurcation result of Rabinowitz [15]. Choose $a > 0$ such that $(\lambda_1,\ \lambda_1 + a)$ does not contain any eigenvalue of (ES). Then the variational characterization of λ_1 and lemma 2.2 yield

$$Deg[L - \lambda G; B_\epsilon(0), 0] = 1, \tag{3.5}$$

when $\lambda \in (\lambda_1 - a, \lambda_1)$. To evaluate (3.4) for $\lambda \in (\lambda_1,\ \lambda_1 + a)$ we use the following trick. Fix a number $K > 0$ and define a function $\psi : I\!R \longrightarrow I\!R$ by

$$\psi(s) = \begin{cases} 0 & \text{for} \quad s \leq K \\ \frac{2a}{\lambda_1}(s - 2K) & \text{for} \quad s \geq 3K \end{cases}$$

and ψ is positive and strictly convex in (K, 3K). Define a functional

$$\bar{\psi}_\lambda(u, v) = \langle L(u, v), (u, v)\rangle - \lambda\langle G(u, v), (u, v)\rangle + \psi(\langle L(u, v), (u, v)\rangle).$$

Then $\bar{\psi}_\lambda$ is continuously Frèchet differentiable and its critical point $(u_0, v_0) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ corresponds to a solution of the following system

$$\frac{\partial\overline{\psi}_\lambda}{\partial u} = \frac{\partial\overline{\psi}_\lambda}{\partial v} = 0,$$

where

$$\langle\frac{\partial\overline{\psi}_\lambda}{\partial u}(u,v),(u,v)\rangle = (\alpha+1)\,[\,\langle A_u(u,v),(u,v)\rangle(1+\psi'(\langle L(u,v),(u,v)\rangle)-\lambda\langle B_u(u,v),(u,v)\rangle\,]\,,$$

and

$$\langle\frac{\partial\overline{\psi}_\lambda}{\partial v}(u,v),(u,v)\rangle = (\beta+1)\,[\,\langle A_v(u,v),(u,v)\rangle(1+\psi'(\langle L(u,v),(u,v)\rangle)-\lambda\langle B_v(u,v),(u,v)\rangle\,]\,,$$

However, since $\lambda \in (\lambda_1 - a, \lambda_1)$, only nontrivial critical points of the derivative of $\bar{\psi}$ noted $D\bar{\psi}_\lambda$ occur if

$$\psi'(\langle L(u_0,v_0),(u_0,v_0)\rangle) = \frac{\lambda}{\lambda_1} - 1. \tag{3.6}$$

Due to the definition of ψ we have $\langle L(u_0,v_0),(u_0,v_0)\rangle \in (K, 3K)$; by (3.6) and the simplicity of λ_1, we conclude that $(u_0,v_0) \in \{(u_1,v_1),(-u_1,-v_1)\}$, where (u_1,v_1) is an eigenvector associated to λ_1. Therefore, for $\lambda \in (\lambda_1, \lambda_1 + a)$, the derivative $D\bar{\psi}_\lambda$ has precisely three isolated critical point .

The existence of such isolated critical points is ensured by the fact $\bar{\psi}_\lambda$ is weakly lower semicontinuous and

$$\lim_{\|(u,v)\|\to+\infty} \|\bar{\psi}_\lambda\|_{1,p} = \infty,$$

due to the definition of ψ. So, $\bar{\psi}_\lambda$ attains local minima at (u_1,v_1) and $(-u_1,-v_1)$. Note that $(0,0)$ is an obvious isolated critical point. It follows from Lemma 2.1 that

$$ind(D\bar{\psi}_\lambda,\ (u_1,v_1)) = ind(D\bar{\psi}_\lambda,\ (-u_1,-v_1)) = 1. \tag{3.7}$$

Since also

$$\langle D\bar{\psi}_\lambda((u,v)),(u,v)\rangle\ > 0$$

for $\|u\|_p + \|v\|_q = R$, with $R > 0$, sufficiently large, we have according to Lemma 2.2 that

$$Deg[D\bar{\psi}_\lambda; B_R(0), 0] = 1. \tag{3.8}$$

Additivity property of the degree, (3.7) and (3.8) yield

$$Deg[L - \lambda G; B_\epsilon(0), 0] = -1. \tag{3.9}$$

for any $\lambda \in (\lambda_1,\, \lambda_1 + a)$ and $\epsilon > 0$ sufficiently small. Since (3.5) and (3.9) establish the "jump" of the degree the proof is completed. ■

Remarks 3.1. (i) *For the case of one equation and when $\partial\Omega$ of class $C^{2,\nu}$, a similar bifurcation theorem has proved by Del Pino and Manasevich* [11] *and Binding and Huang* [2]. *The first authors used the continuity of $\lambda_1(p)$ with respect to p; on the other hand, the second authors considered the following particular case $f \equiv f(x,s)$ which satisfies $f(x,s) \in C(\bar{\Omega} \times I\!R)$ an odd function in s and $|f(x,s)| \leq c|s|^{q-1}$ uniformly in Ω; where $p < q < \bar{q} = p + \frac{p^2}{N}$. They use $C^{1,\nu}(\bar{\Omega})$ regularity of solutions.*

(ii) *Fleckinger, Manásevich and De Thelin show a similar bifurcation theorem for the following system*

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + b|v|^{p-2}v + f(\lambda, |u|^{p-2}u) \text{ in } \Omega \\ -\Delta_p v = c|u|^{p-2}u + d|v|^{p-2}v \text{ in } \Omega \\ u_{|\partial\Omega} = v_{|\partial\Omega} = 0, \end{cases}$$

where Ω is sufficiently regular bounded open subset of $I\!R^N$. The function $f : I\!R \times I\!R \to I\!R$ is continuous and λ is a parameter; b,c and d are constants satisfying $bc > 0,\ d \leq 0$.

Note that the methods used in all papers are not applicable in our case (any bounded domain). So the method used to evaluate the degree is different.

References

[1] A. Anane, *Simplicité et isolation de la première valeur propre du p-Laplacien*, C.R. Acad. Sci. Paris, **305**, 725-728 (1987).

[2] P.A.Binding and Y.X.Huang, Bifurcation from eigencurves of the p-Laplacian , Differential Integral Equations, **8**, 415-428 (1995).

[3] F.E.Browder and W.F.Petryshyn, *Approximation methods and the generalized topological for nonlinear mapping in Banach spaces*, J. Funct. Anal.**3**, 217-245 (1969).

[4] E. Dibenedetto, $C^{1+\alpha}$*-local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Analysis T.M.A.**7**, 827-859 (1983).

[5] F. De Thélin, *Première valaur propre d'un système elliptique non linéaire*, C. R. Acad. Paris, t.**311**, serie I, 603-606 (1990).

[6] K. Chaïb, *Extension of Diaz-Saa's inequality in $I\!R^n$ and application to a system of p-Laplacian* (preprint).

[7] P. Drábek, A. Elkhalil and A. Touzani, *A Result on the Bifurcation from the Principal Eigenvalue of the A_p-Laplacian*, Abstract and Applied Analysis, 185-195 (1997).

[8] P. Drábek, A. Elkhalil and A. Touzani, *A bifurcation problem for the principal eigencurve of the p-Laplacian*, Applicable Analysis **72**, N.3-4, 399-410 (1999).

[9] P. Drábek, *Solvability and bifurcation of nonlinear equations*, Pitman Research Notes in Math. **264**, Longman, Hrlow (1992).
[10] P. Drábek and Y.X. Huang: *Bifurcation problems for the p-Laplacian in* $\mathbb{R}^n$, Trans. Amer. Math. Soc. **349** ,171-188 (1997).
[11] M. A. Del Pino, R.F. Manasèvich, *Global Bifurcation from the eigenvalue of the p-Laplacian*, J. Diff. Equations, **92**, 1-29 (1991).
[12] J. Fleckinger, R.F.Manasevich, N.M.Stavrakakis and F. De Thélin, *Principal eigenvalue for some quasilinear elliptic equations on* $\mathbb{R}^N$, Advances in Differential Equations, **2**, N.6, 981-1003 (1997).
[13] J. Fleckinger, R. Manasevich, and F. de Thélin: *Global Bifurcation from the First Eigenvalue for a System of p-Laplacians*, Math. Nach. **182**, 217-242 (1996).
[14] A. Elkhalil and A. Touzani:*On the first eigencurve of p-Laplacian*, Partial Differential Equations, Lecture Notes in Pure and Applied Mathematics, **222**, Marcel Dekker Inc., 195-205 (2002).
[15] P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal.**7**, 487-513 (1971).
[16] A. Szulkin, *Ljustenik-Schnirelmann theory on* C^1*-manifolds*, Ann. Inst. Henri Poincaré, Anal. No.**5**, 119-139 (1988).
[17] J.L Vazquez, *A Strong Maximum Principle for some Quasilinear Elliptic Equations*, Appl. Math. Optim., **12**, 191-202 (1984).

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