

Perturbed integral equations in modular function spaces

By

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Abstract. We focus our attention on a class of perturbed integral equations in modular spaces, by using fixed point Theorem I.1 (see [1]).

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1 Introduction

In the present work, we focus our attention on a class of perturbed integral equation which can be written as

$$u(t) = \exp(-tA)f_0 + \int_0^t \exp((s-t)A)Tu(s)ds \quad (I)$$

in the modular space $C^\varphi = C([0, b], L^\varphi)$ (see [1]), where L^φ is the Musielak-Orlicz space, f_0 is a fixed element in L^φ , $A : L^\varphi \rightarrow L^\varphi$ is a linear operator and $T : L^\varphi \rightarrow L^\varphi$ is $\rho - c$ -Lipschitz, i.e. there exists $k > 0$ such that $\rho(c(Tx - Ty)) \leq k\rho(x - y)$ for any x, y in L^φ (ρ being a modular). Since ρ is not subadditive, then the sum of these operators is not necessarily ρ -Lipschitz and the convexity of the integral presents a more delicate problem. Therefore, it is natural in our study to introduce c_0 constant c_0 and assume some hypotheses on A , T , and b .

For more details about the concepts of the above mentioned modular spaces, we refer the reader to the books by Musielak [4] and Kozłowski [3].

We begin by recalling the definition below.

Definition 1.1 *Let X be an arbitrary vector space over $K = (\mathbb{R} \text{ or } \mathbb{C})$*

a) A functional $\rho : X \rightarrow [0, +\infty]$ is called a pseudomodular if

i) $\rho(0) = 0$.

ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, $\forall x \in X$.

iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. If in place of iii) there holds also:

iii') $\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$ for $\alpha, \beta \geq 0$ and $\alpha^s + \beta^s = 1$, with an $s \in (0, 1[$, then the pseudomodular ρ is called s -convex. 1-convex pseudomodular are called convex. If besides i) there holds also.

i') $\rho(x) = 0$ implies $x = 0$, then ρ is called a modular.

b) If ρ is a pseudomodular in X , then .

$X_\rho = \{x \in X / \rho(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0\}$ is called a modular space.

c) If ρ is a convex modular, then $\|x\|_\rho = \inf\{u > 0, \rho(\frac{x}{u}) \leq 1\}$ is called the Luxemburg norm.

Recall that ρ has the Fatou property if: $\rho(x - y) \leq \liminf \rho(x_n - y_n)$, whenever $x_n \xrightarrow{\rho} x$ and $y_n \xrightarrow{\rho} y$.

And we say that ρ satisfies the Δ_2 -condition if:

$\rho(2x_n) \rightarrow 0$ as $n \rightarrow +\infty$ whenever $\rho(x_n) \rightarrow 0$ as $n \rightarrow +\infty$, for any sequence $(x_n)_{n \in \mathbb{N}}$ in X_ρ .

2 Perturbed integral equation class

In this section, we will study the existence of solution of the following perturbed integral equation:

$$u(t) = \exp(-tA)f_0 + \int_0^t \exp((s-t)A) Tu(s)ds \quad (I)$$

We present the general hypotheses of the equation (I).

H_1) Let ρ be a modular of the Musielak-Orlicz space L^φ , convex satisfying the Δ_2 -condition and $\rho_a(u) = \sup_{t \in [0, b]} \exp(-at)\rho(u(t))$ is a modular of $C([0, b], L^\varphi)$ with $a > 0$ (see [1]).

H_2) Let $A : L^\varphi \rightarrow L^\varphi$ be a linear application, assume that there exist $\alpha_0 > \max(e^{-1}, eb^2)$ and $M > 0$ such that $\rho(\alpha_0 Ax) \leq M\rho(x)$ for any $x \in L^\varphi$.

H_3) Let $T : L^\varphi \rightarrow L^\varphi$ be $\rho - c$ -Lipschitz with $c > 0$, i.e there exists $k > 0$ such that $\rho(c(Tx - Ty)) \leq k\rho(x - y)$ for any $x, y \in L^\varphi$.

H_4) Let f_0 be fixed element in L^φ .

Theorem 2.1 Under these conditions $H_1 - H_4$ and for all $b > 0$, the perturbed integral equation (I) has a solution $u \in C([0, b], L^\varphi)$.

Remark.

If we restrict our attention to the Banach space $(L^\varphi, \|\cdot\|_\rho)$. Then the equation (I) can be written as follows:

$$u'(t) + Au(t) = Tu(t) \quad (*).$$

Thus, if $A \equiv I$ then (*) becomes

$$u'(t) + (I - T)u(t) = 0.$$

But the latter equation has been treated before in [1] and [4]. This let us to reduce the study to the case $A \neq I$ when (*) can be written in the form below:

$$u'(t) + (I - [T + (I - A)])u(t) = 0.$$

Set $B = I - A$. It follows from the fact that ρ is not subadditive that $T + B$ is not necessarily ρ -Lipschitz contrary to the situation in [1] and [2].

We cite first the theorem below which we shall use in the proof of Theorem 2.1.

Theorem 2.2 . (See [1])

Let X_ρ be a ρ -complete modular space. Assume that ρ is an s -convex, satisfying the Δ_2 -condition and having the Fatou property. Let B be a ρ -closed subset of X_ρ and $T : B \rightarrow B$ a mapping such that

$$(*) \quad \exists c, k \in \mathbb{R}^+ : c > \max(1, k), \quad \rho(c(Tx - Ty)) \leq k^s \rho(x - y) \text{ for any } x, y \in B.$$

Then T has a fixed point.

Proof of Theorem 2.1.

1st) step.

We use the following property. Under the hypotheses of Theorem 2.1, the operator A is continuous from $(L^\varphi, \|\cdot\|_\rho)$ to itself. Indeed, we have $\rho(\alpha_0 Ax) \leq M\rho(x)$ for any $x \in L^\varphi$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in L^φ such that $\|x_n\|_\rho \rightarrow 0$ as $n \rightarrow +\infty$. So $\rho(x_n) \rightarrow 0$ as $n \rightarrow +\infty$, which implies that $\rho(\alpha_0 Ax_n) \rightarrow 0$ as $n \rightarrow +\infty$. By Δ_2 -condition, $\|\alpha_0 Ax_n\|_\rho \rightarrow 0$ as $n \rightarrow +\infty$. Hence $\|Ax_n\|_\rho \rightarrow 0$ as $n \rightarrow +\infty$. Thus, there exists a constant $c > 0$ such that $\|Ax\|_\rho \leq c\|x\|_\rho$, for any $x \in L^\varphi$.

Therefore, $\exp(A)(x) = \sum_{m=0}^{+\infty} \frac{A^m}{m!}(x)$ make a sense.

2^{end}) step.

We claim that $\frac{eb}{\alpha_0} < \frac{1}{b}$. Indeed, since $\alpha_0 > \max\{e^{-1}, eb^2\}$ we have:

a) If $e^{-1} \geq eb^2$ then $e^2b^2 \leq 1$ therefore $\frac{eb}{\alpha_0} < \frac{e^2b^2}{b} \leq \frac{1}{b}$.

b) If $eb^2 \geq e^{-1}$ then $e^2b^2 \geq 1$ therefore $\frac{eb}{\alpha_0} < \frac{eb}{eb^2} = \frac{1}{b}$.

Hence in both cases we have $\frac{eb}{\alpha_0} < \frac{1}{b}$, we choose c_0 such that $\frac{eb}{\alpha_0} \leq c_0 < \frac{1}{b}$ and $c = \frac{e}{c_0}$.

Then $c_0b < 1$. Let $\lambda > 1$ such that $1 < \lambda < \frac{1}{c_0b}$.

We consider $S : C([0, b], L^\varphi) \rightarrow C([0, b], L^\varphi)$ defined by.

$Su(t) = \exp(-tA)f_0 + \int_0^t \exp((s-t)A) Tu(s)ds$ for any $u \in C([0, b], L^\varphi)$. It is clear that $Su(t) \in L^\varphi$ for each $t \in [0, b]$. As Su is continuous from $[0, b]$ into $(L^\varphi, \|\cdot\|_\rho)$, then, Su is ρ -continuous from $[0, b]$ into (L^φ, ρ) . Let $u, v \in C([0, b], L^\varphi)$, we have

$$\lambda(Su(t) - Sv(t)) = \int_0^t \lambda \exp((s-t)A) (Tu - Tv)(s)ds . \text{ We put } Tu - Tv = x.$$

Let $K = \{t_0, t_1, \dots, t_n\}$ be any subdivision of $[0, t]$. $\sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) \exp((t_i - t)A)x(t_i)$ is $\|\cdot\|_\rho$ -convergent, and consequently, ρ -convergent to $\int_0^t \lambda \exp((s-t)A)x(s)ds$ in L^φ when, $|K| = \sup\{|t_{i+1} - t_i|, i = 0, \dots, n-1\} \rightarrow 0$ as $n \rightarrow +\infty$. By Fatou property we have

$$\rho(\int_0^t \lambda \exp((s-t)A)x(s)ds) \leq \liminf \rho(\sum_{i=0}^{n-1} \lambda(t_{i+1}-t_i) \exp((t_i-t)A)x(t_i)).$$

$$\text{And } \sum_{i=0}^{n-1} \lambda(t_{i+1}-t_i) \exp((t_i-t)A)x(t_i) = \sum_{i=0}^{n-1} \lambda(t_{i+1}-t_i)c_0 \frac{1}{c_0} \exp((t_i-t)A)x(t_i).$$

$$\text{Moreover } \sum_{i=0}^{n-1} \lambda(t_{i+1}-t_i)c_0 \leq \lambda c_0 b \leq 1$$

$$\text{Then } \rho(\sum_{i=0}^{n-1} \lambda(t_{i+1}-t_i) \exp((t_i-t)A)x(t_i)) \leq \sum_{i=0}^{n-1} \lambda(t_{i+1}-t_i)c_0 \rho(\frac{1}{c_0} \exp((t_i-t)A)x(t_i)).$$

3rd step. In this part, we show that

$$\rho(\frac{1}{c_0} \exp((t_i-t)A)x(t_i)) \leq \exp(M-1)\rho(\frac{e}{c_0}x(t_i))$$

$$\text{We have } \frac{1}{c_0} \exp((t_i-t)A)x(t_i) = \sum_{m=0}^{+\infty} \frac{1}{c_0} \frac{(t-t_i)^m}{m!} A^m((-1)^m x(t_i)).$$

$$\text{And since } \sum_{m=0}^{+\infty} \frac{\exp(-1)}{m!} = 1, \text{ then } \rho(\frac{1}{c_0} \exp((t-t_i)A)x(t_i)) \leq \sum_{m=0}^{+\infty} \frac{\exp(-1)}{m!} \rho(\frac{e}{c_0} b^m A^m x(t_i)).$$

We have $\alpha_0 \geq \frac{eb}{c_0} > 0$, and since $\alpha_0 > \max(e^{-1}, eb^2)$, then $\alpha_0 > b$. Indeed,

i) if $e^{-1} \geq eb^2$, then $e^2b^2 \leq 1$ which implies that $eb \leq 1$. Therefore $b \leq e^{-1} < \alpha_0$.

ii) if $eb^2 \geq e^{-1}$, then $e^2b^2 \geq 1$ which implies that $eb \geq 1$. Therefore $eb^2 \geq b$ and $\alpha_0 > b$.

From the hypothesis $\rho(\alpha_0 Ax(t_i)) \leq M\rho(x(t_i))$,

we have

$$\begin{aligned} \rho(\alpha_0 b A^2 x(t_i)) &\leq M\rho(bAx(t_i)) \\ &\leq M\rho(\alpha_0 Ax(t_i)) \\ &\leq M^2\rho(x(t_i)) \end{aligned}$$

Which implies that $\rho(\frac{e}{c_0} b^m A^m x(t_i)) \leq M^m \rho(x(t_i)) \leq M^m \rho(\frac{e}{c_0} x(t_i))$ for any m in \mathbb{N}^* .

Therefore,

$$\begin{aligned} \rho(\frac{1}{c_0} \exp((t_i-t)A)x(t_i)) &\leq \sum_{m=0}^{+\infty} \frac{\exp(-1)M^m}{m!} \rho(\frac{e}{c_0} x(t_i)) \\ &= \exp(M-1)\rho(\frac{e}{c_0} x(t_i)). \end{aligned}$$

4th Step. We have

$$\begin{aligned} \rho(\lambda(Su(t) - Sv(t))) &\leq \liminf(\sum_{i=0}^{n-1} \lambda(t_{i+1}-t_i)c_0 \exp(M-1)k\rho(u-v)(t_i)) \\ &\leq k\lambda \exp(M-1) \liminf(\sum_{i=0}^{n-1} (t_{i+1}-t_i)c_0 \exp(at_i))\rho_a(u-v) \\ &= \lambda k \exp(M-1) \int_0^t c_0 \exp(as)ds \quad \rho_a(u-v) \end{aligned}$$

therefore

$$\exp(-at)\rho(\lambda(Su(t) - Sv(t))) \leq k\lambda \exp(M-1) \int_0^t c_0 \exp(a(s-t))ds \quad \rho_a(u-v)$$

Hence,

$$\rho_a(\lambda(su - sv)) \leq k\lambda \exp(M-1) \frac{c_0}{a} (1 - e^{-ab}) \rho_a(u-v).$$

It suffices to take $a > ke^{M-1}c_0$, then we have $\lambda k \exp(M-1) \frac{c_0}{a} (1 - e^{-ab}) < \lambda$.

By Theorem 2.2, S has a fixed point which is a solution of the equation (I).

Remark

In third step, instead of the combination convex $\sum_{m=0}^{\infty} \frac{e^{-1}}{m!} = 1$, we may choose the combination convex

$\sum_{m=0}^{\infty} \frac{e^{-1}b^m}{m!} = 1$, which gives the conclusion of theorem under the following hypotheses:

H'_2 $A : L^\varphi \rightarrow L^\varphi$ is a linear application, and there exists $M > 0$ such that : $\rho(Ax) \leq M\rho(x)$ for any $x \in L^\varphi$.

H'_3 $T : L^\varphi \rightarrow L^\varphi$ is an application and for $\alpha_0 = \frac{\exp(b)}{c_0}$ with $c_0b < 1$ there exists $k > 0$ such that: $\rho(\alpha_0(Tx - Ty)) \leq k\rho(x - y)$.

Consider now the following perturbed integral equation.

$$u(t) = \exp(-t) \exp(-tA) f_0 + \int_0^t \exp(s-t) \exp((s-t)A) Tu(s)ds \quad (II).$$

The same techniques than in the proof of Theorem 2.1 are used to establish Theorem 2.3 below by taking care of the choose of λ in $(1, \frac{1}{1-e^{-b}}]$, which gives

$$\rho(\int_0^t \lambda e^{s-t} e^{(s-t)A} x(s)ds) \leq \liminf_{i=0}^{n-1} (\sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) e^{t_i-t} \rho(e^{(t_i-t)A} x(t_i))) \text{ and}$$

$$\sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) e^{t_i-t} \leq \lambda \int_0^t e^{s-t} ds \leq 1.$$

Theorem 2.3 Assume that for $\alpha_1 \geq eb$, there exists $M > 0$ such that $\rho(\alpha_1 Ax) \leq M\rho(x)$ for any $x \in L^\varphi$ and there exists $k > 0$ such that $\rho(e(Tx - Ty)) \leq k\rho(x - y)$ for any x, y in L^φ . Then, the perturbed integral equation (II) has a solution $u \in C([0, b], L^\varphi)$.

Remark.

By using the same technics as in the proof of Theorem 2.3, we can prove the existence of a solution of the equation below:

$$u(t) = e^{-t} f_0 + \int_0^t \varphi(s-t) e^{(s-t)} Tu(s)ds,$$

where $\varphi : \mathbb{R} \rightarrow \mathbb{R}_*^+$ is a continuous function satisfying $\int_0^b \varphi(-s)ds < 1$.

Conclusion

Concerning the equations (I) and (II), Theorem 2.1 and Theorem 2.3 give local solutions

because of the constraint on b . In this frame, we notice that if A is ρ -Lipschitz i.e. if there exists $M > 0$ such that $\rho(Ax) \leq M\rho(x)$ for any $x \in L^\varphi$, then the equation (I) and the equation (II) have a solution in $[0, \frac{1}{e}]$.

Example of the equation (I).

Let φ be a Musielak-Orlicz function on a measurable space $([0, 1], \mathcal{A}, \mu)$, ρ_φ be a modular defined by

$$\rho_\varphi(u) = \int_0^1 \varphi(s, |u(s)|) ds,$$

for any $u \in L^\varphi$ and $\alpha_0 > \max(e^{-1}, eb^2)$, $c_0 \in [\frac{eb}{\alpha_0}, \frac{1}{b}[$. Assume that ρ_φ is convex satisfying the Δ_2 -condition.

In this example, we study the existence of a solution of the following integral equation

$$u(t) = \exp(-tA)f_0 + \int_0^t \exp[(s-t)A] (\int_0^1 K_1(\xi, u(s)) d\xi) ds \quad (I'),$$

where $K_1 : [0, 1] \times L^\varphi \rightarrow L^\varphi$ is a measurable function satisfying

- 1) $\lim_{\lambda \rightarrow 0^+} \int_0^1 \varphi(\xi, \lambda |(\int_0^1 K_1(s, u) ds) \cdot \xi|) d\xi = 0$ for any $u \in L^\varphi$.
 - 2) $|(\int_0^1 (K_1(\xi, u(s)) - K_1(\xi, v(s))) d\xi)| \leq k|(u - v)(s)|$, for any u, v in L^φ , with $k \in]0, 1[$.
- f_0 is a fixed element in L^φ and the operator A is equal to k_0I , where I is the identity function of L^φ with $k_0 \leq \frac{1}{\alpha_0}$.

Let T be a mapping from L^φ into L^φ defined by

$$Tu = \int_0^1 \frac{c_0}{e} K_1(s, u) ds.$$

Hence, we have $\rho_\varphi(\alpha_0 k_0 x) \leq \alpha_0 k_0 \rho_\varphi(x)$ for any $x \in L^\varphi$, .i.e. $\rho(\alpha_0 Ax) \leq \alpha_0 k_0 \rho(x)$ for any $x \in L^\varphi$.

Now, we show that T is $\rho - \frac{e}{c_0}$ -Lipschitz.

At first, by 1), we have $\int_0^1 \varphi(\xi, \lambda |Tu(\xi)|) d\xi \rightarrow 0$ as $\lambda \rightarrow 0^+$. Hence, by the definition of L^φ , $Tu \in L^\varphi$ for any $u \in L^\varphi$.

On the other hand, let $x, y \in L^\varphi$

$$\begin{aligned} \rho_\varphi\left(\frac{e}{c_0}(Tx - Ty)\right) &= \int_0^1 \varphi\left(s, \frac{e}{c_0}|(Tx - Ty)(s)|\right) ds \\ &= \int_0^1 \varphi\left(s, \left|\int_0^1 (K_1(\xi, x(s)) - K_1(\xi, y(s))) d\xi\right|\right) ds \end{aligned}$$

Therefore, by 2)

$$\begin{aligned} \rho_\varphi\left(\frac{e}{c_0}(Tx - Ty)\right) &\leq \int_0^1 \varphi\left(s, k|(x - y)(s)|\right) ds \\ &= \rho_\varphi(k(u - v)) \\ &= k\rho_\varphi(u - v). \end{aligned}$$

Hence T is $\rho - \frac{\varepsilon}{c_0}$ -Lipschitz. So by Theorem 2.1 the equation (I') has a solution in $C([0, b], L^\varphi)$.

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