Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **93**, 1-13; **http://www.math.u-szeged.hu/ejqtde/**

On the Growth of Solutions of Some Higher Order Linear Differential Equations With Entire Coefficients

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Abstract. In this paper, we investigate the order and the hyper-order of solutions of the linear differential equation

$$f^{(k)} + \left(D_{k-1} + B_{k-1}e^{b_{k-1}z}\right) f^{(k-1)} + ... + \left(D_1 + B_1 e^{b_1 z}\right) f'$$

$$+ \left(D_0 + A_1 e^{a_1 z} + A_2 e^{a_2 z}\right) f = 0,$$

where $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$, $B_l(z)$ $(\not\equiv 0)$ $(l = 1, ..., k-1)$, D_m $(m = 0, ..., k-1)$ are entire functions with $\max\{\sigma(A_j), \sigma(B_l), \sigma(D_m)\} < 1$, a_1, a_2, b_l $(l = 1, ..., k-1)$ are complex numbers. Under some conditions, we prove that every solution $f(z) \not\equiv 0$ of the above equation is of infinite order and with hyper-order 1.

2010 *Mathematics Subject Classification*: 34M10, 30D35.
Key words: Linear differential equations, Entire solutions, Order of growth, Hyper-order.

1 Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [9], [14]). Let $\sigma(f)$ denote the order of growth of an entire function f and the hyper-order $\sigma_2(f)$ of f is defined by (see [10], [14])

$$\sigma_2(f) = \lim_{r\to+\infty}\sup \frac{\log\log T(r,f)}{\log r} = \lim_{r\to+\infty}\sup \frac{\log\log\log M(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f and $M(r,f) = \max_{|z|=r}|f(z)|$.

For the second order linear differential equation

$$f'' + e^{-z}f' + B(z)f = 0, \tag{1.1}$$

where $B(z)$ is an entire function, it is well-known that each solution f of the equation (1.1) is an entire function, and that if f_1, f_2 are two linearly independent solutions of (1.1), then by [4], there is at least one of f_1, f_2 of infinite order. Hence, "most" solutions of (1.1) will have infinite order. But the equation (1.1) with $B(z) = -(1+e^{-z})$ possesses a solution $f(z) = e^z$ of finite order.

A natural question arises: What conditions on $B(z)$ will guarantee that every solution $f \not\equiv 0$ of (1.1) has infinite order? Many authors, Frei [5], Ozawa [12], Amemiya-Ozawa [1] and Gundersen [6], Langley [11] have studied this problem. They proved that when $B(z)$ is a nonconstant polynomial or $B(z)$ is a transcendental entire function with order $\rho(B) \neq 1$, then every solution $f \not\equiv 0$ of (1.1) has infinite order. In [3], Chen has considered equation (1.1) and obtained different results concerning the growth of its solutions when $\rho(B) = 1$.

Recently in [13], Peng and Chen have investigated the order and the hyper-order of solutions of some second order linear differential equations and have proved the following result.

Theorem A ([13]) *Let $A_j(z)$ $(\not\equiv 0)$ $(j = 1,2)$ be entire functions with $\sigma(A_j) < 1$, a_1, a_2 be complex numbers such that $a_1a_2 \neq 0$, $a_1 \neq a_2$ (suppose that $|a_1| \leqslant |a_2|$). If $\arg a_1 \neq \pi$ or $a_1 < -1$, then every solution $f \not\equiv 0$ of the equation*

$$f'' + e^{-z}f' + (A_1e^{a_1z} + A_2e^{a_2z})f = 0$$

has infinite order and $\sigma_2(f) = 1$.

In this paper, we continue the research in this type of problems, the main purpose of this paper is to extend and improve the results of Theorem A to some higher order linear differential equations. In fact we will prove the following results.

Theorem 1.1 *Let* $A_j(z)\ (\not\equiv 0)\ (j=1,2)$, $B_l(z)\ (\not\equiv 0)\ (l=1,...,k-1)$, D_m $(m=0,...,k-1)$ *be entire functions with* $\max\{\sigma(A_j),\sigma(B_l),\sigma(D_m)\}<1$, $b_l\ (l=1,...,k-1)$ *be complex constants such that* (i) $\arg b_l=\arg a_1$ *and* $b_l=c_la_1\ (0<c_l<1)\ (l\in I_1)$ *and* (ii) b_l *is a real constant such that* $b_l\leqslant 0$ $(l\in I_2)$, *where* $I_1\neq\varnothing$, $I_2\neq\varnothing$, $I_1\cap I_2=\varnothing$, $I_1\cup I_2=\{1,2,...,k-1\}$, *and* a_1, a_2 *are complex numbers such that* $a_1a_2\neq 0$, $a_1\neq a_2$ *(suppose that* $|a_1|\leqslant|a_2|$*). If* $\arg a_1\neq\pi$ *or* a_1 *is a real number such that* $a_1<\frac{b}{1-c}$, *where* $c=\max\{c_l:l\in I_1\}$ *and* $b=\min\{b_l:l\in I_2\}$, *then every solution* $f\not\equiv 0$ *of the equation*

$$f^{(k)}+\left(D_{k-1}+B_{k-1}e^{b_{k-1}z}\right)f^{(k-1)}+...+\left(D_1+B_1e^{b_1z}\right)f'$$
$$+\left(D_0+A_1e^{a_1z}+A_2e^{a_2z}\right)f=0 \tag{1.2}$$

satisfies $\sigma(f)=+\infty$ *and* $\sigma_2(f)=1$.

Corollary 1.1 *Let* $A_j(z)\ (\not\equiv 0)\ (j=1,2)$, $B_l(z)\ (\not\equiv 0)\ (l=1,...,k-1)$, D_m $(m=0,...,k-1)$ *be entire functions with* $\max\{\sigma(A_j),\sigma(B_l),\sigma(D_m)\}<1$, $b_l\ (l=1,...,k-1)$ *be complex constants such that* $\arg b_l=\arg a_1$ *and* $b_l=c_la_1\ (0<c_l<1)\ (l=1,...,k-1)$, *and* a_1, a_2 *be complex numbers such that* $a_1a_2\neq 0$, $a_1\neq a_2$ *(suppose that* $|a_1|\leqslant|a_2|$*). If* $\arg a_1\neq\pi$ *or* a_1 *is a real number such that* $a_1<0$, *then every solution* $f\not\equiv 0$ *of equation* (1.2) *satisfies* $\sigma(f)=+\infty$ *and* $\sigma_2(f)=1$.

Corollary 1.2 *Let* $A_j(z)\ (\not\equiv 0)\ (j=1,2)$, $B_l(z)\ (\not\equiv 0)\ (l=1,...,k-1)$, D_m $(m=0,...,k-1)$ *be entire functions with* $\max\{\sigma(A_j),\sigma(B_l),\sigma(D_m)\}<1$, $b_l\ (l=1,...,k-1)$ *be real constants such that* $b_l\leqslant 0$, *and* a_1, a_2 *be complex numbers such that* $a_1a_2\neq 0$, $a_1\neq a_2$ *(suppose that* $|a_1|\leqslant|a_2|$*). If* $\arg a_1\neq\pi$ *or* a_1 *is a real number such that* $a_1<b$, *where* $b=\min\{b_l:l=1,...,k-1\}$, *then every solution* $f\not\equiv 0$ *of equation* (1.2) *satisfies* $\sigma(f)=+\infty$ *and* $\sigma_2(f)=1$.

2 Preliminary lemmas

To prove our theorem, we need the following lemmas.

Lemma 2.1 ([7]) *Let f be a transcendental meromorphic function with $\sigma(f)=\sigma<+\infty$, $H=\{(k_1,j_1),(k_2,j_2),...,(k_q,j_q)\}$ be a finite set of distinct pairs of integers satisfying $k_i>j_i\geqslant 0$ $(i=1,...,q)$ and let $\varepsilon>0$ be a given constant. Then,*
(i) *there exists a set $E_1\subset\left[-\frac{\pi}{2},\frac{3\pi}{2}\right)$ with linear measure zero, such that, if $\psi\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right)\setminus E_1$, then there is a constant $R_0=R_0(\psi)>1$, such that for all z satisfying $\arg z=\psi$ and $|z|\geqslant R_0$ and for all $(k,j)\in H$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leqslant|z|^{(k-j)(\sigma-1+\varepsilon)}, \tag{2.1}$$

(ii) *there exists a set $E_2\subset(1,+\infty)$ with finite logarithmic measure, such that for all z satisfying $|z|\notin E_2\cup[0,1]$ and for all $(k,j)\in H$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leqslant|z|^{(k-j)(\sigma-1+\varepsilon)}, \tag{2.2}$$

(iii) *there exists a set $E_3\subset(0,\infty)$ with finite linear measure, such that for all z satisfying $|z|\notin E_3$ and for all $(k,j)\in H$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leqslant|z|^{(k-j)(\sigma+\varepsilon)}. \tag{2.3}$$

Lemma 2.2 ([3]) *Suppose that $P(z)=(\alpha+i\beta)z^n+...$ $(\alpha,\beta$ are real numbers, $|\alpha|+|\beta|\neq 0)$ is a polynomial with degree $n\geqslant 1$, that $A(z)$ $(\not\equiv 0)$ is an entire function with $\sigma(A)<n$. Set $g(z)=A(z)e^{P(z)}$, $z=re^{i\theta}$, $\delta(P,\theta)=\alpha\cos n\theta-\beta\sin n\theta$. Then for any given $\varepsilon>0$, there is a set $E_4\subset[0,2\pi)$ that has linear measure zero, such that for any $\theta\in[0,2\pi)\diagdown(E_4\cup E_5)$, there is $R>0$, such that for $|z|=r>R$, we have*
(i) *if $\delta(P,\theta)>0$, then*

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\}\leqslant\left|g\left(re^{i\theta}\right)\right|\leqslant\exp\{(1+\varepsilon)\delta(P,\theta)r^n\}; \tag{2.4}$$

(ii) *if $\delta(P,\theta)<0$, then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\}\leqslant\left|g\left(re^{i\theta}\right)\right|\leqslant\exp\{(1-\varepsilon)\delta(P,\theta)r^n\}, \tag{2.5}$$

where $E_5 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ *is a finite set.*

Lemma 2.3 ([13]) *Suppose that* $n \geqslant 1$ *is a positive entire number. Let* $P_j(z) = a_{jn} z^n + ...$ $(j = 1, 2)$ *be nonconstant polynomials, where* a_{jq} $(q = 1, ..., n)$ *are complex numbers and* $a_{1n} a_{2n} \neq 0$*. Set* $z = re^{i\theta}$, $a_{jn} = |a_{jn}| e^{i\theta_j}$, $\theta_j \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\delta(P_j, \theta) = |a_{jn}| \cos(\theta_j + n\theta)$, *then there is a set* $E_6 \subset \left[-\frac{\pi}{2n}, \frac{3\pi}{2n}\right)$ *that has linear measure zero. If* $\theta_1 \neq \theta_2$*, then there exists a ray* $\arg z = \theta$, $\theta \in \left(-\frac{\pi}{2n}, \frac{\pi}{2n}\right) \setminus (E_6 \cup E_7)$*, such that*

$$\delta(P_1, \theta) > 0 \ , \ \delta(P_2, \theta) < 0 \tag{2.6}$$

or

$$\delta(P_1, \theta) < 0 \ , \ \delta(P_2, \theta) > 0, \tag{2.7}$$

where $E_7 = \left\{\theta \in \left[-\frac{\pi}{2n}, \frac{3\pi}{2n}\right) : \delta(P_j, \theta) = 0\right\}$ *is a finite set, which has linear measure zero.*

Remark 2.1 ([13]) In Lemma 2.3, if $\theta \in \left(-\frac{\pi}{2n}, \frac{\pi}{2n}\right) \setminus (E_6 \cup E_7)$ is replaced by $\theta \in \left(\frac{\pi}{2n}, \frac{3\pi}{2n}\right) \setminus (E_6 \cup E_7)$, then we obtain the same result.

Lemma 2.4 ([2]) *Suppose that* $k \geqslant 2$ *and* $B_0, B_1, ..., B_{k-1}$ *are entire functions of finite order and let* $\sigma = \max\{\sigma(B_j) : j = 0, ..., k-1\}$*. Then every solution* f *of the equation*

$$f^{(k)} + B_{k-1} f^{(k-1)} + ... + B_1 f' + B_0 f = 0 \tag{2.8}$$

satisfies $\sigma_2(f) \leqslant \sigma$.

Lemma 2.5 ([7]) *Let* $f(z)$ *be a transcendental meromorphic function, and let* $\alpha > 1$ *be a given constant. Then there exist a set* $E_8 \subset (1, \infty)$ *with finite logarithmic measure and a constant* $B > 0$ *that depends only on* α *and* i, j $(0 \leqslant i < j \leqslant k)$*, such that for all* z *satisfying* $|z| = r \notin [0, 1] \cup E_8$*, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leqslant B \left\{\frac{T(\alpha r, f)}{r} (\log^\alpha r) \log T(\alpha r, f)\right\}^{j-i}. \tag{2.9}$$

Lemma 2.6 ([8]) *Let* $\varphi : [0, +\infty) \to \mathbb{R}$ *and* $\psi : [0, +\infty) \to \mathbb{R}$ *be monotone non-decreasing functions such that* $\varphi(r) \leqslant \psi(r)$ *for all* $r \notin E_9 \cup [0, 1]$*, where* $E_9 \subset (1, +\infty)$ *is a set of finite logarithmic measure. Let* $\gamma > 1$ *be a given constant. Then there exists an* $r_1 = r_1(\gamma) > 0$ *such that* $\varphi(r) \leqslant \psi(\gamma r)$ *for all* $r > r_1$.

3 Proof of Theorem 1.1

Assume that $f\,(\not\equiv 0)$ is a solution of equation (1.2).

First step: We prove that $\sigma(f) = +\infty$. Suppose that $\sigma(f) = \sigma < +\infty$. Set $\max\{\sigma(A_j), \sigma(B_l), \sigma(D_m)\} = \beta < 1$ where $(j = 1, 2)$, $(l = 1, ..., k-1)$, $(m = 0, ..., k-1)$. Then, for any given ε $(0 < \varepsilon < 1-\beta)$ and for sufficiently large r, we have

$$|A_j(z)| \leqslant \exp\left\{r^{\beta+\varepsilon}\right\}, \ |B_l(z)| \leqslant \exp\left\{r^{\beta+\varepsilon}\right\}, \ |D_m(z)| \leqslant \exp\left\{r^{\beta+\varepsilon}\right\}. \tag{3.1}$$

By Lemma 2.1 (i), for the above ε, there exists a set $E_1 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ of linear measure zero, such that if $\theta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus E_1$, then there is a constant $R_0 = R_0(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| = r \geqslant R_0$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant r^{j(\sigma-1+\varepsilon)} \ \ (j = 1, ..., k). \tag{3.2}$$

Let $z = re^{i\theta}$, $a_1 = |a_1| e^{i\theta_1}$, $a_2 = |a_2| e^{i\theta_2}$, $\theta_1, \theta_2 \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$. We know that $\delta(b_l z, \theta) = \delta(c_l a_1 z, \theta) = c_l \delta(a_1 z, \theta)$ $(l \in I_1)$.

Case 1: $\arg a_1 \neq \pi$, which is $\theta_1 \neq \pi$.
(i) Assume that $\theta_1 \neq \theta_2$. By Lemma 2.3, for any given ε $(0 < \varepsilon < \min\{\frac{|a_2|-|a_1|}{|a_2|+|a_1|}, 1-\beta, \frac{1-c}{2(1+c)}\})$, there is a ray $\arg z = \theta$ such that $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$ (where E_6 and E_7 are defined as in Lemma 2.3, $E_1 \cup E_6 \cup E_7$ is of the linear measure zero), and satisfying

$$\delta(a_1 z, \theta) > 0, \ \delta(a_2 z, \theta) < 0 \text{ or } \delta(a_1 z, \theta) < 0, \ \delta(a_2 z, \theta) > 0.$$

a) When $\delta(a_1 z, \theta) > 0$, $\delta(a_2 z, \theta) < 0$, for sufficiently large r, we get by Lemma 2.2

$$|A_1 e^{a_1 z}| \geqslant \exp\{(1-\varepsilon)\,\delta(a_1 z, \theta)\, r\}, \tag{3.3}$$

$$|A_2 e^{a_2 z}| \leqslant \exp\{(1-\varepsilon)\,\delta(a_2 z, \theta)\, r\} < 1. \tag{3.4}$$

By (3.3) and (3.4), we have

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geqslant |A_1 e^{a_1 z}| - |A_2 e^{a_2 z}|$$

$$\geqslant \exp\{(1-\varepsilon)\,\delta(a_1 z, \theta)\, r\} - 1$$

$$\geqslant (1-o(1))\exp\{(1-\varepsilon)\delta(a_1z,\theta)r\}. \tag{3.5}$$

By (1.2), we get

$$|A_1e^{a_1z}+A_2e^{a_2z}| \leqslant \left|\frac{f^{(k)}(z)}{f(z)}\right| + \left(|D_{k-1}|+\left|B_{k-1}(z)e^{b_{k-1}z}\right|\right)\left|\frac{f^{(k-1)}(z)}{f(z)}\right|$$

$$+ ... + \left(|D_1|+\left|B_1(z)e^{b_1z}\right|\right)\left|\frac{f'(z)}{f(z)}\right| + |D_0(z)|. \tag{3.6}$$

For $l \in I_1$, we have

$$\left|B_l(z)e^{b_lz}\right| \leqslant \exp\{(1+\varepsilon)c_l\delta(a_1z,\theta)r\} \leqslant \exp\{(1+\varepsilon)c\delta(a_1z,\theta)r\}. \tag{3.7}$$

For $l \in I_2$, we have

$$\left|B_l(z)e^{b_lz}\right| = |B_l(z)|\left|e^{b_lz}\right| \leqslant \exp\left\{r^{\beta+\varepsilon}\right\}e^{b_lr\cos\theta} \leqslant \exp\left\{r^{\beta+\varepsilon}\right\} \tag{3.8}$$

because $b_l \leqslant 0$ and $\cos\theta > 0$. Substituting (3.1), (3.2), (3.5), (3.7) and (3.8) into (3.6), we obtain

$$\begin{aligned}
&(1-o(1))\exp\{(1-\varepsilon)\delta(a_1z,\theta)r\} \\
&\leqslant r^{k(\sigma-1+\varepsilon)} + \left(\exp\left\{r^{\beta+\varepsilon}\right\} + \left|B_{k-1}(z)e^{b_{k-1}z}\right|\right)r^{(k-1)(\sigma-1+\varepsilon)} \\
&+ ... + \left(\exp\left\{r^{\beta+\varepsilon}\right\} + \left|B_1(z)e^{b_1z}\right|\right)r^{\sigma-1+\varepsilon} + \exp\left\{r^{\beta+\varepsilon}\right\} \\
&\leqslant M_0r^{k(\sigma-1+\varepsilon)}\exp\left\{r^{\beta+\varepsilon}\right\}\exp\{(1+\varepsilon)c\delta(a_1z,\theta)r\},
\end{aligned} \tag{3.9}$$

where $M_0 > 0$ is a some constant. From (3.9) and $0 < \varepsilon < \frac{1-c}{2(1+c)}$, we get

$$(1-o(1))\exp\left\{\frac{1-c}{2}\delta(a_1z,\theta)r\right\} \leqslant M_0r^{k(\sigma-1+\varepsilon)}\exp\left\{r^{\beta+\varepsilon}\right\}. \tag{3.10}$$

By $\delta(a_1z,\theta) > 0$ and $\beta+\varepsilon < 1$ we know that (3.10) is a contradiction.

b) When $\delta(a_1z,\theta) < 0$, $\delta(a_2z,\theta) > 0$, for sufficiently large r, we get by Lemma 2.2

$$|A_1e^{a_1z}| \leqslant \exp\{(1-\varepsilon)\delta(a_1z,\theta)r\} < 1, \tag{3.11}$$

$$|A_2e^{a_2z}| \geqslant \exp\{(1-\varepsilon)\delta(a_2z,\theta)r\}. \tag{3.12}$$

By (3.11) and (3.12), we have

$$|A_1e^{a_1z}+A_2e^{a_2z}| \geqslant (1-o(1))\exp\{(1-\varepsilon)\delta(a_2z,\theta)r\}. \tag{3.13}$$

For $l \in I_1$, we have

$$\left|B_l(z) e^{b_l z}\right| \leqslant \exp\{(1+\varepsilon) c_l \delta(a_1 z, \theta) r\} < 1. \tag{3.14}$$

Substituting (3.1) , (3.2) , (3.8) , (3.13) and (3.14) into (3.6), we obtain

$$(1 - o(1)) \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\} \leqslant M_0 r^{k(\sigma-1+\varepsilon)} \exp\left\{r^{\beta+\varepsilon}\right\}. \tag{3.15}$$

By $\delta(a_2 z, \theta) > 0$ and $\beta + \varepsilon < 1$ we know that (3.15) is a contradiction.

(ii) Assume that $\theta_1 = \theta_2$. By Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$ and $\delta(a_1 z, \theta) > 0$. Since $|a_1| \leqslant |a_2|$, $a_1 \neq a_2$ and $\theta_1 = \theta_2$, then $|a_1| < |a_2|$, thus $\delta(a_2 z, \theta) > \delta(a_1 z, \theta) > 0$. For sufficiently large r, we have by Lemma 2.2

$$|A_1 e^{a_1 z}| \leqslant \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\}, \tag{3.16}$$

$$|A_2 e^{a_2 z}| \geqslant \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\} \tag{3.17}$$

and (3.7) , (3.8) hold. By (3.16) and (3.17), we get

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geqslant |A_2 e^{a_2 z}| - |A_1 e^{a_1 z}|$$

$$\geqslant \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\} - \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\}$$
$$= \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\}[\exp\{\alpha r\} - 1], \tag{3.18}$$

where

$$\alpha = (1-\varepsilon)\delta(a_2 z, \theta) - (1+\varepsilon)\delta(a_1 z, \theta).$$

Since $0 < \varepsilon < \frac{|a_2|-|a_1|}{|a_2|+|a_1|}$, then

$$\alpha = (1-\varepsilon)|a_2|\cos(\theta_2 + \theta) - (1+\varepsilon)|a_1|\cos(\theta_1 + \theta)$$

$$= \cos(\theta_1 + \theta)[(1-\varepsilon)|a_2| - (1+\varepsilon)|a_1|]$$
$$= \cos(\theta_1 + \theta)[|a_2| - |a_1| - \varepsilon(|a_2| + |a_1|)] > 0.$$

Then, by $\alpha > 0$ and from (3.18), we get

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geqslant (1 - o(1)) \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\} \exp\{\alpha r\}. \tag{3.19}$$

Substituting (3.1), (3.2), (3.7), (3.8) and (3.19) into (3.6), we obtain

$$(1 - o(1)) \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\} \exp\{\alpha r\}$$

$$\leqslant M_1 r^{k(\sigma-1+\varepsilon)} \exp\left\{r^{\beta+\varepsilon}\right\} \exp\left\{(1+\varepsilon) c\delta(a_1 z, \theta) r\right\}, \tag{3.20}$$

where $M_1 > 0$ is a some constant. By (3.20), we have

$$(1-o(1)) \exp\left\{[(1+\varepsilon)(1-c)\delta(a_1 z, \theta) + \alpha] r\right\} \leqslant M_1 r^{k(\sigma-1+\varepsilon)} \exp\left\{r^{\beta+\varepsilon}\right\}. \tag{3.21}$$

By $\delta(a_1 z, \theta) > 0$, $\alpha > 0$ and $\beta + \varepsilon < 1$ we know that (3.21) is a contradiction.

Case 2: $a_1 < \frac{b}{1-c}$, which is $\theta_1 = \pi$.
(i) Assume that $\theta_1 \neq \theta_2$, then $\theta_2 \neq \pi$. By Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$ and $\delta(a_2 z, \theta) > 0$. Because $\cos\theta > 0$, we have $\delta(a_1 z, \theta) = |a_1| \cos(\theta_1 + \theta) = -|a_1| \cos\theta < 0$. For sufficiently large r, we obtain by Lemma 2.2

$$|A_1 e^{a_1 z}| \leqslant \exp\left\{(1-\varepsilon)\delta(a_1 z, \theta) r\right\} < 1, \tag{3.22}$$

$$|A_2 e^{a_2 z}| \geqslant \exp\left\{(1-\varepsilon)\delta(a_2 z, \theta) r\right\} \tag{3.23}$$

and (3.8), (3.14) hold. By (3.22) and (3.23), we obtain

$$\begin{aligned} |A_1 e^{a_1 z} + A_2 e^{a_2 z}| &\geqslant |A_2 e^{a_2 z}| - |A_1 e^{a_1 z}| \\ &\geqslant \exp\left\{(1-\varepsilon)\delta(a_2 z, \theta) r\right\} - 1 \\ &\geqslant (1-o(1)) \exp\left\{(1-\varepsilon)\delta(a_2 z, \theta) r\right\}. \end{aligned} \tag{3.24}$$

Using the same reasoning as in **Case 1**(i), we can get a contradiction.

(ii) Assume that $\theta_1 = \theta_2$, then $\theta_1 = \theta_2 = \pi$. By Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$, then $\cos\theta < 0$, $\delta(a_1 z, \theta) = |a_1| \cos(\theta_1 + \theta) = -|a_1| \cos\theta > 0$, $\delta(a_2 z, \theta) = |a_2| \cos(\theta_2 + \theta) = -|a_2| \cos\theta > 0$. Since $|a_1| \leqslant |a_2|$, $a_1 \neq a_2$ and $\theta_1 = \theta_2$, then $|a_1| < |a_2|$, thus $\delta(a_2 z, \theta) > \delta(a_1 z, \theta) > 0$. For sufficiently large r, we get (3.7), (3.16), (3.17) and (3.19) holds. For $l \in I_2$, we have

$$\begin{aligned} \left|B_l(z) e^{b_l z}\right| = |B_l(z)| \left|e^{b_l z}\right| &\leqslant \exp\left\{r^{\beta+\varepsilon}\right\} \exp\left\{b_l r \cos\theta\right\} \\ &\leqslant \exp\left\{r^{\beta+\varepsilon}\right\} \exp\left\{b r \cos\theta\right\} \end{aligned} \tag{3.25}$$

because $b_l \leqslant 0$, $b = \min\{b_l : l \in I_2\}$ and $\cos\theta < 0$. Substituting (3.1), (3.2), (3.7), (3.19) and (3.25) into (3.6), we obtain

$$(1-o(1)) \exp\left\{(1+\varepsilon)\delta(a_1 z, \theta) r\right\} \exp\left\{\alpha r\right\}$$

$$\leqslant M_2 r^{k(\sigma-1+\varepsilon)} \exp\left\{r^{\beta+\varepsilon}\right\} \exp\left\{(1+\varepsilon) c\delta\left(a_1 z, \theta\right) r\right\} \exp\left\{br\cos\theta\right\},$$

where $M_2 > 0$ is a some constant. Thus

$$(1 - o(1)) \exp\{\gamma r\} \leqslant M_2 r^{k(\sigma-1+\varepsilon)} \exp\left\{r^{\beta+\varepsilon}\right\}, \tag{3.26}$$

where $\gamma = (1+\varepsilon)(1-c)\delta(a_1 z, \theta) + \alpha - b\cos\theta$. Since $\alpha > 0$, $\cos\theta < 0$, $\delta(a_1 z, \theta) = -|a_1|\cos\theta$, $a_1 < \frac{b}{1-c}$ and $b \leqslant 0$, then

$$\begin{aligned}
\gamma &= -(1+\varepsilon)(1-c)|a_1|\cos\theta - b\cos\theta + \alpha \\
&= -\left[(1+\varepsilon)(1-c)|a_1| + b\right]\cos\theta + \alpha \\
&> -\left[(1+\varepsilon)(1-c)\frac{|b|}{1-c} + b\right]\cos\theta + \alpha \\
&= -\left[-(1+\varepsilon)b + b\right]\cos\theta + \alpha = \alpha + b\varepsilon\cos\theta > 0.
\end{aligned}$$

By $\beta + \varepsilon < 1$ and $\gamma > 0$, we know that (3.26) is a contradiction. Concluding the above proof, we obtain $\sigma(f) = +\infty$.

Second step: We prove that $\sigma_2(f) = 1$. By

$$\max\left\{\sigma\left(D_l + B_l e^{b_l z}\right) \;\; (l = 1, ..., k-1),\; \sigma\left(D_0 + A_1 e^{a_1 z} + A_2 e^{a_2 z}\right)\right\} = 1$$

and Lemma 2.4, we obtain $\sigma_2(f) \leqslant 1$. By Lemma 2.5, we know that there exists a set $E_8 \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0,1] \cup E_8$, we get

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant B\left[T(2r, f)\right]^{j+1} \;\; (j = 1, ..., k). \tag{3.27}$$

Case 1: $\arg a_1 \neq \pi$.
(i) $(\theta_1 \neq \theta_2)$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying

$$\delta(a_1 z, \theta) > 0,\; \delta(a_2 z, \theta) < 0 \text{ or } \delta(a_1 z, \theta) < 0,\; \delta(a_2 z, \theta) > 0.$$

a) When $\delta(a_1 z, \theta) > 0$, $\delta(a_2 z, \theta) < 0$, for sufficiently large r, we get (3.5) holds. Substituting (3.1), (3.5), (3.7), (3.8) and (3.27) into (3.6), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0,1] \cup E_8$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$

$$(1 - o(1)) \exp\left\{(1-\varepsilon)\delta(a_1 z, \theta) r\right\}$$

$$\leqslant B\left[T(2r,f)\right]^{k+1}+B\left[\exp\left\{r^{\beta+\varepsilon}\right\}+\left|B_{k-1}(z)\,e^{b_{k-1}z}\right|\right]\left[T(2r,f)\right]^{k}$$
$$+...+B\left[\exp\left\{r^{\beta+\varepsilon}\right\}+\left|B_{1}(z)\,e^{b_{1}z}\right|\right]\left[T(2r,f)\right]^{2}+\exp\left\{r^{\beta+\varepsilon}\right\}$$
$$\leqslant M_0\exp\left\{r^{\beta+\varepsilon}\right\}\exp\left\{(1+\varepsilon)\,c\delta\left(a_1z,\theta\right)r\right\}\left[T(2r,f)\right]^{k+1}, \tag{3.28}$$

where $M_0>0$ is a some constant. From (3.28) and $0<\varepsilon<\frac{1-c}{2(1+c)}$, we get

$$(1-o(1))\exp\left\{\frac{1-c}{2}\delta\left(a_1z,\theta\right)r\right\}\leqslant M_0\exp\left\{r^{\beta+\varepsilon}\right\}\left[T(2r,f)\right]^{k+1}. \tag{3.29}$$

Since $\delta(a_1z,\theta)>0$, $\beta+\varepsilon<1$, then by using Lemma 2.6 and (3.29), we obtain $\sigma_2(f)\geqslant 1$, hence $\sigma_2(f)=1$.
b) When $\delta(a_1z,\theta)<0$, $\delta(a_2z,\theta)>0$, for sufficiently large r, we get (3.13) holds. Substituting (3.1), (3.8) , (3.13), (3.14) and (3.27) into (3.6), we obtain for all $z=re^{i\theta}$ satisfying $|z|=r\notin[0,1]\cup E_8$, $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\setminus\left(E_1\cup E_6\cup E_7\right)$

$$(1-o(1))\exp\left\{(1-\varepsilon)\,\delta\left(a_2z,\theta\right)r\right\}\leqslant M_0\exp\left\{r^{\beta+\varepsilon}\right\}\left[T(2r,f)\right]^{k+1}, \tag{3.30}$$

where $M_0>0$ is a some constant. By $\delta(a_2z,\theta)>0$, $\beta+\varepsilon<1$ and (3.30), we have $\sigma_2(f)\geqslant 1$, then $\sigma_2(f)=1$.
(ii) $(\theta_1=\theta_2)$. In first step, we have proved that there is a ray $\arg z=\theta$ where $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\setminus\left(E_1\cup E_6\cup E_7\right)$, satisfying $\delta(a_2z,\theta)>\delta(a_1z,\theta)>0$ and for sufficiently large r, we get (3.19) holds. Substituting (3.1), (3.7) , (3.8), (3.19) and (3.27) into (3.6), we obtain for all $z=re^{i\theta}$ satisfying $|z|=r\notin[0,1]\cup E_8$, $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\setminus\left(E_1\cup E_6\cup E_7\right)$

$$(1-o(1))\exp\left\{(1+\varepsilon)\,\delta\left(a_1z,\theta\right)r\right\}\exp\left\{\alpha r\right\}$$
$$\leqslant M_1\exp\left\{r^{\beta+\varepsilon}\right\}\exp\left\{(1+\varepsilon)\,c\delta\left(a_1z,\theta\right)r\right\}\left[T(2r,f)\right]^{k+1}, \tag{3.31}$$

where $M_1>0$ is a some constant. By (3.31), we have

$$(1-o(1))\exp\left\{\left[(1+\varepsilon)\,(1-c)\,\delta\left(a_1z,\theta\right)+\alpha\right]r\right\}\leqslant M_1\exp\left\{r^{\beta+\varepsilon}\right\}\left[T(2r,f)\right]^{k+1}. \tag{3.32}$$

Since $\delta(a_1z,\theta)>0$, $\alpha>0$, $\beta+\varepsilon<1$, then by using Lemma 2.6 and (3.32), we obtain $\sigma_2(f)\geqslant 1$, hence $\sigma_2(f)=1$.
Case 2: $a_1<\frac{b}{1-c}$.
(i) $(\theta_1\neq\theta_2)$. In first step, we have proved that there is a ray $\arg z=\theta$ where $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\setminus\left(E_1\cup E_6\cup E_7\right)$, satisfying $\delta(a_2z,\theta)>0$ and $\delta(a_1z,\theta)<0$ and for sufficiently large r, we get (3.24) holds. Using the same reasoning as in second step (**Case 1** (i)), we can get $\sigma_2(f)=1$.

(ii) $(\theta_1 = \theta_2)$ In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying $\delta(a_2 z, \theta) > \delta(a_1 z, \theta) > 0$ and for sufficiently large r, we get (3.19) holds. Substituting (3.1), (3.7), (3.19), (3.25) and (3.27) into (3.6), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0,1] \cup E_8$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$

$$(1 - o(1)) \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\} \exp\{\alpha r\}$$

$$\leqslant M_2 \exp\left\{r^{\beta+\varepsilon}\right\} \exp\{(1+\varepsilon) c\delta(a_1 z, \theta) r\} \exp\{br\cos\theta\} [T(2r, f)]^{k+1},$$

where $M_2 > 0$ is a some constant. Thus

$$(1 - o(1)) \exp\{\gamma r\} \leqslant M_2 \exp\left\{r^{\beta+\varepsilon}\right\} [T(2r, f)]^{k+1}, \tag{3.33}$$

where $\gamma = (1+\varepsilon)(1-c)\delta(a_1 z, \theta) + \alpha - b\cos\theta$. Since $\gamma > 0$, $\beta + \varepsilon < 1$, then by using Lemma 2.6 and (3.33), we have $\sigma_2(f) \geqslant 1$, hence $\sigma_2(f) = 1$. Concluding the above proof, we obtain that every solution $f \not\equiv 0$ of (1.2) satisfies $\sigma_2(f) = 1$. The proof of Theorem 1.1 is complete.

4 Proofs of Corollary 1.1 and Corollary 1.2

Using the same reasoning as in the proof of Theorem 1.1, we can obtain Corollary 1.1 and Corollary 1.2.

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(Received August 15, 2011)