INITIAL AND BOUNDARY VALUE PROBLEMS FOR SECOND ORDER IMPULSIVE FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract

In this paper we investigate the existence of solutions for initial and boundary value problems for second order impulsive functional differential inclusions. We shall rely on a fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

Key words and phrases: Impulsive functional differential inclusions, measurable selection, contraction multivalued map, existence, fixed point, Banach space.
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1 Introduction

This paper is concerned with the existence of solutions for initial and boundary value problems for second order impulsive functional differential inclusions in Banach spaces. More precisely, in Section 3 we consider the second order initial value problem for the impulsive functional differential equations of the form

$$y'' \in F(t, y_t), \ a.e. \ t \in J = [0, T], \ t \neq t_k, \quad k = 1, \dots, m, \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{2}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{3}$$

$$y(t) = \phi(t), \quad t \in J_0 = [-r, 0], \ y'(0) = \eta, \tag{4}$$

where $F : J \times D \to P(E)$ is a multivalued map, $D = \{\psi : [-r, 0] \to E;\ \psi$ is continuous everywhere except for a finite number of points $\tilde{t}$ at which $\psi(\tilde{t}^-)$ and $\psi(\tilde{t}^+)$ exist and $\psi(\tilde{t}^-) = \psi(\tilde{t})\}$, $\phi \in D$, $P(E)$ is the family of all nonempty subsets of E, $0 < r < \infty$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, $I_k : E \to E$ $(k = 1, 2, \dots, m)$, $\overline{I}_k : E \to E$ and $\eta \in E$. $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively, and E a real separable Banach space with norm $|\cdot|$.

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For any continuous function y defined on $[-r,T]-\{t_1,\dots,t_m\}$ and any $t\in J$, we denote by y_t the element of D defined by $y_t(\theta)=y(t+\theta),\ \ \theta\in[-r,0]$. $y_t(\cdot)$ represents the history of the state from time $t-r$, up to the present time t. In Section 4 we study the boundary value problem for the second order impulsive differential inclusion

$$y''\in F(t,y_t),\ a.e.\ t\in J=[0,T],\ t\neq t_k,\ \ k=1,\dots,m, \tag{5}$$

$$\Delta y|_{t=t_k}=I_k(y(t_k^-)),\ \ k=1,\dots,m, \tag{6}$$

$$\Delta y'|_{t=t_k}=\overline{I}_k(y(t_k^-)),\ \ k=1,\dots,m, \tag{7}$$

$$y(t)=\phi(t),\ \ t\in J_0=[-r,0],\ y(T)=y_T, \tag{8}$$

where $y_T\in E, F, I_k, \overline{I}_k$ and ϕ are as in problem (1)–(4). The study of impulsive functional differential equations is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena. That is why the perturbations are considered to take place "instantaneously" in the form of impulses. The theory of impulsive differential equations has seen considerable development; see the monographs of Bainov and Simeonov [1], Lakshmikantham, et al. [16], and Samoilenko and Perestyuk [17] where numerous properties of their solutions are studied, and detailed bibliographies are given. Recently, by means of the Leray-Schauder alternative for convex valued multivalued maps and a fixed point theorem due to Martelli for condensing multivalued maps existence, results of solutions for first and second order impulsive functional differential inclusions were given by Benchohra, et al. in [6], [7]. Different tools such as Martelli's fixed point theorem [2], the topological transversality theorem of Granas [12], the Leray-Schauder alternative [3], [4], [13], and the lower and upper solutions method [5], [8] have been used recently for various initial and boundary value problems for impulsive differential inclusions. However, in all the above works, the right hand side, $F(t,y_t)$, was assumed to be convex valued. Here we drop this restriction and consider problems with a nonconvex valued right-hand side. This paper will be divided into four sections. In Section 2 we will recall briefly some basic definitions and preliminary facts which will be used in the following sections. In Sections 3 and 4 we shall establish existence theorems for (1)–(4) and (5)–(8), repectively. Our method involves reducing the existence of solutions to problems (1)-(4) and (5)–(8) to a search for fixed points of suitable multivalued maps on appropriate Banach space. In order to prove the existence of fixed points, we shall rely on a fixed point theorem for contraction multivalued maps due to Covitz and Nadler [10] (see also Deimling [11]).

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$AC^i(J,E)$ is the space of i-times differentiable functions $y : J \to E$, whose i^{th} derivative, $y^{(i)}$, is absolutely continuous.
Let (X,d) be a metric space. We use the notations:
$P(X) = \{Y \subset X : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in P(X) : Y \text{ compact}\}$.

Consider $H_d : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$.

Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space.

Definition 2.1 *A multivalued operator $G : X \to P_{cl}(X)$ is called*

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(G(x), G(y)) \leq \gamma d(x,y), \quad \textit{for each } x,\ y \in X,$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

G has *a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. For more details on multivalued maps we refer to the books of Deimling [11], Gorniewicz [14], Hu and Papageorgiou [15] and Tolstogonov [18].

Our considerations are based on the following fixed point theorem for contraction multivalued operators given by Covitz and Nadler in 1970 [10] (see also Deimling, [11] Theorem 11.1).

Lemma 2.1 *Let (X,d) be a complete metric space. If $G : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

3 Initial Value Problem for Impulsive FDIs

In this section we give an existence result for the problem IVP (1)–(4). In order to define the solution of IVP (1)–(4) we shall consider the following space

$$\begin{aligned} PC &= \{y : [0,T] \to E : y_k \in C(J_k, E), k = 0, \ldots, m \text{ and there exist } y(t_k^-) \text{ and} \\ &\quad y(t_k^+), k = 1, \ldots, m \text{ with } y(t_k^-) = y(t_k)\}, \end{aligned}$$

which is a Banach space with the norm

$$\|y\|_{PC} = \max\{\|y_k\|_{J_k}, k = 0, \ldots, m\},$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}],\ k = 0, \ldots, m.$

Set $\Omega = D \cup PC$. Then Ω is a Banach space with norm

$$\|y\|_\Omega = \max\{\|y\|_D, \|y\|_{PC}\}, \text{ for each } y \in \Omega.$$

For any $y \in \Omega$ we define the set

$$S_{F,y} = \{v \in L^1(J, E) : \ v(t) \in F(t, y_t) \text{ for a.e. } \ t \in J\}.$$

Definition 3.1 *A function $y \in \Omega \cap \cup_{k=0}^m AC^1((t_k, t_{k+1}), E)$ is said to be a solution of (1)–(4) if y satisfies the differential inclusion $y''(t) \in F(t, y_t)$ a.e. on $J - \{t_1, \ldots, t_m\}$ and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad \Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-)),\ k = 1, \ldots, m,\ y(t) = \phi(t)$ for all $t \in [-r, 0]$ and $y(0) = \eta$.*

Let us introduce the following hypotheses:

(H1) $F : J \times D \longrightarrow P_{cp}(E)$; $(., u) \longmapsto F(., u)$ is mesurable for each $u \in D$;

(H2) There exists constants $c_k,\ d_k$ such that

$$|I_k(y) - I_k(\overline{y})| \le c_k |y - \overline{y}|, \text{ for each } k = 1, \ldots, m, \text{ and for all } y, \overline{y} \in E;$$

and

$$|\bar{I}_k(y) - \bar{I}_k(\overline{y})| \le d_k |y - \overline{y}|, \text{ for each } k = 1, \ldots, m, \text{ and for all } y, \overline{y} \in E;$$

(H3) There exists a function $l \in L^1(J, \mathbb{R}^+)$ such that

$$H_d(F(t, u), F(t, \overline{u})) \le l(t)\|u - \overline{u}\|, \text{ for a.e. } t \in [0, T] \text{ and all } u, \overline{u} \in D;$$

and $d(0, F(t, 0)) \le l(t)$ for a.e. $t \in [0, T]$.

Theorem 3.1 *Suppose that hypotheses (H1)–(H3) are satisfied. If*

$$T\|l\|_{L^1} + \sum_{k=1}^m [c_k + (T - t_k)d_k] < 1,$$

then the impulsive initial value problem (1)–(4) has at least one solution.

Proof. Transform the problem into a fixed point problem. Consider the multivalued map, $G : \Omega \longrightarrow P(\Omega)$ defined by:

$$G(y) = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & t \in J_0, \\ \phi(0) + t\eta + \displaystyle\int_0^t (t - s)g(s)ds & \\ + \displaystyle\sum_{0<t_k<t} [I_k(y(t_k)) + (t - t_k)\bar{I}_k(y(t_k))] & t \in J,\ g \in S_{F,y} \end{cases} \right\}$$

where

$$S_{F,y} = \Big\{g \in L^1(J, E) : g(t) \in F(t, y_t) \ \text{ for a.e. } \ t \in J\Big\},$$

Remark 3.1 *(i) For each $y \in \Omega$ the set $S_{F,y}$ is nonempty since by (H1) F has a measurable selection (see [9], Theorem III.6).*
(ii) Clearly the fixed points of G are solutions to (1)-(4).

We shall show that G satisfies the assumptions of Lemma 2.1. The proof will be given in two steps.

Step 1: *$G(y) \in P_{cl}(\Omega)$ for each $y \in \Omega$.*

Indeed, let $(y_n)_{n\geq 0} \in G(y)$ such that $y_n \longrightarrow \tilde{y}$ in Ω. Then $\tilde{y} \in \Omega$ and there exists $g_n \in S_{F,y}$ such that for each $t \in [0,T]$

$$y_n(t) = \phi(0) + t\eta + \int_0^t (t-s)g_n(s)ds + \sum_{0<t_k<t} [I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k))].$$

Since F has compact values and from the second part of (H3) we may pass to a subsequence if necessary to get that g_n converges to g in $L^1(J,E)$ and hence $g \in S_{F,y}$. Then for each $t \in [0,T]$

$$y_n(t) \longrightarrow \tilde{y}(t) = \phi(0) + t\eta + \int_0^t (t-s)g(s)ds + \sum_{0<t_k<t} [I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k))].$$

So $\tilde{y} \in G(y)$.

Step 2: *There exists $\gamma < 1$, such that $H_d(G(y), G(\overline{y})) \leq \gamma \|y - \overline{y}\|_\Omega$ for each $y, \overline{y} \in \Omega$.*

Let $y, \overline{y} \in \Omega$ and $h \in G(y)$. Then there exists $g(t) \in F(t, y_t)$ such that for each $t \in [0,T]$

$$h(t) = \phi(0) + t\eta + \int_0^t (t-s)g(s)ds + \sum_{0<t_k<t} [I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k))].$$

From (H3) it follows that

$$H_d(F(t,y_t), F(t,\overline{y}_t)) \leq l(t)|y_t - \overline{y}_t|.$$

Hence there is $w \in F(t, \overline{y}_t)$ such that

$$|g(t) - w| \leq l(t)|y_t - \overline{y}_t|, \quad t \in [0,T].$$

Consider $U : [0,T] \to P(E)$, given by

$$U(t) = \{w \in E : |g(t) - w| \leq l(t)|y_t - \overline{y}_t|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}_t)$ is measurable (see Proposition III.4 in [9]), there exists a function $\overline{g}(t)$, which is a measurable selection for V. So, $\overline{g}(t) \in F(t, \overline{y}_t)$ and

$$|g(t) - \overline{g}(t)| \leq l(t)|y - \overline{y}|, \quad \text{for each} \quad t \in [0,T].$$

Let us define for each $t \in [0,T]$

$$\overline{h}(t) = \phi(0) + t\eta + \int_0^t (t-s)\overline{g}(s)ds + \sum_{0<t_k<t} [I_k(\overline{y}(t_k)) + (t-t_k)\bar{I}_k(\overline{y}(t_k))].$$

Then we have

$$\begin{aligned}
|h(t) - \overline{h}(t)| &\leq \int_0^t T|g(s) - \overline{g}(s)|\, ds \\
&+ \sum_{0<t_k<t} |I_k(y(t_k)) - I_k(\overline{y}(t_k))| \\
&+ \sum_{0<t_k<t} (t-t_k)|\bar{I}_k(y(t_k)) - \bar{I}_k(\overline{y}(t_k))| \\
&\leq \int_0^T Tl(s)|y_s - \overline{y}_s|\, ds \\
&+ \sum_{k=1}^m [c_k|y(t_k) - \overline{y}(t_k)| + (T-t_k)d_k|y(t_k) - \overline{y}(t_k)|] \\
&\leq \|y - \overline{y}\|_\Omega \int_0^T Tl(t)ds \\
&+ \sum_{k=1}^m [c_k + (T-t_k)d_k]\|y - \overline{y}\|_\Omega.
\end{aligned}$$

Then

$$\|h - \overline{h}\|_\Omega \leq \left(T\|l\|_{L^1} + \sum_{k=1}^m [c_k + (T-t_k)d_k]\right) \|y - \overline{y}\|_\Omega.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(G(y), G(\overline{y})) \leq \left(T\|l\|_{L^1} + \sum_{k=1}^m [c_k + (T-t_k)d_k]\right) \|y - \overline{y}\|_\Omega.$$

So, G is a contraction and thus, by Lemma 2.1, N has a fixed point which is solution to the IVP (1)–(4).

4 Boundary Value Problem for Impulsive FDIs

In this section we study the bounday value problem BVP (5–(8).

Definition 4.1 *A function $y \in \Omega \cap \cup_{k=0}^m AC^1((t_k, t_{k+1}), E)$ is said to be a solution of (5)–(8) if y satisfies the differential inclusion $y''(t) \in F(t, y_t)$ a.e. on $J - \{t_1, \ldots, t_m\}$ and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-))$, $k = 1, \ldots, m$, $y(t) = \phi(t)$ for all $t \in [-r, 0]$ and $y(T) = y_T$.*

Theorem 4.1 *Suppose that hypotheses (H1)–(H3) are satisfied. If*

$$T\|l\|_{L^1} + 2\sum_{k=1}^m [c_k + (T-t_k)d_k] < 1,$$

then the impulsive boundary value problem (5)–(8) has at least one solution.

Proof. Transform the problem into a fixed point problem. Consider the multivalued map, $N : \Omega \longrightarrow P(\Omega)$ defined by:

$$N(y) = \left\{ h \in \Omega : h(t) = \left\{ \begin{array}{ll} \phi(t), & t \in J_0, \\ \frac{T-t}{T}\phi(0) + \frac{t}{T}y_T + \int_0^T H(t,s)g(s)ds & \\ + \sum_{0<t_k<t} \left[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k)) \right] & \\ -\frac{t}{T}\sum_{k=1}^{m} \left[I_k(y(t_k)) + (T-t_k)\bar{I}_k(y(t_k)) \right], \ g \in S_{F,y} & t \in J, \end{array} \right. \right\}$$

where

$$S_{F,y} = \left\{ g \in L^1(J,E) : g(t) \in F(t,y_t) \ \text{ for a.e. } \ t \in J \right\},$$

and

$$H(t,s) = \left\{ \begin{array}{ll} \frac{t}{T}(s-T), & 0 \le s \le t \le T, \\ \frac{s}{T}(t-T), & 0 \le t < s \le T. \end{array} \right.$$

Remark 4.1 *A standard argument shows that the fixed points of G are solutions to the BVP (4)-(7).*

As in Theorem 3.1 we can show that N has closed values. Here we repeat the proof that N is a contraction. i.e There exists $\gamma < 1$, such that

$$H_d(N(y), N(\overline{y})) \le \gamma \|y - \overline{y}\|_\Omega \text{ for each } y, \overline{y} \in \Omega.$$

Let $y, \overline{y} \in \Omega$ and $h \in G(y)$. Then there exists $g(t) \in F(t, y_t)$ such that for each $t \in [0,T]$

$$\begin{aligned} h(t) &= \frac{T-t}{T}\phi(0) + \frac{t}{T}y_T + \int_0^T H(t,s)g(s)ds \\ &\quad + \sum_{0<t_k<t} \left[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k)) \right] \\ &\quad -\frac{t}{T}\sum_{k=1}^{m} \left[I_k(y(t_k)) + (T-t_k)\bar{I}_k(y(t_k)) \right]. \end{aligned}$$

From (H3) it follows that

$$H_d(F(t,y_t), F(t,\overline{y}_t)) \le l(t)|y_t - \overline{y}_t|.$$

Hence there is $w \in F(t, \overline{y}_t)$ such that

$$|g(t) - w| \le l(t)|y_t - \overline{y}_t|, \quad t \in [0,T].$$

Consider $U : [0, T] \to P(E)$, given by

$$U(t) = \{w \in E : |g(t) - w| \le l(t)|y_t - \overline{y}_t|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}_t)$ is measurable (see Proposition III.4 in [9]), there exists a function $\overline{g}(t)$, which is a measurable selection for V. So, $\overline{g}(t) \in F(t, \overline{y}_t)$ and

$$|g(t) - \overline{g}(t)| \le l(t)|y - \overline{y}|, \quad \text{for each} \quad t \in [0, T].$$

Let us define for each $t \in [0, T]$

$$\begin{aligned}
\overline{h}(t) &= \frac{T-t}{T}\phi(0) + \frac{t}{T}y_T + \int_0^T H(t,s)\overline{g}(s)ds \\
&\quad + \sum_{0<t_k<t} \left[I_k(\overline{y}(t_k)) + (t - t_k)\bar{I}_k(\overline{y}(t_k))\right] \\
&\quad - \frac{t}{T}\sum_{k=1}^{m} \left[I_k(\overline{y}(t_k)) + (T - t_k)\bar{I}_k(\overline{y}(t_k))\right].
\end{aligned}$$

Then we have

$$\begin{aligned}
|h(t) - \overline{h}(t)| &\le \int_0^T |H(t,s)||g(s) - \overline{g}(s)|\, ds \\
&+ \sum_{0<t_k<t} |I_k(y(t_k)) - I_k(\overline{y}(t_k))| \\
&+ \sum_{0<t_k<t} (t - t_k)|\bar{I}_k(y(t_k)) - \bar{I}_k(\overline{y}(t_k))| \\
&+ \frac{t}{T}\sum_{k=1}^{m} |I_k(y(t_k)) - I_k(\overline{y}(t_k))| \\
&+ \frac{t}{T}\sum_{k=1}^{m} (T - t_k)|\bar{I}_k(y(t_k)) - \bar{I}_k(\overline{y}(t_k))| \\
&\le \int_0^T Tl(s)|y_s - \overline{y}_s|\, ds \\
&+ \sum_{k=1}^{m} [c_k|y(t_k) - \overline{y}(t_k)| + (T - t_k)d_k|y(t_k) - \overline{y}(t_k)|] \\
&+ \sum_{k=1}^{m} [c_k|y(t_k) - \overline{y}(t_k)| + (T - t_k)d_k|y(t_k) - \overline{y}(t_k)|] \\
&\le \|y - \overline{y}\|_\Omega \int_0^T Tl(t)ds \\
&+ \sum_{k=1}^{m} [c_k + (T - t_k)d_k]\|y - \overline{y}\|_\Omega \\
&+ \sum_{k=1}^{m} [c_k + (T - t_k) + d_k]\|y - \overline{y}\|_\Omega.
\end{aligned}$$

Then

$$\|h - \overline{h}\|_\Omega \le \left(T\|l\|_{L^1} + 2\sum_{k=1}^{m} [c_k + (T - t_k)d_k]\right)\|y - \overline{y}\|_\Omega.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y), N(\overline{y})) \leq \left(T\|l\|_{L^1} + 2\sum_{k=1}^{m}[c_k + (T - t_k)d_k] \right) \|y - \overline{y}\|_{\Omega}.$$

So, N is a contraction and thus, by Lemma 2.1, N has a fixed point which is solution to problem (5)–(8).

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