Electronic Journal of Qualitative Theory of Differential Equations
2012, No. **15**, 1-13; **http://www.math.u-szeged.hu/ejqtde/**

Existence of multiple positive solutions of a nonlinear arbitrary order boundary value problem with advanced arguments

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Abstract

In this paper, we investigate nonlinear fractional differential equations of arbitrary order with advanced arguments

$$\begin{cases} D_{0+}^{\alpha}u(t)+a(t)f(u(\theta(t)))=0, & 0<t<1,\ n-1<\alpha\leq n,\\ u^{(i)}(0)=0, & i=0,1,2,\cdots,n-2,\\ [D_{0+}^{\beta}u(t)]_{t=1}=0, & 1\leq\beta\leq n-2,\end{cases}$$

where $n>3$ $(n\in\mathbb{N})$, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α, $f:[0,\infty)\to[0,\infty)$, $a:[0,1]\to(0,\infty)$ and $\theta:(0,1)\to(0,1]$ are continuous functions. By applying fixed point index theory and Leggett-Williams fixed point theorem, sufficient conditions for the existence of multiple positive solutions to the above boundary value problem are established.

Keywords: Positive solution; advanced arguments; fractional differential equations; fixed point index theory; Leggett-Williams fixed point theorem.
MSC 2010: 34A08; 34B18; 34K37.

1 Introduction

Fractional order differential equations have proved to be better for the description of hereditary properties of various materials and processes than integer order differential equations. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc.[22, 23, 31, 32]. Recently, there are some papers dealing with the

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existence and multiplicity of solutions (or positive solutions) of boundary value problems for nonlinear fractional differential equations [3, 11, 12, 15, 16, 24]. The interest in the study of fractional differential equations lies in the fact that fractional order models are more accurate than integer order models, that is, there are more degrees of freedom in the fractional order models. For some new development on the topic, see [1, 4, 7, 8, 10, 13, 18, 25, 27, 28, 29].

Differential equations with deviated arguments are found to be important mathematical tools for the better understanding of several real world problems in physics, mechanics, engineering, economics, etc. [2, 14]. As a matter of fact, the theory of integer order differential equations with deviated arguments has found its extensive applications in realistic mathematical modelling of a wide variety of practical situations and has emerged as an important area of investigation. For the general theory and applications of integer order differential equations with deviated arguments, we refer the reader to the references [6, 20, 21, 33, 36, 37, 38]. However, fractional order differential equations with deviated arguments have not been much studied and many aspects of these equations are yet to be explored. For some recent work on equations of fractional order with deviated arguments, see [9, 30, 34, 35] and the references therein.

Motivated by some recent work on advanced arguments and boundary value problems of fractional order, in this paper, we investigate the following nonlinear fractional-order differential equation with advanced arguments

$$\begin{cases} D_{0+}^{\alpha}u(t) + a(t)f(u(\theta(t))) = 0, & 0 < t < 1,\ n-1 < \alpha \leq n, \\ u^{(i)}(0) = 0, & i = 0, 1, 2, \cdots, n-2, \\ [D_{0+}^{\beta}u(t)]_{t=1} = 0, & 1 \leq \beta \leq n-2. \end{cases} \tag{1.1}$$

where $n > 3$ $(n \in \mathbb{N})$, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α, $f : [0, \infty) \to [0, \infty)$, $a : [0, 1] \to (0, \infty)$ and $\theta : (0, 1) \to (0, 1]$ are continuous functions.

By a positive solution of (1.1), one means a function $u(t) \in C[0, 1]$ that is positive on $0 < t < 1$ and satisfies (1.1).

Throughout this paper we assume that:

(H_1) $a \in C([0, 1], [0, \infty))$ and a does not vanish identically on any subinterval.

(H_2) The advanced argument θ satisfies $t \leq \theta(t) \leq 1, \forall t \in (0, 1)$.

By applying the well-known Banach contraction principle and Guo-Krasnoselskii fixed point theorem, Ntouyas, Wang and Zhang [30] have successfully investigated the existence of at least one positive solutions to the nonlinear fractional boundary value problem (1.1). Here, we show that under certain sufficient conditions, the nonlinear advanced fractional boundary value problem (1.1) has at least two and at least three positive solutions. The main tools employed are the fixed point index theory (Theorem 2.8) and the well-known Leggett-Williams fixed point theorem (Theorem 2.9).

2 Preliminaries

For the reader's convenience, we present some necessary definitions and preliminary results from fractional differential equations and fixed point theory.

Definition 2.1 *The Riemann-Liouville fractional integral of order q is defined as*

$$I^q y(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s) ds, \qquad q > 0,$$

provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2.2 *The Riemann-Liouville fractional derivative of order q for a function y is defined by*

$$D^q y(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} y(s) ds, \qquad n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q, provided the right hand side is pointwise defined on $(0, \infty)$.

Lemma 2.3 *[18] Assume $y(t) \in C[0,1]$, then the following problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, & 0 < t < 1, \ n-1 < \alpha \leq n, \\ u^{(i)}(0) = 0, & i = 0, 1, 2, \cdots, n-2, \\ [D_{0+}^{\beta} u(t)]_{t=1} = 0, & 1 \leq \beta \leq n-2. \end{cases} \tag{2.1}$$

has the unique solution

$$u(t) = \int_0^1 G(t,s) y(s) ds$$

where

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Lemma 2.4 *[18] There exists a constant $\gamma \in (0,1)$ such that*

$$\min_{t \in [\frac{1}{2}, 1]} G(t,s) \geq \gamma \max_{t \in [0,1]} G(t,s) = \gamma G(1,s),$$

where $G(t,s)$ is given by (2.2).

Remark 2.5 *[18] γ has the expression*

$$\gamma = \min \left\{ \frac{\left(\frac{1}{2}\right)^{\alpha-\beta-1}}{2^{\beta}-1}, \left(\frac{1}{2}\right)^{\alpha-1} \right\}. \tag{2.3}$$

Remark 2.6 *Since* $u(t)=\int_0^1 G(t,s)y(s)ds$*, by Lemma 2.4, we can get*

$$\inf_{t\in[\frac{1}{2},1]} u(t)=\int_0^1 \inf_{t\in[\frac{1}{2},1]} G(t,s)y(s)ds \geq \gamma \sup_{t\in[0,1]} \int_0^1 G(t,s)y(s)ds=\gamma\|u\|.$$

Now we present some results from fixed point theory. Firstly we list some properties about the fixed point index of compact maps (Lemma 2.7) and the fixed point index theory (Theorem 2.8) which is needed to prove the existence of at least two solutions of (1.1).

Lemma 2.7 [5, 19] *Let* S *be a closed convex set in a Banach space and let* D *be a bounded open set such that* $D_S=D\bigcap S\neq\emptyset$*. Let* $T:\overline{D}_S\to S$ *be a compact map. Suppose that* $x\neq Tx$ *for all* $x\in\partial D_S$*.*

(i) (Existence) If $i(T,D_S,S)\neq 0$*, then* T *has a fixed point in* D_S*.*

(ii) (Normalization) If $u\in D_S$*, then* $i(\widetilde{u},D_S,S)=1$*, where* $\widetilde{u}(x)=u$ *for* $x\in\overline{D}_S$*.*

(iii) (Homotopy) Let $\zeta: J\times\overline{D}_S\to S$*,* $J=[0,1]$*, be a compact map such that* $x\neq\zeta(t,x)$ *for* $x\in\partial D_S$ *and* $t\in J$*. Then* $i(\zeta(0,\cdot),D_S,S)=i(\zeta(1,\cdot),D_S,S)$

(iv) (Additivity) If U_1,U_2 *are disjoint open subsets of* D_S *such that* $x\neq Tx$ *for* $x\in\overline{D}_S\setminus(U_1\bigcup U_2)$*, then* $i(T,D_S,S)=i(T,U_1,S)+i(T,U_2,S)$*,*
where $i(T,U_j,S)=i(T|_{\overline{U}_j},U_j,S)$*,* $\quad j=1,2$*.*

Theorem 2.8 [5, 17] *Let* P *be a cone in a Banach space* E*. For* $\rho>0$*, define* $\Omega_\rho=\{x\in P\mid \|x\|<\rho\}$*. Assume that* $T:\overline{\Omega}_\rho\to P$ *is a compact map such that* $x\neq Tx$ *for* $x\in\partial\Omega_\rho$*.*

(i) If $\|x\|<\|Tx\|$ *for* $x\in\partial\Omega_\rho$*, then* $i(T,\Omega_\rho,P)=0$*.*

(ii) If $\|x\|>\|Tx\|$ *for* $x\in\partial\Omega_\rho$*, then* $i(T,\Omega_\rho,P)=1$*.*

Next, we state a known result due to Leggett and Williams [26] which is needed to prove the existence of at least three solutions of (1.1).

Theorem 2.9 [26] *Suppose* $T:\overline{P}_c\to\overline{P}_c$ *is completely continuous and suppose there exists a nonnegative continuous concave functional* q *on* P *such that* $q(u)\leq\|u\|$ *for* $u\in\overline{P}_c$*. Suppose there exist constants* $0<a<b<d\leq c$ *such that*

(B1) $\{u\in P(q,b,d):q(u)>b\}\neq\emptyset$ *and* $q(Tu)>b$ *if* $u\in P(q,b,d)$*;*

(B2) $\|Tu\|<a$ *if* $u\in P_a$*;*

(B3) $q(Tu)>b$ *for* $u\in P(q,b,c)$ *with* $\|Tu\|>d$*.*

Then T has at least three fixed points $u_1,\ u_2$ and u_3 such that $\|u_1\| < a,\ b < q(u_2)$ and $\|u_3\| > a$ with $q(u_3) < b$.

Here, $P_c = \{u \in P : \|u\| < c\}$, $P(q, b, d) = \{u \in P : b \leq q(u),\ \|u\| \leq d\}$ and the map q is a nonnegative continuous concave functional on a cone P of a real Banach space, that is to say, $q : P \to [0, +\infty)$ is continuous and

$$q(tu + (1-t)v) \leq tq(u) + (1-t)q(v)$$

for all $u,\ v \in P$ and $0 \leq t \leq 1$.

3 Main results

Let $E = C[0, 1]$ be the Banach space endowed with the sup-norm. Let us introduce the cone $P = \{u | u \in C[0, 1], u \geq 0, \inf_{t\in[\frac{1}{2},1]} u(\theta(t)) \geq \gamma\|u\|\}$, where $\gamma \in (0, 1)$. Define the operator $T : C[0, 1] \to C[0, 1]$ as follows,

$$Tu(t) = \int_0^1 G(t, s)a(s)f(u(\theta(s)))ds. \tag{3.1}$$

By applying Lemma 2.3 with $y(t) = a(t)f(u(\theta(t)))$, the problem (1.1) has a solution if and only if the operator T has a fixed point, where T is given by (3.1).

Since $t \leq \theta(t) \leq 1,\ t \in (0, 1)$, we have

$$\inf_{t\in[\frac{1}{2},1]} u(\theta(t)) \geq \inf_{t\in[\frac{1}{2},1]} u(t) \geq \gamma\|u\| \tag{3.2}$$

by Remark 2.6, which plays an important role in proving our main theorems. This also show that $TP \subset P$, i.e. $T : P \to P$. By using Ascoli-Arzelá theorem, it is easy to prove that $T : P \to P$ is completely continuous.

For convenience, we introduce the following notations:

$$f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}.$$

Theorem 3.1 *Let $f_0 = f_\infty = \infty$. Suppose that $(H_1), (H_2)$ and the following condition holds:*

(H_3) *There exists a constant $r > 0$ such that*

$$f(u) < \frac{r}{m_1}, \quad \textit{for} \quad u \in [0, r], \quad \textit{where} \quad m_1 = \int_0^1 G(1, s)a(s)ds.$$

Then problem (1.1) *has at least two positive solutions* u_1 *and* u_2 *with* $0 < \|u_1\| < r < \|u_2\|$.

Proof. Since $f_0 = \infty$, we can choose a constant $r_1 \in (0, r)$ such that for $0 < u < r_1$ it holds $f(u) \geq \tau_1 u$, where $\tau_1 > 0$ satisfies

$$\gamma^2 \tau_1 \int_{\frac{1}{2}}^{1} G(1,s)a(s)ds \geq 1.$$

Let $\Omega_{r_1} = \{u \in P \mid \|u\| < r_1\}$. Take $u \in P$, such that $\|u\| = r_1$, so $u \in \partial\Omega_{r_1}$. Then, we have

$$\begin{aligned}
\|Tu\| &= \sup_{t\in[0,1]} \int_0^1 G(t,s)a(s)f(u(\theta(s)))ds \\
&\geq \int_0^{\frac{1}{2}} G(t,s)a(s)f(u(\theta(s)))ds + \int_{\frac{1}{2}}^{1} G(t,s)a(s)f(u(\theta(s)))ds \\
&> \int_{\frac{1}{2}}^{1} G(t,s)a(s)f(u(\theta(s)))ds \\
&\geq \int_{\frac{1}{2}}^{1} \min_{t\in[\frac{1}{2},1]} G(t,s)a(s)f(u(\theta(s)))ds \\
&\geq \int_{\frac{1}{2}}^{1} \gamma G(1,s)a(s)f(u(\theta(s)))ds \\
&\geq \int_{\frac{1}{2}}^{1} \gamma G(1,s)a(s)\tau_1 u(\theta(s))ds \\
&\geq \gamma^2\tau_1 \int_{\frac{1}{2}}^{1} G(1,s)a(s)ds\|u\| \\
&\geq \|u\|,
\end{aligned}$$

which implies $\|Tu\| > \|u\|$ for $u \in \partial\Omega_{r_1}$. Thus, $i(T, \Omega_{r_1}, P) = 0$ by Theorem 2.8.

Next, we consider the condition $f_\infty = \infty$. It implies that there exists a constant $R_0 > r$ such that $f(u) \geq \tau_2 u$ for $u \geq R_0$, where $\tau_2 > 0$ satisfies

$$\gamma^2 \tau_2 \int_{\frac{1}{2}}^{1} G(1,s)a(s)ds \geq 1.$$

Let $\Omega_{r_2} = \{u \in P \mid \|u\| < r_2\}$, where $r_2 > \max\left\{\frac{R_0}{\gamma}, r\right\}$. Then for $u \in \partial\Omega_{r_2}$, we have

$$\inf_{t\in[\frac{1}{2},1]} u(\theta(t)) \geq \gamma\|u\| > R_0.$$

By the same method as above, we have

$$\|Tu\| > \gamma^2\tau_2 \int_{\frac{1}{2}}^{1} G(1,s)a(s)ds\|u\| \geq \|u\|.$$

This implies for $u \in \partial\Omega_{r_2}$, we have $\|Tu\| > \|u\|$. Thus, $i(T, \Omega_{\rho_2}, P) = 0$.

Finally, let $\Omega_r = \{u \in P \mid \|u\| < r\}$. Then for $u \in \partial\Omega_r$, by (H_3), we have

$$\begin{aligned}
\|Tu\| &= \sup_{t\in[0,1]} \int_0^1 G(t,s)a(s)f(u(\theta(s)))ds \\
&< \int_0^1 G(1,s)a(s)\frac{r}{m_1}ds \\
&= r = \|u\|.
\end{aligned}$$

Theorem 2.8 implies that $i(T, \Omega_r, P) = 1$.

Since $r_1 < r < r_2$, it holds that $i(T, \Omega_r \setminus \overline{\Omega}_{r_1}, P) = i(T, \Omega_r, P) - i(T, \Omega_{r_1}, P) = 1$ and $i(T, \Omega_{r_2} \setminus \overline{\Omega}_r, P) = i(T, \Omega_{r_2}, P) - i(T, \Omega_r, P) = -1$, which imply that the operator T has at least two positive fixed points $u_1 \in \Omega_r \setminus \overline{\Omega}_{r_1}$, $u_2 \in \Omega_{r_2} \setminus \overline{\Omega}_r$ such that $0 < \|u_1\| < r < \|u_2\|$. □

Theorem 3.2 *Let $f_0 = f_\infty = 0$. Suppose that $(H_1), (H_2)$ and the following condition holds:*

(H_4) *There exists a constant $\rho > 0$ such that*

$$f(u) > \frac{\rho}{m_2}, \quad \text{for} \quad u \in [\gamma\rho, \rho], \quad \text{where} \quad m_2 = \int_0^{\frac{1}{2}} \gamma G(1,s)a(s)ds.$$

Then problem (1.1) *has at least two positive solutions u_1 and u_2 with $0 < \|u_1\| < \rho < \|u_2\|$.*

Proof. Firstly, since $f_0 = 0$, there exists a constant $\rho_1 \in (0, \rho)$ such that for $0 < u \leq \rho_1$ it holds $f(u) \leq \delta_1 u$, where $\delta_1 > 0$ satisfies

$$\delta_1 \int_1^1 G(1,s)a(s)ds \leq 1.$$

Let $\Omega_{\rho_1} = \{u \in P \mid \|u\| < \rho_1\}$. For $u \in \partial\Omega_{\rho_1}$, we have

$$\begin{aligned}
\|Tu\| &= \sup_{t\in[0,1]} \int_0^1 G(t,s)a(s)f(u(\theta(s)))ds \\
&< \int_0^1 G(1,s)a(s)\delta_1 u(\theta(s))ds
\end{aligned}$$

$$\begin{aligned} &\leq \delta_1 \int_0^1 G(1,s)a(s)ds\|u\| \\ &\leq \|u\|. \end{aligned}$$

Theorem 2.8 implies $i(T, \Omega_{\rho_1}, P) = 1$.

Next, since $f_\infty = 0$, there exists a constant $R_0^{'} > \rho$ such that $f(u) \leq \delta_2 u$ for $u \geq R_0^{'}$, where $\delta_2 > 0$ satisfies

$$\delta_2 \int_0^1 G(1,s)a(s)ds < 1.$$

We consider the following two cases:

Case I: f is bounded. Then there exists $M_1 > 0$ such that $f(u) < M_1$ for $u \in [0, \infty)$.

Let $\mu = \int_0^1 G(1,s)a(s)dsM_1$. Choose $\rho_2 > \max\{\mu, R_0^{'}\}$ and define $\Omega_{\rho_2} = \{u \in P \mid \|u\| < \rho_2\}$. Then for $u \in \partial\Omega_{\rho_2}$, we have

$$\begin{aligned} \|Tu\| &= \sup_{t\in[0,1]} \int_0^1 G(t,s)a(s)f(u(\theta(s)))ds \\ &< \int_0^1 G(1,s)a(s)dsM_1 \\ &= \mu < \rho_2 = \|u\|. \end{aligned}$$

Case II: f is unbounded. Since f is continuous, there exists $\rho_2 > \max\left\{\frac{R_0^{'}}{\gamma}, \rho\right\}$ such that $f(u) < f(\rho_2)$ for $0 < u \leq \rho_2$. Let $\Omega_{\rho_2} = \{u \in P \mid \|u\| < \rho_2\}$. Then for $u \in \partial\Omega_{\rho_2}$, we have

$$\begin{aligned} \|Tu\| &= \sup_{t\in[0,1]} \int_0^1 G(t,s)a(s)f(u(\theta(s)))ds \\ &< \int_0^1 G(1,s)a(s)f(\rho_2)ds \\ &\leq \int_0^1 G(1,s)a(s)\delta_2\rho_2 ds \\ &\leq \delta_2 \int_0^1 G(1,s)a(s)ds\rho_2 \\ &< \rho_2 = \|u\|. \end{aligned}$$

Combine Case I and Case II, we can get for $u \in \partial\Omega_{\rho_2}$, we have $\|Tu\| < \|u\|$. Therefore, $i(T, \Omega_{\rho_2}, P) = 1$.

Finally, let $\Omega_\rho = \{u \in P \mid \|u\| < \rho\}$. Since $\partial\Omega_\rho \subset P$, it follows

$$\inf_{t\in[\frac{1}{2},1]} u(\theta(t)) \geq \gamma\|u\| = \gamma\rho$$

for any $u \in \partial\Omega_\rho$. Then by (H_4), we have

$$\begin{aligned} \|Tu\| &= \sup_{t\in[0,1]} \int_0^1 G(t,s)a(s)f(u(\theta(s)))ds \\ &> \int_0^{\frac{1}{2}} \gamma G(1,s)a(s)\frac{\rho}{m_2}ds \\ &= \rho = \|u\|. \end{aligned}$$

Theorem 2.8 shows that $i(T,\Omega_\rho,P)=0$.

Since $\rho_1 < \rho < \rho_2$, it holds that $i(T,\Omega_\rho \setminus \overline{\Omega}_{\rho_1},P) = i(T,\Omega_\rho,P) - i(T,\Omega_{\rho_1},P) = 1$ and $i(T,\Omega_{\rho_2} \setminus \overline{\Omega}_\rho,P) = i(T,\Omega_{\rho_2},P) - i(T,\Omega_\rho,P) = -1$, which imply that the operator T has at least two positive fixed points $u_1 \in \Omega_\rho \setminus \overline{\Omega}_{\rho_1}$, $u_2 \in \Omega_{\rho_2} \setminus \overline{\Omega}_\rho$ such that $0 < \|u_1\| < \rho < \|u_2\|$. □

Theorem 3.3 *Let a, b and c be constants such that $0 < a < b < c$. In addition we suppose that $(H_1), (H_2)$ hold and there exist constants A and B such that*

$$0 < A \le \frac{1}{\int_0^1 G(1,s)a(s)ds} \quad \text{and} \quad B > \frac{1}{\int_{\frac{1}{2}}^1 \gamma G(1,s)a(s)ds}.$$

Assume that the following conditions are satisfied.

(H_5) $f(u) < Aa$ *for all* $u \in [0,a]$;

(H_6) $f(u) > Bb$ *for all* $u \in [b,c]$;

(H_7) $f(u) \le Ac$ *for all* $u \in [0,c]$.

Then the problem (1.1) *has at least three positive solutions $u_1, u_2, u_3 \in P$ satisfying*

$$\|u_1\| < a, \quad b < q(u_2), \quad a < u_3 \text{ with } q(u_3) < b.$$

Proof. Under assumptions $(H_1), (H_2)$ operator T is completely continuous.

Let $q(u) = \min\limits_{\frac{1}{2}\le t\le 1} |u(t)|$, it is obvious that $q(u)$ is a nonnegative continuous concave functional. Note that $q(u) \le \|u\|$ for $u \in \overline{P_c}$. We will show that the conditions of Theorem 2.9 are satisfied.

Put $u \in P_c$. Then $\|u\| \le c$, and

$$\begin{aligned} \|Tu\| &= \sup_{t\in[0,1]} \int_0^1 G(t,s)a(s)f(u(\theta(s)))ds \\ &< \int_0^1 G(1,s)a(s)dsAc \\ &< c. \end{aligned}$$

This implies $T : P_c \to P_c$.

By the same method, if $u \in P_a$, then we can get $\|Tu\| < a$, and therefore $(B2)$ is satisfied.

Let d be a fixed constant such that $b < d \le c$. Then $q(d) \ge d > b$ and $\|d\| = d$, it means $P(q, b, d) \neq \emptyset$.

For any $u \in P(q, b, d)$, it holds that $\|u\| \le d$ and $q(u) = \min_{\frac{1}{2} \le t \le 1} u(t) \ge b$. Then we have

$$\begin{aligned} q(Tu) &= \min_{\frac{1}{2} \le t \le 1} \int_0^1 G(t,s)a(s)f(u(\theta(s)))ds \\ &> \int_{\frac{1}{2}}^1 \gamma G(1,s)a(s)dsBb \\ &> b. \end{aligned}$$

Thus $(B1)$ is satisfied.

Finally, for any $u \in P(q, b, c)$ with $\|Tu\| > d$, then $\|u\| \le c$ and $\min_{\frac{1}{2} \le t \le 1} u(t) \ge b$, by the same method, we can also show that $q(Tu) > b$ easily, which means that $(B3)$ holds.

Therefore, by the conclusion of Theorem 2.9, the operator T has at least three fixed points. This implies that (1.1) has at least three solutions. □

4 Example

Example 4.1 *Consider the fractional differential equation with advanced arguments*

$$\begin{cases} D_{0+}^{\alpha} u(t) + \Gamma(\alpha)(1-t)f(u(\theta(t))) = 0, & 0 < t < 1, \ n-1 < \alpha \le n, \\ u^{(i)}(0) = 0, & i = 0, 1, 2, \cdots, n-2, \\ [D_{0+}^{\beta} u(t)]_{t=1} = 0, & 1 \le \beta \le n-2. \end{cases} \tag{4.1}$$

where $\theta(t) = t^{\nu}, \ 0 < \nu < 1$ *and*

$$f(u) = \begin{cases} \dfrac{1}{2}(u^{\frac{1}{3}} + u^2), \ 0 \le u \le 1, \\ e^{u-1}, u > 1. \end{cases}$$

Obviously, it's not difficult to verify conditions (H_1) and (H_2) of Theorem 3.1 hold. Through a simple calculation we can get $f_0 = f_\infty = \infty$.

Note that if $a(t) = \Gamma(\alpha)(1-t)$, then

$$m_1 = \int_0^1 G(1,s)a(s)ds = \int_0^1 [(1-s)^{\alpha-\beta} - (1-s)^{\alpha}]ds < 1.$$

Take $r = 1$, then it holds $f(u) = \dfrac{1}{2}(u^{\frac{1}{3}} + u^2) < \dfrac{1}{m_1}$, for $u \in [0, 1]$, then condition (H_3) of Theorem 3.1 holds. Thus, by Theorem 3.1, we can get that the above problem (4.1) has at least two positive solutions u_1 and u_2 with $0 < \|u_1\| < 1 < \|u_2\|$.

Acknowledgment

We would like to express our gratitude to the anonymous reviewers and editors for their valuable comments and suggestions which improve the quality of the present paper.

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(Received October 11, 2011)