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C^n-almost periodic and almost periodic solutions for some nonlinear integral equations*

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Abstract

In this paper, we investigate the existence of C^n-almost periodic solution for a class of nonlinear Fredholm integral equation, and the existence of almost periodic solution for a class of more general nonlinear integral equation. Our existence theorems extend some earlier results. Two examples are given to illustrate our results.

Keywords: almost periodic; C^n-almost periodic; nonlinear integral equations.

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1 Introduction

In [1], the author initiated the study on C^n-almost periodic functions, which turns out to be one of the most important generalizations of the concept of almost periodic functions in the sense of Bohr. C^n-almost periodic functions are very interesting since their properties are better than almost periodic functions to some extent as well as they have wide applications in differential equations. Recently, C^n-almost periodic functions has attracted more and

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more attentions. We refer the reader to [6, 7, 11, 12, 14] and references therein for some recent development in this topic.

On the other hand, the existence of almost periodic type solutions for various kinds of integral equations has been of great interest for many authors (see, e.g., [2–5, 8, 10, 15] and references therein). Especially, in [15], the authors studied the existence of almost periodic solutions for the following Fredholm integral equation:

$$y(t) = h(t) + \int_{\mathbb{R}} k(t,s) f(s, y(s)) ds, \quad t \in \mathbb{R}. \tag{1.1}$$

Stimulated by the above works, we will make further study on these topics, i.e., we will study the existence of C^n-almost periodic solutions for Eq. (1.1), and we will also investigate the existence of almost periodic solutions for the following more general integral equation:

$$y(t) = e(t, y(\alpha(t))) + g(t, y(\beta(t))) \left[h(t) + \int_{\mathbb{R}} k(t,s) f(s, y(\gamma(s))) ds \right], \quad t \in \mathbb{R}. \tag{1.2}$$

It is easy to see that Eq. (1.1) is a special case of Eq. (1.2).

In fact, to the best of our knowledge, there is no results in the literature concerning the existence of C^n-almost periodic solutions for Eq. (1.1) and the existence of almost periodic solutions for Eq. (1.2). Therefore, in this paper, we will extend the results in [15] to the C^n-almost periodic case and to a more general integral equation, i.e., Eq. (1.2).

Throughout the rest of this paper, if there is no special statement, we denote by $\mathbb{R}$ the set of real numbers, by X a Banach space, by $C^n(\mathbb{R}, X)$ (briefly $C^n(X)$) the space of all functions $\mathbb{R} \to X$ which have a continuous $n-th$ derivative on $\mathbb{R}$, and by $C_b^n(\mathbb{R}, X)$ (briefly $C_b^n(X)$) be the subspace of $C^n(\mathbb{R}, X)$ consisting of such functions satisfying

$$\sup_{t\in\mathbb{R}} \sum_{i=0}^{n} |f^{(i)}(t)| < +\infty$$

where $f^{(i)}$ denote the $i-th$ derivative of f and $f^{(0)} := f$. Clearly $C_b^n(X)$ turns out to be a Banach space with the norm

$$\|f\|_n = \sup_{t\in\mathbb{R}} \sum_{i=0}^{n} |f^{(i)}(t)|.$$

First, let us recall some definitions and notations about almost periodicity and C^n-almost periodicity (for more details, see [6, 7, 9, 13]).

Definition 1.1. *A continuous function $f : \mathbb{R} \to X$ is called almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that*

$$\|f(t+\tau) - f(t)\| < \varepsilon \quad \text{for all} \quad t \in \mathbb{R}.$$

We denote by $AP(\mathbb{R}, X)$ (briefly $AP(X)$) the set of all such functions.

Definition 1.2. *$\mathcal{F} \subseteq AP(X)$ is said to be equi-almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that for all $f \in \mathcal{F}$ and $t \in \mathbb{R}$,*

$$\|f(t+\tau) - f(t)\| < \varepsilon.$$

Definition 1.3. *Let $\Omega \subseteq X$. A continuous function $f : \mathbb{R} \times \Omega \to X$ is called almost periodic in t uniformly for $x \in \Omega$ if for each $\varepsilon > 0$ and for each compact subset $K \subset \Omega$ there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that*

$$\|f(t+\tau, x) - f(t, x)\| < \varepsilon \quad \text{for all} \quad t \in \mathbb{R},\ x \in K.$$

We denote by $AP(\mathbb{R} \times \Omega, X)$ the set of all such functions.

Definition 1.4. *A function $f \in C^n(\mathbb{R}, X)$ is called C^n-almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that*

$$\|f(t+\tau) - f(t)\|_n < \varepsilon \quad \text{for all} \quad t \in \mathbb{R}.$$

We denote by $AP^n(\mathbb{R}, X)$ (briefly $AP^n(X)$) the set of all such functions.

Remark 1.5. By [7], we know that $AP^n(X)$ turns out to be a Banach space equipped with the $\|\cdot\|_n$ norm. In addition, we usually denote $AP^0(X)$ by $AP(X)$, which is the classical Banach space of all X-valued almost periodic functions in Bohr's sense.

2 C^n-almost periodic solution for nonlinear Fredholm integral equation

Lemma 2.1. *The following two statements are equivalent:*

(a) for each $k \in \{0, 1, 2, \ldots, n\}$, $\mathcal{F}^{(k)} \subseteq AP(\mathbb{R})$ is precompact,

(b) $\mathcal{F} \subseteq AP^n(\mathbb{R})$ *is precompact,*

where $\mathcal{F}^{(k)} := \{f^{(k)} : f \in \mathcal{F}\}$.

Proof. By noting that

$$\|f^{(k)}\|_0 \leq \|f\|_n, \quad k = 0, 1, 2, \ldots, n$$

and

$$\|f\|_n \leq \sum_{k=0}^{n} \|f^{(k)}\|_0,$$

it is not difficult to get the conclusion. □

Combining Lemma 2.1 and the compactness criteria for $AP(\mathbb{R})$ (cf. [9, Theorem 6.10]), we get the following compactness criteria for $AP^n(\mathbb{R})$:

Theorem 2.2. *The necessary and sufficient condition that* $\mathcal{F} \subseteq AP^{(n)}(\mathbb{R})$ *be precompact is that the following properties hold true:*

(i) for each $t \in \mathbb{R}$ *and* $k \in \{0, 1, \cdots, n\}$, $\{f^{(k)}(t) : f \in \mathcal{F}\}$ *is precompact in* X*;*

(ii) for $k \in \{0, 1, \cdots, n\}$, $\mathcal{F}^{(k)}$ *is equi-continuous;*

(iii) for $k \in \{0, 1, \cdots, n\}$, $\mathcal{F}^{(k)}$ *is equi-almost periodic.*

Now, let $1 \leq p \leq \infty$ and q be such that $\frac{1}{p} + \frac{1}{q} = 1$. For convenience, we list some assumptions.

(H1) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a L^p-`Carathéodory function`, i.e., the following two conditions hold:
(i) the map $t \to f(t, y)$ is measurable for all $y \in \mathbb{R}$, and the map $y \to f(t, y)$ is continuous for almost all $t \in \mathbb{R}$;
(ii) for each $r > 0$, there exists a function $\mu_r \in L^p(\mathbb{R})$ such that $|y| \leq r$ implies that $|f(t, y)| \leq \mu_r(t)$ for almost all $t \in \mathbb{R}$.

(H2) Let $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be such that $\frac{\partial^m k(t,s)}{\partial t^m}$ exists for $m = 1, 2, \ldots, n$; and
(i) there exist functions $a_m \in L^q(\mathbb{R})$ such that

$$|k_t^m(s)| \leq a_m(s), \quad m = 0, 1, 2, \ldots, n,$$

for all $t \in \mathbb{R}$ and almost all $s \in \mathbb{R}$, where $k_t^m(s) := \frac{\partial^m k(t,s)}{\partial t^m}$ is measurable for each $t \in \mathbb{R}$;
(ii) the map $t \longmapsto k_t^m$ is almost periodic from $\mathbb{R}$ to $L^q(\mathbb{R})$, $m = 0, 1, 2, \ldots, n$.

(H3) there exists a constant $r_0 > 0$ such that

$$\|h\|_n + \sum_{m=0}^{n} \|a_m\|_q \cdot \|\mu_{r_0}\|_p \le r_0.$$

Now, we are ready to establish one of our main results.

Theorem 2.3. *Assume that (H1)-(H3) hold and $h \in AP^n(\mathbb{R})$. Then Eq.* (1.1) *has a C^n-almost periodic solution.*

Proof. We give the proof by three steps.

Step 1. $F : AP^n(\mathbb{R}) \to L^p(\mathbb{R})$ is bounded and continuous, where

$$(Fy)(t) := f(t, y(t)), \quad t \in \mathbb{R},\ y \in AP^n(\mathbb{R}).$$

Let $E \subset AP^n$ be a bounded subset and

$$r = \sup_{y \in E} \|y\|_n.$$

Then, by (H1), there exists a function $\mu_r \in L^p(\mathbb{R})$ such that

$$|f(t, y(t))| \le \mu_r(t)$$

for almost all $t \in \mathbb{R}$ and all $y \in E$, which yields that

$$\|Fy\|_p = \left[\int_{\mathbb{R}} |f(t, y(t))|^p dt\right]^{\frac{1}{p}} \le \left[\int_{\mathbb{R}} |\mu_r(t)|^p dt\right]^{\frac{1}{p}} < +\infty, \quad \forall y \in E.$$

Thus $F(E)$ is bounded. Next, we show that F is continuous. Let $y_k \to y$ in $AP^n(\mathbb{R})$. Then

$$r' := \sup_k \|y_k\| + 1 < +\infty.$$

By using (H1) again, we know that

$$\lim_{k \to \infty} f(t, y_k(t)) = f(t, y(t))$$

for almost all $t \in \mathbb{R}$, and

$$|f(t, y_k(t)) - f(t, y(t))| \le 2\mu_{r'}(t)$$

for almost all $t \in \mathbb{R}$. Then, by the Lebesgue's dominated convergence theorem, we get

$$\|Fy_k - Fy\|_p^p = \int_{\mathbb{R}} |f(t, y_k(t)) - f(t, y(t))|^p dt \to 0, \quad k \to \infty,$$

i.e., $Fy_k \to Fy$ in $L^p(\mathbb{R})$.

Step 2. $K : L^p(\mathbb{R}) \longrightarrow AP^n(\mathbb{R})$ is continuous and compact, where

$$(Ky)(t) = \int_{\mathbb{R}} k(t,s)y(s)ds, \quad t \in \mathbb{R},\ y \in L^p(\mathbb{R}).$$

First, let us show that K is well-defined, i.e., $Ky \in AP^n(\mathbb{R})$ for $y \in L^p(\mathbb{R})$. Noting that

$$\left| \frac{k(t+\Delta t, s) - k(t,s)}{\Delta t} \right| \leq a_1(s)$$

we get

$$(Ky)'(t) = \lim_{\Delta t \to 0} \int_{\mathbb{R}} \frac{k(t+\Delta t, s) - k(t,s)}{\Delta t} y(s)ds = \int_{\mathbb{R}} k_t^1(s)y(s)ds.$$

Similarly, one can show that

$$(Ky)^{(m)}(t) = \int_{\mathbb{R}} k_t^m(s)y(s)ds, \quad m = 0, 1, 2, \ldots, n.$$

Now, fix $y \in L^p(\mathbb{R})$ and $m \in \{0, 1, 2, \ldots, n\}$. We have

$$\begin{aligned} |(Ky)^{(m)}(t_1) - (Ky)^{(m)}(t_2)| &= \left| \int_{\mathbb{R}} k_{t_1}^m(s)y(s)ds - \int_{\mathbb{R}} k_{t_2}^m(s)y(s)ds \right| \\ &\leq \int_{\mathbb{R}} |k_{t_1}^m(s) - k_{t_2}^m(s)| \cdot |y(s)|ds \\ &\leq \|k_{t_1}^m - k_{t_2}^m\|_q \cdot \|y\|_p \end{aligned} \tag{2.1}$$

for all $t_1, t_2 \in \mathbb{R}$. By the almost periodicity of k_t^m, the map $t \longmapsto k_t^m$ is uniformly continuous on $\mathbb{R}$. Combining this with (2.1), we conclude that $(Ky)^{(m)}$ is uniformly continuous on $\mathbb{R}$. Again by the almost periodicity of k_t^m, for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that

$$\|k_{t+\tau}^m - k_t^m\|_q < \varepsilon, \quad \forall t \in \mathbb{R},$$

which and (2.1) yields that

$$|(Ky)^{(m)}(t+\tau) - (Ky)^{(m)}(t)| \leq \|k_{t+\tau}^m - k_t^m\|_q \cdot \|y\|_p \leq \varepsilon \|y\|_p, \quad \forall t \in \mathbb{R}.$$

Thus, $(Ky)^{(m)} \in AP(\mathbb{R})$. By the definition of $AP^n(\mathbb{R})$, we know that $Ky \in AP^n(\mathbb{R})$.

Next, we show that $K : L^p(\mathbb{R}) \longrightarrow AP^n(\mathbb{R})$ is compact. Let $E \subset L^p(\mathbb{R})$ be a bounded subset. For each $m \in \{0, 1, 2, \ldots, n\}$, we claim that the follow properties hold:

(a) $\{(Ky)^{(m)}(t) : y \in E\}$ is precompact for each $t \in \mathbb{R}$;

(b) $\{(Ky)^{(m)} : y \in E\}$ is equi-continuous on $\mathbb{R}$;

(c) $\{(Ky)^{(m)} : y \in E\}$ is equi-almost periodic.

In fact, the property (a) follows directly from

$$\begin{aligned}\left|(Ky)^{(m)}(t)\right| &= \left|\int_{\mathbb{R}} k_t^m(s)y(s)ds\right| \\ &\leq \int_{\mathbb{R}} |a_m(s)| \cdot |y(s)|ds \\ &\leq \|a_m\|_q \cdot \|y\|_p \\ &\leq \|a_m\|_q \cdot \sup_{y\in E} \|y\|_p < +\infty\end{aligned}$$

for all $y \in E$ and $t \in \mathbb{R}$. In addition, by some direct calculations, it follows from (2.1) that the properties (b) and (c) hold. Thus, by Theorem 2.2, we know that $K(E)$ is precompact in $AP^n(\mathbb{R})$. Moreover, noting that K is linear, we conclude that K is continuous.

Step 3. Eq. (1.1) has a C^n-almost periodic solution.

We denote

$$(Sy)(t) = h(t) + [K(Fy)](t) = h(t) + \int_{\mathbb{R}} k(t,s)f(s,y(s))ds, \quad y \in AP^n(\mathbb{R}), t \in \mathbb{R}.$$

Noting that $h \in AP^n(\mathbb{R})$, it follows from Step 1 and Step 2 that S is from $AP^n(\mathbb{R})$ to $AP^n(\mathbb{R})$, and $S : AP^n(\mathbb{R}) \to AP^n(\mathbb{R})$ is continuous and compact. Now, let

$$\mathcal{E} = \{y \in AP^n(\mathbb{R}) : \|y\|_n \leq r_0\}.$$

Then, for all $y \in \mathcal{E}$, we have

$$\begin{aligned}\|Sy\|_n &\leq \|h\|_n + \|K(Fy)\|_n \\ &\leq \|h\|_n + \sum_{m=0}^{n} \|[K(Fy)]^{(m)}\|_0 \\ &\leq \|h\|_n + \sum_{m=0}^{n} \|a_m\|_q \cdot \|Fy\|_p \\ &\leq \|h\|_n + \sum_{m=0}^{n} \|a_m\|_q \cdot \|\mu_{r_0}\|_p \\ &\leq r_0,\end{aligned}$$

which means that $S(\mathcal{E}) \subseteq \mathcal{E}$. Noting that $S : \mathcal{E} \to \mathcal{E}$ is continuous and $S(\mathcal{E})$ is precompact, by the classical Schauder's fixed point theorem, S has a fixed point in $\mathcal{E}$, i.e., Eq. (1.1) has a C^n-almost periodic solution. □

Next, we present an example to illustrate our result.

Example 2.4. Let $n = 1$, $p = 1$, $q = \infty$, and

$$h(t) = \cos \pi t, \quad k(t,s) = (\sin t + \sin \sqrt{2}t)e^{-s^2}, \quad f(t,y) = \frac{y \sin(ye^{t^2})}{20(1+t^2)}.$$

By some direct calculations, one can show that (H1) holds with $\mu_r(t) = \frac{r}{20(1+t^2)}$; (H2) holds with $a_0(s) = 2e^{-s^2}$ and $a_1(s) = (\sqrt{2}+1)e^{-s^2}$; (H3) follows from

$$\limsup_{r \to +\infty} \frac{\sum_{m=0}^{n} \|a_m\|_q \cdot \|\mu_r\|_p}{r} \leq \frac{(3+\sqrt{2})\pi}{20} < 1.$$

In addition, it is easy to see that $h \in AP^1(\mathbb{R})$. Thus, by Theorem 2.3, the following integral equation

$$y(t) = \cos \pi t + \int_{\mathbb{R}} \frac{\sin t + \sin \sqrt{2}t}{20(1+s^2)} e^{-s^2} y(s) \sin[e^{s^2} y(s)] ds$$

has a C^1-almost periodic solution.

3 Almost periodic solution for a class of integral equation

In this section, we consider the existence of almost periodic solution for Eq. (1.2). In the case of no confusion, we will denote the norm of $AP(\mathbb{R})$ by $\|\cdot\|$ instead of $\|\cdot\|_0$ for convenience.

Theorem 3.1. *Assume that (H1)-(H2) hold with $n = 0$ and $h \in AP(\mathbb{R})$. Moreover, the following assumptions hold:*

(H4) *$\alpha, \beta, \gamma : \mathbb{R} \to \mathbb{R}$ are three functions such that $y \in AP(\mathbb{R})$ implies that*

$$y(\alpha(\cdot))), y(\beta(\cdot))) \in AP(\mathbb{R}).$$

(H5) *$e, g \in AP(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exist two constants L_e, L_g such that*

$$|e(t,u) - e(t,v)| \leq L_e|u-v|, \quad |g(t,u) - g(t,v)| \leq L_g|u-v|, \quad \forall u, v \in \mathbb{R}.$$

(H6) *There exists a constant $r_0 > 0$ such that $ML_g + L_e < 1$ and*

$$M \cdot \sup_{t \in \mathbb{R}, |u| \leq r} |g(t,u)| + \sup_{t \in \mathbb{R}, |u| \leq r} |e(t,u)| < r, \quad \forall r > r_0,$$

where $M = \|h\| + \|a_0\|_q \cdot \|\mu_{r_0}\|_p$.

Then Eq. (1.2) has an almost periodic solution.

Proof. Let B be defined as follows

$$(By)(t) = h(t) + \int_{\mathbb{R}} k(t,s)f(s, y(\gamma(s)))ds, \quad y \in AP(\mathbb{R}),\ t \in \mathbb{R}.$$

By a similar proof to that of Theorem 2.3, one can also show that $B : AP(\mathbb{R}) \to AP(\mathbb{R})$ is continuous and compact.

In addition, we denote

$$(Ay)(t) = g(t, y(\beta(t))), \quad y \in AP(\mathbb{R}),\ t \in \mathbb{R};$$

and

$$(Cy)(t) = e(t, y(\alpha(t))), \quad y \in AP(\mathbb{R}),\ t \in \mathbb{R}.$$

Since $e, g \in AP(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $y(\alpha(\cdot)), y(\beta(\cdot)) \in AP(\mathbb{R})$ for each $y \in AP(\mathbb{R})$, we conclude that

$$Ay, Cy \in AP(\mathbb{R}), \quad \forall y \in AP(\mathbb{R}),$$

i.e., A, C are two operators from $AP(\mathbb{R})$ to $AP(\mathbb{R})$.

Denote $\mathcal{E} = \{y \in AP(\mathbb{R}) : \|y\| \le r_0\}$. For each $y \in \mathcal{E}$, define an operator $S(y)$ on $AP(\mathbb{R})$ by

$$[S(y)]x = Ax \cdot By + Cx, \quad x \in AP(\mathbb{R}).$$

Then $S(y)$ ia an operator from $AP(\mathbb{R})$ to $AP(\mathbb{R})$. For all $x_1, x_2 \in AP(\mathbb{R})$, by (H5), we have

$$\begin{aligned}
& \|[S(y)]x_1 - [S(y)]x_2\| \\
= \ & \|Ax_1 \cdot By + Cx_1 - Ax_2 \cdot By - Cx_2\| \\
\le \ & \|Ax_1 - Ax_2\| \cdot \|By\| + \|Cx_1 - Cx_2\| \\
\le \ & (L_g \cdot \|By\| + L_e) \cdot \|x_1 - x_2\| \\
\le \ & (ML_g + L_e) \cdot \|x_1 - x_2\|,
\end{aligned}$$

where

$$\|By\| \le \sup_{t \in \mathbb{R}} |h(t)| + \sup_{t \in \mathbb{R}} \int_{\mathbb{R}} |k_t(s)| \cdot \mu_{r_0}(s)ds \le \|h\| + \|a_0\|_q \cdot \|\mu_{r_0}\|_p = M$$

since $\|y\| \le r_0$. Noting that $ML_g + L_e < 1$, by the Banach contraction principle, we know that there exists a unique fixed point x_y of $S(y)$ in $AP(\mathbb{R})$.

Now, we define an operator on $\mathcal{E}$ by

$$\mathfrak{S}y = x_y,$$

where x_y is the unique fixed point of $S(y)$ in $AP(\mathbb{R})$. Then

$$\mathfrak{S}y = [S(y)]x_y = Ax_y \cdot By + Cx_y.$$

In addition, we claim that $\mathfrak{S}y \in \mathcal{E}$ for each $y \in \mathcal{E}$. In fact, letting $\|x_y\| = r_y$, if $r_y > r_0$, then by (H6), we get

$$\begin{aligned} r_y &= \|x_y\| = \|Ax_y \cdot By + Cx_y\| \\ &\leq M\|Ax_y\| + \|Cx_y\| \\ &\leq M \cdot \sup_{t\in\mathbb{R},|u|\leq r_y} |g(t,u)| + \sup_{t\in\mathbb{R},|u|\leq r_y} |e(t,u)| \\ &< r_y, \end{aligned}$$

which is a contradiction. So $\mathfrak{S}(\mathcal{E}) \subseteq \mathcal{E}$.

Next, let us show that $\mathfrak{S} : \mathcal{E} \to \mathcal{E}$ is continuous and $\mathfrak{S}(\mathcal{E})$ is precompact. For all $y_1, y_2 \in \mathcal{E}$, we have

$$\begin{aligned} & \|\mathfrak{S}y_1 - \mathfrak{S}y_2\| \\ = {} & \|Ax_{y_1} \cdot By_1 + Cx_{y_1} - Ax_{y_2} \cdot By_2 - Cx_{y_2}\| \\ \leq {} & L_g \cdot \|x_{y_1} - x_{y_2}\| \cdot \|By_1\| + \|Ax_{y_2}\| \cdot \|By_1 - By_2\| + L_e \cdot \|x_{y_1} - x_{y_2}\| \\ \leq {} & (ML_g + L_e)\|\mathfrak{S}y_1 - \mathfrak{S}y_2\| + \|Ax_{y_2}\| \cdot \|By_1 - By_2\| \\ \leq {} & (ML_g + L_e)\|\mathfrak{S}y_1 - \mathfrak{S}y_2\| + (\sup_{t\in\mathbb{R}} |g(t,0)| + L_g r_0) \cdot \|By_1 - By_2\|, \end{aligned}$$

which

$$\|\mathfrak{S}y_1 - \mathfrak{S}y_2\| \leq \frac{\sup_{t\in\mathbb{R}} |g(t,0)| + L_g r_0}{1 - ML_g - L_e} \cdot \|By_1 - By_2\|. \tag{3.1}$$

Let $y_k \to y$ in $\mathcal{E}$. Combining (3.1) with the continuity of B, we conclude that $\mathfrak{S}y_k \to \mathfrak{S}y$. In addition, letting $\{y_k\} \subset \mathcal{E}$, since $B(\mathcal{E})$ is precompact, there exists a subsequence $\{y_i\} \subset \{y_k\}$ such that $B(y_i)$ is convergent. Then, (3.1) yields that $\mathfrak{S}y_i$ is convergent, which means that $\mathfrak{S}(\mathcal{E})$ is also precompact.

At last, by using Schauder's fixed point theorem, we conclude that there exists a fixed point $y^* \in \mathcal{E}$ of $\mathfrak{S}$. Then, we have

$$y^* = \mathfrak{S}y^* = x_{y^*} = Ax_{y^*} \cdot By^* + Cx_{y^*} = Ay^* \cdot By^* + Cy^*,$$

which yields that Eq. (1.2) has an almost periodic solution. □

Remark 3.2. We remark that Theorem 3.1 is a generalization of [15, Theorem 2.4] to some extent. In fact, in [15, Theorem 2.4], the authors established the existence of almost periodic solution for Eq. (1.1) by using nonlinear alternative of Leray-Schauder type; Here, we deal with a more general integral equation, i.e., Eq. (1.2), by using the classical Schauder's fixed point theorem directly.

Example 3.3. Let $n = 1$, $p = 1$, $q = \infty$,

$$h(t) = \cos \pi t, \quad k(t,s) = (\sin t + \sin \sqrt{2}t)e^{-s^2}, \quad f(t,y) = \frac{y \sin(ye^{t^2})}{20(1+t^2)},$$

and

$$\alpha(t) = t-1, \quad \beta(t) = t - \sin t, \quad \gamma(t) = |t|, \quad g(t,u) \equiv \frac{1}{2(1+u^2)}, \quad e(t,u) = \frac{\cos t + \cos \pi t}{6} u.$$

By Example 2.4, we know that that (H1) holds with $\mu_r(t) = \frac{r}{20(1+t^2)}$, and (H2) holds with $n = 0$ and $a_0(s) = 2e^{-s^2}$.

It is easy to see that (H4) holds, and (H5) holds with $L_g = \frac{1}{2}, L_e = \frac{1}{3}$. In addition, since

$$\|h\| + \|a_0\|_q \cdot \|\mu_r\|_p = 1 + \frac{\pi r}{10}$$

and

$$\sup_{t \in \mathbb{R}, |u| \le r} |g(t,u)| \le \frac{1}{2}, \quad \sup_{t \in \mathbb{R}, |u| \le r} |e(t,u)| \le \frac{r}{3},$$

we conclude that (H6) holds $r_0 = 1$. Thus, by Theorem 3.1, the following integral equation

$$y(t) = \frac{\cos t + \cos \pi t}{6} y(t-1) + \frac{\cos \pi t + \int_{\mathbb{R}} \frac{\sin t + \sin \sqrt{2}t}{20(1+s^2)} e^{-s^2} y(|s|) \sin[e^{s^2} y(|s|)] ds}{2 + 2[y(t - \sin t)]^2}$$

has an almost periodic solution.

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