

# Regularity of Weak Solutions for Nonlinear Parabolic Problem with $p(x)$ -Growth

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## Abstract

In this paper, we study the nonlinear parabolic problem with  $p(x)$ -growth conditions in the space  $W^{1,x}L^{p(x)}(Q)$ , and give a regularity theorem of weak solutions for the following equation

$$\frac{\partial u}{\partial t} + A(u) = 0$$

where  $A(u) = -\operatorname{div}a(x, t, u, \nabla u) + a_0(x, t, u, \nabla u)$ ,  $a(x, t, u, \nabla u)$  and  $a_0(x, t, u, \nabla u)$  satisfy  $p(x)$ -growth conditions with respect to  $u$  and  $\nabla u$ .

**Keywords:** nonlinear parabolic problem, regularity,  $W^{1,x}L^{p(x)}(Q)$  space,  $p(x)$ -growth condition.

**Mathematics Subject Classification:** 35K15, 35K20.

## 1 Introduction

In recent years, the research of variational problems with nonstandard growth conditions is an interesting topic.  $p(x)$ -growth problems can be regarded as a kind of nonstandard growth problems and they appear in nonlinear elastic, electrorheological fluids and other physics phenomena. Many results have been obtained on this kind of problems, for examples [1-9].

In this paper, we will qualitatively study the properties of weak solutions. For more information about qualitative analysis, we refer to [10-11]. Let  $Q$  be  $\Omega \times (0, T)$  where  $T > 0$  is given. In [8], the authors studied the following equation in the space  $W_{loc}^{1,p(x,t)}(Q) \cap C(0, T; L_{loc}^2(\Omega))$ ,

$$u_t - \operatorname{div}(|Du|^{p(x,t)-2}Du) = 0,$$

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where  $\max\{1; \frac{2N}{N+2}\} < p_1 = \inf_{(x,t) \in Q} p(x,t) \leq p(x,t) \leq \sup_{(x,t) \in Q} p(x,t) = p_2 < \infty$ ,

$p(x,t)$  is dependent on the space variable  $x$  and the time variable  $t$ , and satisfies the following Logarithmic Hölder condition

$$|p(x,t) - p(y,s)| \leq \frac{C_1}{-\ln(|x-y| + C_2|t-s|^{p_2})}$$

for all  $(x,t), (y,s) \in Q$ ,  $|x-y| < \frac{1}{2}, |t-s| < \frac{1}{2}$ , where  $C_1, C_2 > 0$  are constants. The authors proved the Hölder continuity of the local weak solution with the scale transformation method. In this paper, we will study the following more general problem

$$\frac{\partial u}{\partial t} + A(u) = 0, \quad \text{in } Q, \tag{1.1}$$

$$u(x,t) = 0, \quad \text{on } \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = \psi(x), \quad \text{in } \Omega, \tag{1.3}$$

where  $\psi(x)$  is a given function in  $L^2(\Omega)$  and  $A : W_0^{1,x}L^{p(x)}(Q) \rightarrow W^{-1,x}L^{q(x)}(Q)$  is an elliptic operator of the form  $A(u) = -\text{div}a(x,t,u,\nabla u) + a_0(x,t,u,\nabla u)$  with the coefficients  $a$  and  $a_0$  satisfying the classical Leray-Lions conditions. In [12-13] we have proved the existence and the local boundedness of the solutions of (1.1)-(1.3) and have obtained  $u \in W^{1,x}L^{p(x)}(Q) \cap L^\infty(0,T;L^2(\Omega))$ . In this paper we will give the regularity theorem of the weak solutions in the framework space  $W^{1,x}L^{p(x)}(Q)$ , which can be considered as a special case of the space  $W^{1,p(x,t)}(Q)$ .

The space  $W^{1,x}L^{p(x)}(Q)$  provides a suitable framework to discuss some physical problems. In [14], the authors studied a functional with variable exponent,  $1 \leq p(x) \leq 2$ , which provided a model for image denoising, enhancement, and restoration. Because in [14] the direction and speed of diffusion at each location depended on the local behavior,  $p(x)$  only depended on the location  $x$  in the image. Consider that the space  $W^{1,x}L^{p(x)}(Q)$  was introduced and discussed in [12] and [15], we think that the space  $W^{1,x}L^{p(x)}(Q)$  is a reasonable framework to discuss the  $p(x)$ -growth problem (1.1)-(1.3), where  $p(x)$  only depends on the space variable  $x$  similar to [14].

In this paper, let  $a : Q \times R \times R^N \rightarrow R^N$  and  $a_0 : Q \times R \times R^N \rightarrow R$  be the operators such that for any  $s \in R$  and  $\xi \in R^N$ ,  $a(x,t,s,\xi)$  and  $a_0(x,t,s,\xi)$  are both continuous in  $(t,s,\xi)$  for a.e.  $x \in \Omega$  and measurable in  $x$  for all  $(t,s,\xi) \in (0,T) \times R \times R^n$ . They also satisfy that for a.e.  $(x,t) \in Q$ , any  $s \in R$  and  $\xi \neq \xi^* \in R^N$ :

$$|a(x,t,s,\xi)| \leq \alpha(|s|^{p(x)-1} + |\xi|^{p(x)-1}), \tag{1.4}$$

$$|a_0(x,t,s,\xi)| \leq \alpha(|s|^{p(x)-1} + |\xi|^{p(x)-1}), \tag{1.5}$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi^*)](\xi - \xi^*) > 0, \tag{1.6}$$

$$a(x,t,s,\xi)\xi \geq \beta(|\xi|^{p(x)} + |s|^{p(x)}), \tag{1.7}$$

$$a_0(x,t,s,\xi)s \geq \beta(|\xi|^{p(x)} + |s|^{p(x)}), \tag{1.8}$$

where  $\alpha, \beta > 0$  are constants.

Throughout this paper, unless special statement, we always suppose that  $p(x)$  is Lipschitz continuous on  $\bar{\Omega}$ , and satisfies

$$1 < p^- = \inf_{\bar{\Omega}} p(x) \leq p(x) \leq \sup_{\bar{\Omega}} p(x) = p^+ < \infty. \quad (1.9)$$

Because  $p(x)$  is Lipschitz continuous, there exists a constant  $C > 0$  such that

$$\rho^{-(p^+ - p^-)} \leq C, \quad \forall Q_\rho \subset Q, \quad (1.10)$$

where  $Q_\rho = K_\rho \times (-\rho^{p^+}, 0)$ ,  $0 < \rho < 1$ ,  $K_\rho = \{x \in \Omega \mid \max_{1 \leq i \leq N} |x_i| < \rho\}$ ,  $p_\rho^+ = \sup_{K_\rho} p(x)$ ,  $p_\rho^- = \inf_{K_\rho} p(x)$ .

**Definition 1.1** A function  $u \in W^{1,x}L^{p(x)}(Q) \cap L^\infty(0, T; L^2(\Omega))$  is called a weak solution of (1.1)-(1.3) if

$$\begin{aligned} & - \int_Q u \frac{\partial \varphi}{\partial t} dx dt + \int_\Omega u \varphi dx \Big|_0^T \\ & + \int_Q [a(x, t, u, \nabla u) \nabla \varphi + a_0(x, t, u, \nabla u) \varphi] dx dt = 0 \end{aligned}$$

for all  $\varphi \in C^1(0, T; C_0^\infty(\Omega))$ .

**Definition 1.2** The functions  $u_n \in C(0, T; C_0^\infty(\Omega))$  are called the Galerkin solutions of (1.1)-(1.3) if

$$\int_{Q^t} \frac{\partial u_n}{\partial \tau} \varphi dx d\tau + \int_{Q^t} a(x, \tau, u_n, \nabla u_n) \varphi dx d\tau + \int_{Q^t} a_0(x, \tau, u_n, \nabla u_n) \varphi dx d\tau = 0 \quad (1.11)$$

for all  $\varphi \in C^1(0, T; C_0^\infty(\Omega))$  and  $Q^t = \Omega \times (0, t)$ ,  $t \in (0, T]$ .

We will prove the following regularity theorem:

**Theorem 1** Let  $p^- > 2$ . If  $u \in W^{1,x}L^{p(x)}(Q) \cap L^\infty(0, T; L^2(\Omega))$  is a local weak solution of (1.1)-(1.3), then  $u$  is local Hölder continuous in  $Q$ .

## 2 Preliminaries

We first recall some facts on spaces  $L^{p(x)}(\Omega)$ ,  $W^{m,p(x)}(\Omega)$ ,  $W^{m,x}L^{p(x)}(Q)$  and parabolic space. For the details see [15-18].

Although we assume (1.9) holds in this paper, in this section we introduce the general spaces  $L^{p(x)}(\Omega)$ ,  $W^{m,p(x)}(\Omega)$  and  $W^{m,x}L^{p(x)}(Q)$ .

Denote

$$E = \{\omega : \omega \text{ is a measurable function on } \Omega\},$$

where  $\Omega \subset R^N$  is an open subset.

Let  $p(x) : \Omega \rightarrow [1, \infty]$  be an element in  $E$ . Denote  $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$ . For  $u \in E$ , we define

$$\rho(u) = \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} dx + \operatorname{ess\,sup}_{x \in \Omega_\infty} |u(x)|.$$

The space  $L^{p(x)}(\Omega)$  is

$$L^{p(x)}(\Omega) = \{u \in E : \exists \lambda > 0, \rho(\lambda u) < \infty\}$$

endowed with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\{\lambda > 0 : \rho\left(\frac{u}{\lambda}\right) \leq 1\}.$$

We define the conjugate function  $q(x)$  of  $p(x)$  by

$$q(x) = \begin{cases} \infty, & \text{if } p(x) = 1; \\ 1, & \text{if } p(x) = \infty; \\ \frac{p(x)}{p(x)-1}, & \text{if } 1 < p(x) < \infty. \end{cases}$$

**Lemma 2.1** (see [18]) (1) The dual space of  $L^{p(x)}(\Omega)$  is  $L^{q(x)}(\Omega)$ , if  $1 \leq p(x) < \infty$ .

(2) The space  $L^{p(x)}(\Omega)$  is reflexive if and only if (1.9) is satisfied.

**Lemma 2.2** (see [18]) If  $1 \leq p(x) < \infty$ ,  $C_0^\infty(\Omega)$  is dense in the space  $L^{p(x)}(\Omega)$  and  $L^{p(x)}(\Omega)$  is separable.

**Lemma 2.3** (see [18]) Let  $1 \leq p(x) \leq \infty$ , for every  $u(x) \in L^{p(x)}(\Omega)$  and  $v(x) \in L^{q(x)}(\Omega)$ , we have

$$\int_{\Omega} |u(x)v(x)| dx \leq C \|u(x)\|_{L^{p(x)}(\Omega)} \|v(x)\|_{L^{q(x)}(\Omega)},$$

where  $C$  is only dependent on  $p(x)$  and  $\Omega$ , not dependent on  $u(x), v(x)$ .

Next let  $m > 0$  be an integer. For each  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ ,  $\alpha_i$  are nonnegative integers and  $|\alpha| = \sum_{i=1}^n \alpha_i$ , and denote by  $D^\alpha$  the distributional derivative of order  $\alpha$  with respect to the variable  $x$ .

We now introduce the generalized Lebesgue-Sobolev space  $W^{m,p(x)}(\Omega)$  which is defined as

$$W^{m,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq m\}.$$

$W^{m,p(x)}(\Omega)$  is a Banach space endowed with the norm

$$\|u\| = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^{p(x)}(\Omega)}.$$

The space  $W_0^{m,p(x)}(\Omega)$  is defined as the closure of  $C_0^\infty(\Omega)$  in  $W^{m,p(x)}(\Omega)$ . The dual space  $(W_0^{m,p(x)}(\Omega))^*$  is denoted by  $W^{-m,q(x)}(\Omega)$  equipped with the norm

$$\|f\|_{W^{-m,q(x)}(\Omega)} = \inf \sum_{|\alpha| \leq m} \|f_\alpha\|_{L^{q(x)}(\Omega)},$$

where infimum is taken on all possible decompositions

$$f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha, \quad f_\alpha \in L^{q(x)}(\Omega).$$

**Lemma 2.4**(see [18]) (1)  $W^{m,p(x)}(\Omega)$  and  $W_0^{m,p(x)}(\Omega)$  are separable if  $1 \leq p(x) < \infty$ .

(2)  $W^{m,p(x)}(\Omega)$  and  $W_0^{m,p(x)}(\Omega)$  are reflexive if (1.9) holds.

We define the space  $W^{m,x}L^{p(x)}(Q)$  as the following:

$$W^{m,x}L^{p(x)}(Q) = \{u \in L^{p(x)}(Q) : D^\alpha u \in L^{p(x)}(Q), |\alpha| \leq m\}.$$

$W^{m,x}L^{p(x)}(Q)$  is a Banach space with the norm  $\|u\| = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^{p(x)}(Q)}$ ,

where  $p(x)$  is independent of  $t$ .

The space  $W_0^{m,x}L^{p(x)}(Q)$  is defined as the closure of  $C_0^\infty(Q)$  in  $W^{m,x}L^{p(x)}(Q)$  and  $W_0^{m,x}L^{p(x)}(Q) \hookrightarrow L^{p(x)}(Q)$  is continuous embedding. Let  $\bar{M}$  be the number of multiindexes  $\alpha$  which satisfies  $0 \leq |\alpha| \leq m$ , then the space  $W_0^{m,x}L^{p(x)}(Q)$  can be considered as a close subspace of the product space  $\prod_{i=1}^{\bar{M}} L^{p(x)}(Q)$ . So if  $1 < p(x) < \infty$ ,  $\prod_{i=1}^{\bar{M}} L^{p(x)}(Q)$  is reflexive and further we can get that the space  $W_0^{m,x}L^{p(x)}(Q)$  is reflexive. The dual space  $(W_0^{m,x}L^{p(x)}(Q))^*$  is denoted by  $W^{-m,x}L^{q(x)}(Q)$  equipped with the norm

$$\|f\|_{W^{-m,x}L^{q(x)}(Q)} = \sup_{\|u\|_{W_0^{m,x}L^{p(x)}(Q)} \leq 1} |\langle f, u \rangle| = \inf \sum_{|\alpha| \leq m} \|f_\alpha\|_{L^{q(x)}(Q)},$$

where infimum is taken on all possible decompositions

$$f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D_x^\alpha f_\alpha, \quad f_\alpha \in L^{q(x)}(Q).$$

Next, we will introduce the parabolic space and some results in [16]:

**Definition 2.5** Let  $p, r \geq 1$ . A function  $f$  defined in  $Q$  belongs to the space  $L^r(0, T; L^p(\Omega))$ , if

$$\|f\|_{p,r,Q} = \left( \int_0^T \left( \int_\Omega |f|^p dx \right)^{\frac{r}{p}} dt \right)^{\frac{1}{r}} < \infty.$$

**Definition 2.6** Let  $p, r \geq 1$ . We define the function spaces

$$V^{r,p}(Q) = L^\infty(0, T; L^r(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)),$$

$$V_0^{r,p}(Q) = L^\infty(0, T; L^r(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega)),$$

which are both equipped with the norm

$$\|v\|_{V^{r,p}(Q)} = \operatorname{ess\,sup}_{0 < t < T} \|v(x, t)\|_{L^r(\Omega)} + \|\nabla v\|_{L^p(Q)}.$$

**Lemma 2.7** *let  $\{Y_n\}, n = 0, 1, 2, \dots$ , be a sequence of positive numbers, satisfying the inequalities  $Y_{n+1} \leq Cb^n Y_n^{1+\alpha}$ , where  $C, b > 1$  and  $\alpha > 0$  are given numbers. If  $Y_0 \leq C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}}$ , then  $\{Y_n\}$  converges to 0 as  $n \rightarrow \infty$ .*

**Lemma 2.8** *Let  $r > 1$ , there exists a constant  $C$  depending only on  $N, r$ , such that for every  $v \in L^\infty(0, T; L^r(\Omega)) \cap L^r(0, T; W_0^{1,r}(\Omega))$ ,*

$$\|v\|_{L^r(Q)}^r \leq C \| |v| > 0 \|_{V^{r,r}(Q)}^{\frac{r}{r+N}}$$

where  $\| |v| > 0 \| = \text{meas}\{(x, t) : |v| > 0\}$ .

**Lemma 2.9** *Let  $v \in W^{1,1}(K_\rho(x_0)) \cap C(K_\rho(x_0))$  for some  $\rho > 0$  and some  $x_0 \in R^N$ , and let  $k$  and  $h$  be any pair of real numbers such that  $k < h$ , then there exists a constant  $C$  depending only upon  $N$ , such that*

$$(h - k)|A(h)| \leq C \frac{\rho^{N+1}}{|K_\rho(x_0) \setminus A(k)|} \int_{A(k) \setminus A(h)} |\nabla v| dx$$

where  $A(k) = \{x \in K_\rho(x_0) : v(x) > k\}$ ,  $|A(k)| = \text{meas}A(k)$ .

Let  $u \in L^1(Q)$ . For any  $0 < h < T$ , we introduce the Steklov average function

$$u_h(x, t) = \begin{cases} \frac{1}{h} \int_t^{t+h} u(x, \tau) d\tau, & t \in (0, T - h], \\ 0, & t > T - h. \end{cases}$$

**Lemma 2.10** *Let  $u \in L^r(0, T; L^p(\Omega))$ , then as  $h \rightarrow 0$ ,  $u_h \rightarrow u$  in  $L^r(0, T - \varepsilon; L^p(\Omega))$  for every  $\varepsilon \in (0, T)$ . If  $u \in C(0, T; L^2(\Omega))$ , then as  $h \rightarrow 0$ ,  $u_h \rightarrow u$  in  $L^2(\Omega)$  for every  $t \in (0, T - \varepsilon)$ .*

Similarly, we can get the following lemma in variable exponent space.

**Lemma 2.11** *If  $u \in L^{p(x)}(Q)$ , then as  $h \rightarrow 0$ ,  $u_h \rightarrow u$  in  $L^{p(x)}(Q)$ .*

Proof: Because  $p(x)$  is bounded and independent of  $t$ . We only need to notice that there exist  $u_k \in C_0^1(Q)$  such that  $u_k \rightarrow u$  in  $L^{p(x)}(Q)$ , and by the uniform continuity of  $u_k$ , we can conclude the lemma.  $\square$

### 3 Regularity of Weak Solutions

In [12-13], we have obtained that for the Galerkin solution  $u_n \in C^1(0, T; C_0^\infty(\Omega))$ ,  $u_n \rightarrow u$  strongly in  $L^2(Q)$  and  $L^{p(x)}(Q)$ ,  $u_n \rightharpoonup u$  weakly in  $W_0^{1,x} L^{p(x)}(Q)$ ,  $a(x, t, u_n, \nabla u_n) \rightarrow a(x, t, u, \nabla u)$  and  $a_0(x, t, u_n, \nabla u_n) \rightarrow a_0(x, t, u, \nabla u)$  weakly in  $L^{q(x)}(Q)$ ,  $u_n \rightarrow u$  a.e. in  $Q$  and  $\nabla u_n \rightarrow \nabla u$  a.e. in  $Q$ .

For (1.11), integrating by parts, we can get

$$\int_{Q^t} \frac{\partial u_n}{\partial \tau} \varphi dx d\tau = \int_{\Omega} u_n(x, t) \varphi(x, t) dx - \int_{Q^t} u_n \frac{\partial \varphi}{\partial \tau} dx d\tau,$$

therefore

$$\lim_{n \rightarrow \infty} \int_{Q^t} \frac{\partial u_n}{\partial \tau} \varphi dx d\tau = \int_{\Omega} u(x, t) \varphi(x, t) dx - \int_{Q^t} u \frac{\partial \varphi}{\partial \tau} dx d\tau.$$

As  $a(x, t, u_n, \nabla u_n) \rightharpoonup a(x, t, u, \nabla u)$  weakly in  $L^{q(x)}(Q)$  and  $a_0(x, t, u_n, \nabla u_n) \rightharpoonup a_0(x, t, u, \nabla u)$  weakly in  $L^{q(x)}(Q)$ , we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} (\int_{Q^t} a(x, \tau, u_n, \nabla u_n) \varphi + \int_{Q^t} a_0(x, \tau, u_n, \nabla u_n) \varphi dx d\tau) \\ &= \int_{Q^t} a(x, \tau, u, \nabla u) \varphi + a_0(x, \tau, u, \nabla u) \varphi dx d\tau, \end{aligned}$$

then (1.11) can be written as

$$\begin{aligned} & \int_{\Omega} u(x, t) \varphi(x, t) dx - \int_{Q^t} u \frac{\partial \varphi}{\partial \tau} dx d\tau + \int_{Q^t} a(x, \tau, u, \nabla u) \varphi dx d\tau \\ & \quad + \int_{Q^t} a_0(x, \tau, u, \nabla u) \varphi dx d\tau = 0. \end{aligned} \tag{3.1}$$

In (3.1), let  $\varphi$  be independent of  $t$  and  $t = t + h$ , then we get

$$\int_{\Omega} \frac{\partial u_h(x, \tau)}{\partial \tau} \varphi dx + \int_{Q^t} [a(x, \tau, u, \nabla u)]_h \varphi dx d\tau + \int_{Q^t} [a_0(x, \tau, u, \nabla u)]_h \varphi dx d\tau = 0, \tag{3.2}$$

where  $\varphi \in C_0^\infty(\Omega)$ .

**Lemma 3.1** *If  $u$  is a weak solution of (1.1)-(1.3), then  $u \in C(0, T; L^2(\Omega))$ .*

Proof: Because  $u_n \rightharpoonup u$  weakly in  $W_0^{1,x} L^{p(x)}(Q)$ , there exists convex combination of  $u_n$ , denoted by  $v_n$ , such that  $v_n \rightarrow u$  strongly in  $W_0^{1,x} L^{p(x)}(Q)$  and  $v_n(x, 0) \rightarrow \psi(x)$  strongly in  $L^2(\Omega)$ . Take  $\varphi = u_n - v_m$  as the testing function in (1.11),

$$\begin{aligned} & \int_{Q^t} \frac{\partial u_n}{\partial \tau} (u_n - v_m) dx d\tau + \int_{Q^t} a(x, \tau, u_n, \nabla u_n) \nabla (u_n - v_m) dx d\tau \\ & \quad + \int_{Q^t} a_0(x, \tau, u_n, \nabla u_n) (u_n - v_m) dx d\tau = 0, \end{aligned}$$

then for the sufficient large  $m$ , we have

$$\begin{aligned} & \overline{\lim}_{n \rightarrow \infty} \int_{Q^t} \frac{\partial u_n}{\partial \tau} (u_n - v_m) dx d\tau \\ & \leq \int_{Q^t} a(x, \tau, u, \nabla u) \nabla (u - v_m) dx d\tau + \int_{Q^t} a_0(x, \tau, u, \nabla u) \nabla (u - v_m) dx d\tau \\ & \leq 2(\|a\|_{L^{q(x)}(Q)} + \|a_0\|_{L^{q(x)}(Q)}) \|\nabla (u - v_m)\|_{L^{p(x)}(Q)} \leq \varepsilon(m) \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \int_{Q^t} \frac{\partial v_m}{\partial \tau} (v_m - u_n) dx d\tau \leq \varepsilon(m)$$

where  $\varepsilon(m) \rightarrow 0$  as  $m \rightarrow 0$ .

In short,

$$\overline{\lim}_{n \rightarrow \infty} \int_{Q^t} \frac{\partial (u_n - v_m)}{\partial \tau} (u_n - v_m) dx d\tau \leq \varepsilon(m),$$

i.e.

$$\overline{\lim}_{n \rightarrow \infty} \int_{\Omega} |u_n - v_m|^2 dx \leq \varepsilon(m).$$

therefore, for  $k > m$ , we get

$$\max_{0 < t < T} \|v_k - v_m\| \leq \max_{0 < t < T} [\overline{\lim}_{n \rightarrow \infty} (\|v_k - u_n\| + \|u_n - v_m\|)] \leq \varepsilon(k) + \varepsilon(m),$$

namely  $\{v_n\}$  is a Cauchy sequence in  $C(0, T; L^2(\Omega))$ , so we get the result.  $\square$

Next, we will prove the main theorem.

By [13], we know that there exists a constant  $M > 0$ , such that  $\|u\|_{L^\infty(Q)} \leq M$ . Fix a point  $(x_0, t_0)$  in  $Q$ , let  $\rho \in (0, 1)$  be small enough such that

$$Q(\rho^{p_\rho^+ - \varepsilon}, 2\rho) = K_{2\rho}(x_0) \times (t_0 - \rho^{p_\rho^+ - \varepsilon}, t_0) \subset Q,$$

where  $K_{2\rho}(x_0) = \{x \in \Omega \mid \max_{1 \leq i \leq N} |x_i - x_{0,i}| < 2\rho\}$ ,  $p_\rho^+ = \sup_{K_{2\rho}(x_0)} p(x)$ ,  $p_\rho^- = \inf_{K_{2\rho}(x_0)} p(x)$ .

$$\text{Denote } \mu^+ = \text{ess sup}_{Q(\rho^{p_\rho^+ - \varepsilon}, 2\rho)} u, \mu^- = \text{ess inf}_{Q(\rho^{p_\rho^+ - \varepsilon}, 2\rho)} u, \omega = \text{ess osc}_{Q(\rho^{p_\rho^+ - \varepsilon}, 2\rho)} u = \mu^+ - \mu^-.$$

Consider the cylinder  $Q(a\rho^{p_\rho^+}, \rho)$ ,  $\frac{1}{a} = (\frac{\omega}{A})^{p_\rho^- - 2}$ , where  $A > 2$  is a constant to be determined later. We assume that

$$\left(\frac{\omega}{A}\right)^{p_\rho^- - 2} > \rho^\varepsilon, \tag{3.3}$$

where  $\varepsilon \in (0, 1)$  will be determined later. This implies the inclusion

$$Q(a\rho^{p_\rho^+}, \rho) \subset Q(\rho^{p_\rho^+ - \varepsilon}, 2\rho)$$

and

$$\text{ess osc}_{Q(a\rho^{p_\rho^+}, \rho)} u \leq \omega.$$

If (3.3) is not hold,  $\omega \leq A\rho^{\frac{\varepsilon}{p_\rho^- - 2}}$ . Take  $C = A$ , then the first iterative of proposition 3.4 is hold, so the proposition 3.4 is right. therefore we also assume that (3.3) is hold in the following proof.

Let  $[(0, t^*) + Q(l\rho^{p_\rho^+}, \rho)] = \{x \in \Omega \mid \max_{1 \leq i \leq N} |x_i| < \rho\} \times [t^* - l\rho^{p_\rho^+}, t^*]$ ,  $\frac{1}{l} = (\frac{\omega}{2})^{p_\rho^- - 2}$ . For  $[(0, t^*) + Q(l\rho^{p_\rho^+}, \rho)] \subset Q(a\rho^{p_\rho^+}, \rho)$ ,  $-(A^{p_\rho^- - 2} - 2^{p_\rho^- - 2})\rho^{p_\rho^+} \omega^{2 - p_\rho^-} < t^* < 0$ . We assume  $(x_0, t_0) = (0, 0)$  and define  $(u - k)_\pm = \max\{\pm(u - k), 0\}$ .

**Lemma 3.2** *There exists a number  $\sigma \in (0, 1)$  independent of  $\omega, \rho$  such that if (3.3) and*

$$|(x, t) \in [(0, t^*) + Q(l\rho^{p_\rho^+}, \rho)] : u < \mu^- + \frac{\omega}{2}| \leq \sigma |Q(l\rho^{p_\rho^+}, \rho)| \tag{3.4}$$

hold, then  $u > \mu^- + \frac{\omega}{4}$ , a.e.  $(x, t) \in [(0, t^*) + Q(l(\frac{\rho}{2})^{p_\rho^+}, \frac{\rho}{2})]$ .



Proof: Up to a translation we may assume that  $(0, t^*) = (0, 0)$ . Let  $\rho_m = \frac{\rho}{2} + \frac{\rho}{2^{m+1}}, k_m = \mu^- + \frac{\omega}{4} + \frac{\omega}{2^{m+2}}, Q_{\rho_m} = K_{\rho_m} \times (-l\rho_m^{p_\rho^+}, 0), m = 0, 1, 2, \dots$ . We choose smooth cutoff function  $\eta_m = \xi_1(x)\xi_2(t)$ , where  $0 \leq \xi_1 \leq 1, 0 \leq \xi_2 \leq 1$  and

$$\xi_1 = 1, \text{ if } x \in K_{\rho_{m+1}}; \quad \xi_1 = 0, \text{ if } x \in \bar{K}_{\rho_{m+1}}; \quad \text{and } |\nabla \xi_1| \leq \frac{1}{\rho_m - \rho_{m+1}}.$$

$$\xi_2 = 1, \text{ if } t \geq -l\rho_{m+1}^{p_\rho^+}; \quad \xi_2 = 0, \text{ if } t \leq -l\rho_m^{p_\rho^+}; \quad \text{and } 0 \leq \frac{\partial \xi_2}{\partial t} \leq \frac{1}{l(\rho_m^{p_\rho^+} - \rho_{m+1}^{p_\rho^+})}.$$

Take  $\varphi = -(u_n - k_m)_- \eta_m^{p_\rho^+}$  as the testing function in (1.11), then

$$\begin{aligned} \int_{Q_m^t} \frac{\partial u_n}{\partial \tau} [-(u_n - k_m)_- \eta_m^{p_\rho^+}] dx d\tau + \int_{Q_m^t} a(x, \tau, u_n, \nabla u_n) [-\nabla((u_n - k_m)_- \eta_m^{p_\rho^+})] dx d\tau \\ + \int_{Q_m^t} a_0(x, \tau, u_n, \nabla u_n) [-(u_n - k_m)_- \eta_m^{p_\rho^+}] dx d\tau = 0, \end{aligned}$$

where  $Q_m^t = K_{\rho_m} \times (-l\rho_m^{p_\rho^+}, t), t \in (-l\rho_m^{p_\rho^+}, 0)$ .

First, integrating by parts,

$$\begin{aligned} \int_{Q_m^t} \frac{\partial u_n}{\partial \tau} [-(u_n - k_m)_- \eta_m^{p_\rho^+}] dx d\tau \\ = \frac{1}{2} \int_{Q_m^t} \frac{\partial [(u_n - k_m)_- \eta_m^{p_\rho^+}]}{\partial \tau} dx - \frac{p_\rho^+}{2} \int_{Q_m^t} (u_n - k_m)_-^2 \eta_m^{p_\rho^+ - 1} \frac{\partial \eta_m}{\partial \tau} dx d\tau \\ = \frac{1}{2} \int_{K_{\rho_m}} (u_n - k_m)_- \eta_m^{p_\rho^+}(x, t) dx - \frac{1}{2} \int_{K_{\rho_m}} (u_n - k_m)_- \eta_m^{p_\rho^+}(x, -l\rho_m^{p_\rho^+}) dx \\ - \frac{p_\rho^+}{2} \int_{Q_m^t} [(u_n - k_m)_-^2 \eta_m^{p_\rho^+ - 1} \frac{\partial \eta_m}{\partial \tau}] dx d\tau. \end{aligned}$$

Since  $u_n \rightarrow u$  in  $L^2(Q)$  and  $u \in C(0, T; L^2(\Omega))$ ,  $u_n \rightarrow u$  in  $L^2(\Omega)$  for  $\forall t \in (0, T)$ , therefore we can get

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{Q_m^t} \frac{\partial u_n}{\partial \tau} [-(u_n - k_m)_- \eta_m^{p_\rho^+}] dx d\tau \\ = \frac{1}{2} \int_{K_{\rho_m}} (u - k_m)_- \eta_m^{p_\rho^+}(x, t) dx - \frac{1}{2} \int_{K_{\rho_m}} (u - k_m)_- \eta_m^{p_\rho^+}(x, -l\rho_m^{p_\rho^+}) dx \\ - \frac{p_\rho^+}{2} \int_{Q_m^t} [(u - k_m)_-^2 \eta_m^{p_\rho^+ - 1} \frac{\partial \eta_m}{\partial \tau}] dx d\tau. \end{aligned}$$

Since  $\nabla(u_n - k_m)_- \rightarrow \nabla(u - k_m)_-$  and  $a(x, t, u_n, \nabla u_n) \rightarrow a(x, t, u, \nabla u)$  a.e. in  $Q_m^t$ , by Fatou lemma,

$$\begin{aligned} & \underline{\lim}_{n \rightarrow \infty} \int_{Q_m^t} a(x, \tau, u_n, \nabla u_n) [-\nabla(u_n - k_m)_- \eta_m^{p_\rho^+}] dx d\tau \\ & \geq \int_{Q_m^t} a(x, \tau, u, \nabla u) [-\nabla(u - k_m)_- \eta_m^{p_\rho^+}] dx d\tau. \end{aligned}$$

By the fact that  $u_n \rightarrow u$  strongly in  $L^{p(x)}(Q)$ ,  $a(x, t, u_n, \nabla u_n) \rightharpoonup a(x, t, u, \nabla u)$  weakly and  $a_0(x, t, u_n, \nabla u_n) \rightharpoonup a_0(x, t, u, \nabla u)$  weakly in  $L^q(x)(Q)$ , we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{Q_m^t} a(x, \tau, u_n, \nabla u_n) [-(u_n - k_m)_- \eta_m^{p_\rho^+ - 1} \nabla \eta_m] dx d\tau \\ & = \int_{Q_m^t} a(x, \tau, u, \nabla u) [-(u - k_m)_- \eta_m^{p_\rho^+ - 1} \nabla \eta_m] dx d\tau, \end{aligned}$$

and

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{Q_m^t} a_0(x, \tau, u_n, \nabla u_n) [-(u_n - k_m)_- \eta_m^{p_\rho^+}] dx d\tau \\ & = \int_{Q_m^t} a_0(x, \tau, u, \nabla u) [-(u - k_m)_- \eta_m^{p_\rho^+}] dx d\tau. \end{aligned}$$

Let  $I = \underline{\lim}_{n \rightarrow \infty} (\int_{Q_m^t} a(x, \tau, u_n, \nabla u_n) \nabla \varphi dx d\tau + \int_{Q_m^t} a_0(x, \tau, u_n, \nabla u_n) \varphi dx d\tau)$ , so

$$\begin{aligned} I & \geq - \int_{Q_m^t} a(x, \tau, u, \nabla u) [\nabla(u - k_m)_- \eta_m^{p_\rho^+}] dx d\tau \\ & \quad - p_\rho^+ \int_{Q_m^t} a(x, \tau, u, \nabla u) [(u - k_m)_- \eta_m^{p_\rho^+ - 1} \nabla \eta_m] dx d\tau \\ & \quad - \int_{Q_m^t} a_0(x, \tau, u, \nabla u) [(u - k_m)_- \eta_m^{p_\rho^+}] dx d\tau. \end{aligned}$$

By (1.4)-(1.5), (1.7)-(1.8),  $\|u\|_{L_{loc}^\infty(Q_m)} \leq M$  and  $\|(u - k_m)_-\|_{L_{loc}^\infty(Q_m)} \leq \|(k_m - u)\|_{L_{loc}^\infty(Q_m)} \leq \frac{\omega}{2}$ , we have

$$\begin{aligned} I & \geq \beta \int_{Q_m^t} (|\nabla(u - k_m)|^{p(x)} + |u|^{p(x)}) \eta_m^{p_\rho^+} dx d\tau \\ & \quad - \alpha p^+ \int_{Q_m^t} (|\nabla(u - k_m)|^{p(x)-1} + |\nabla(u - k_m)|^{p(x)-1}) (u - k_m)_- \eta_m^{p_\rho^+ - 1} |\nabla \eta_m| dx d\tau \\ & \quad - \alpha \int_{Q_m^t} (|\nabla(u - k_m)|^{p(x)-1} + |u|^{p(x)-1}) (u - k_m)_- \eta_m^{p_\rho^+} dx d\tau \\ & \geq \frac{\beta}{2} \int_{Q_m^t} |\nabla(u - k_m)|^{p(x)} \eta_m^{p_\rho^+} dx d\tau - C 2^{mp_\rho^+} \rho^{-p_\rho^+} |A_m|, \end{aligned}$$

where  $A_m = \{(x, t) \in Q_{\rho_m} : u(x, t) < k_m\}$ ,  $C = C(M, p^+)$ .

So we can get the following inequality

$$\begin{aligned} & \sup_{-l\rho_m^+ < t < 0} \int_{K_{\rho_m}} (u - k_m)_-^2 \eta_m^{p^+} dx + \int_{Q_{\rho_m}} |\nabla(u - k_m)_-|^{p(x)} \eta_m^{p^+} dx dt \\ & \leq C 2^{mp^+} \rho^{-p^+} |A_m|, \end{aligned} \quad (3.5)$$

where  $C = C(M, p^+)$ .

On the other hand, we have

$$\int_{K_{\rho_m}} (u - k_m)_-^{p^-} \eta_m^{p^+} dx \leq \left(\frac{\omega}{2}\right)^{p^- - 2} \int_{K_{\rho_m}} (u - k_m)_-^2 \eta_m^{p^+} dx$$

and

$$\begin{aligned} \int_{Q_{\rho_m}} |\nabla(u - k_m)_-|^{p^-} \eta_m^{p^+} dx dt & \leq \int_{Q_{\rho_m}} |\nabla(u - k_m)_-|^{p(x)} \eta_m^{p^+} dx dt \\ & \quad + \int_{Q_{\rho_m}} \chi[(u - k_m)_- > 0] \eta_m^{p^+} dx dt, \end{aligned}$$

then by (3.5),

$$\begin{aligned} & \sup_{-l\rho_m^+ < t < 0} \int_{K_{\rho_m}} (u - k_m)_-^{p^-} \eta_m^{p^+} dx + \frac{1}{l} \int_{Q_{\rho_m}} |\nabla(u - k_m)_-|^{p^-} \eta_m^{p^+} dx dt \\ & \leq C 2^{mp^+} \rho^{-p^+} \frac{1}{l} |A_m|. \end{aligned}$$

Next, we introduce the change of time-variable  $z = l^{-1}t$  which transforms  $Q_{\rho_m}$  into  $\tilde{Q}_{\rho_m} = K_{\rho_m} \times (-\rho_m^+, 0)$ . Setting also  $v(x, t) = u(x, zl)$ ,  $\tilde{\eta}_m(x, z) = \eta_m(x, zl)$ ,  $|\tilde{A}_m| = \text{meas}\{(x, z) \in \tilde{Q}_{\rho_m} : v(x, z) < k_m\}$ , then

$$\begin{aligned} & \|(v - k_m)_- \tilde{\eta}_m^{p^+}\|_{V^{p^-, p^-}(\tilde{Q}_{\rho_m})}^{p^-} \\ & \leq C \left( \sup_{-\rho_m^+ < z < 0} \int_{K_{\rho_m}} (v - k_m)_-^{p^-} \tilde{\eta}_m^{p^+} dx + \int_{\tilde{Q}_{\rho_m}} |\nabla(v - k_m)_-|^{p^-} \tilde{\eta}_m^{p^+} dx dz \right. \\ & \quad \left. + \int_{\tilde{Q}_{\rho_m}} |(v - k_m)_- \nabla \tilde{\eta}_m^{p^+}|^{p^-} dx dz \right) \leq C 2^{mp^+} \rho^{-p^+} |\tilde{A}_m|. \end{aligned}$$

By lemma 2.8,

$$\begin{aligned} \frac{1}{2^{p^-(m+2)}} \left(\frac{\omega}{2}\right)^{p^-} |\tilde{A}_{m+1}| & = |k_m - k_{m+1}|^{p^-} |\tilde{A}_{m+1}| \\ & \leq \|(v - k_m)_-\|_{L^{p^-}(\tilde{Q}_{\rho_{m+1}})}^{p^-} \leq \|(v - k_m)_- \tilde{\eta}_m^{p^+}\|_{L^{p^-}(\tilde{Q}_{\rho_m})}^{p^-} \\ & \leq \|(v - k_m)_- \tilde{\eta}_m^{p^+}\|_{V^{p^-, p^-}(\tilde{Q}_{\rho_m})}^{p^-} |\tilde{A}_m|^{\frac{p^-}{p^- + N}} \\ & \leq C 2^{mp^+} \rho^{-p^+} |\tilde{A}_m|^{1 + \frac{p^-}{p^- + N}}. \end{aligned}$$

By (3.3), when  $A > 2$ , we choose  $\varepsilon \leq p^- - 2$ , then  $(\frac{\omega}{2})^{-p^-} \leq \rho^{-p^-}$ . Next, denote  $Y_m = \frac{|\tilde{A}_m|}{|\tilde{Q}_{\rho_m}|}$ , then by (1.10) we obtain

$$\begin{aligned}
Y_{m+1} &\leq \frac{C4^{mp^+} \rho^{-p^+} |A_m|^{1+\frac{p^-}{p^-+N}}}{|\tilde{Q}_{\rho_{m+1}}|} \\
&= \frac{C4^{mp^+} \rho^{-p^+} |\tilde{Q}_{\rho_m}|^{1+\frac{p^-}{p^-+N}}}{|\tilde{Q}_{\rho_{m+1}}|} Y_m^{1+\frac{p^-}{p^-+N}} \\
&\leq C4^{mp^+} Y_m^{1+\frac{p^-}{p^-+N}}.
\end{aligned}$$

By lemma 2.7, when  $m \rightarrow \infty$ ,  $Y_m \rightarrow 0$  if  $Y_0 \leq C^{-\frac{N+p^-}{p^-}} 4^{-p^+(\frac{N+p^-}{p^-})^2} \equiv \sigma$  which just satisfies the condition of this lemma, i.e.

$$Y_0 = \frac{|\{(x, t) \in Q(l\rho^{p^+}, \rho) : u < \mu^- + \frac{\omega}{2}\}|}{|Q(l\rho^{p^+}, \rho)|} \leq \sigma.$$

By the fact that  $\rho_m \searrow \frac{\rho}{2}$ ,  $k_m \searrow \mu^- + \frac{\omega}{4}$  and  $|A_m| \rightarrow 0$ , we can get

$$|\{(x, t) \in Q(l(\frac{\rho}{2})^{p^+}, \frac{\rho}{2}) : u(x, t) \leq \mu^- + \frac{\omega}{4}\}| = 0,$$

therefore  $u > \mu^- + \frac{\omega}{4}$ , a.e.  $(x, t) \in Q(l(\frac{\rho}{2})^{p^+}, \frac{\rho}{2})$ .

Let  $\theta = l(\frac{\rho}{2})^{p^+}$ , by lemma 3.2 and  $u \in C(0, T; L^2(\Omega))$ , we obtain  $u(x, -\theta) > \mu^- + \frac{\omega}{4}$  a.e.  $x \in K_{\frac{\rho}{2}}$ .  $\square$

**Lemma 3.3** *Let (3.3)-(3.4) hold, then for every number  $\sigma_1 \in (0, 1)$ , there exists a positive integer  $s$  such that*

$$|x \in K_{\frac{\rho}{4}} : u(x, t) < \mu^- + \frac{\omega}{2^s}| \leq \sigma_1 |K_{\frac{\rho}{4}}|, \quad \forall t \in (-\theta, 0).$$

Proof: Set  $\rho^* = 2^{-1}\rho$ , we will consider the problem in  $Q(\theta, \rho^*) = K_{\rho^*} \times (-\theta, 0)$ . Let  $k = \mu^- + \frac{\omega}{4}$ ,  $H_k^- = \text{ess sup}_{Q(\theta, \rho^*)} (u - k)_-$ , thus  $H_k^- \leq \frac{\omega}{4}$ . Then we take

$$\Psi(u) = \max\{0, \ln \frac{H_k^-}{H_k^- - (u - k)_- + \omega 2^{-(m+2)}}\} = \ln^+ \frac{H_k^-}{H_k^- - (u - k)_- + \omega 2^{-(m+2)}}.$$

By lemma 3.2, we know  $u(x, -\theta) > \mu^- + \frac{\omega}{4}$  a.e.  $x \in K_{\rho^*}$ , so  $(u - k)_- = 0$  a.e. in  $K_{\rho^*} \times \{-\theta\}$ , moreover  $\Psi(u(x, -\theta)) = 0$ , a.e.  $x \in K_{\rho^*}$ . Since  $\frac{\omega}{4} \geq H_k^- \geq (u - k)_-$ , we get  $\Psi(u) \leq \ln \frac{\frac{\omega}{4}}{\omega 2^{-(m+2)}} = m \ln 2$  and

$$\left| \frac{\partial \Psi(u)}{\partial u} \right| = \begin{cases} \frac{1}{H_k^- - (u - k)_- + \omega 2^{-(m+2)}}, & u < k - \omega 2^{-(m+2)}, \\ 0, & u \geq k - \omega 2^{-(m+2)}, \end{cases}$$

therefore when  $u < k - \omega 2^{-(m+2)}$ ,  $\frac{2}{\omega} \leq \left| \frac{\partial \Psi(u)}{\partial u} \right| \leq \frac{2^{(m+2)}}{\omega}$ .

Take  $\varphi = \frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h}$  as the testing function in (3.2), where  $\eta$  is the cutoff function independent of  $t$  and satisfies  $0 < \eta < 1$  in  $K_{\rho^*}$ ,  $\eta = 1$  in  $K_{2^{-1}\rho^*}$ , and  $|\nabla\eta| \leq 4\rho^{-1}$ , then

$$\begin{aligned} & \int_{Q^t(\theta, \rho^*)} \frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h} \frac{\partial u_h}{\partial \tau} dx d\tau \\ & + \int_{Q^t(\theta, \rho^*)} [a(x, \tau, u, \nabla u)]_h \nabla \left[ \frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h} \right] dx d\tau \\ & + \int_{Q^t(\theta, \rho^*)} [a_0(x, \tau, u, \nabla u)]_h \frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h} dx d\tau = 0, \end{aligned} \quad (3.6)$$

where  $Q^t(\theta, \rho^*) = K_{\rho^*} \times (-\theta, t)$ ,  $t \in (-\theta, 0)$ .

Integrating by parts,

$$\begin{aligned} & \int_{Q^t(\theta, \rho^*)} \frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h} \frac{\partial u_h}{\partial \tau} dx d\tau = \int_{Q^t(\theta, \rho^*)} \frac{\partial}{\partial t}[\Psi^2(u_h)]\eta^{p_\rho^+} dx d\tau \\ & = \int_{K_{\rho^*}} \Psi^2(u_h(x, t))\eta^{p_\rho^+} dx - \int_{K_{\rho^*}} \Psi^2(u_h(x, -\theta))\eta^{p_\rho^+} dx, \end{aligned}$$

by  $\Psi(u_h) \leq m \ln 2$ ,  $\Psi(u) \leq m \ln 2$ ,  $|\Psi^2(u_h) - \Psi^2(u)| \leq \frac{m2^{m+3} \ln 2}{\omega} |u_h - u|$ , and  $u_h \rightarrow u$  in  $L^2(K_{\rho^*})$  for  $\forall t \in (-\theta, 0)$ , so

$$\begin{aligned} & \int_{K_{\rho^*}} \Psi^2(u_h(x, t))\eta^{p_\rho^+} dx \rightarrow \int_{K_{\rho^*}} \Psi^2(u(x, t))\eta^{p_\rho^+} dx, \\ & \int_{K_{\rho^*}} \Psi^2(u_h(x, -\theta))\eta^{p_\rho^+} dx \rightarrow \int_{K_{\rho^*}} \Psi^2(u(x, -\theta))\eta^{p_\rho^+} dx, \end{aligned}$$

therefore we obtain

$$\begin{aligned} & \int_{Q^t(\theta, \rho^*)} \frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h} \frac{\partial u_h}{\partial \tau} dx d\tau \\ & \rightarrow \int_{K_{\rho^*}} \Psi^2(u(x, t))\eta^{p_\rho^+} dx - \int_{K_{\rho^*}} \Psi^2(u(x, -\theta))\eta^{p_\rho^+} dx, \end{aligned} \quad (3.7)$$

Denote  $\Psi'(u) = \frac{\partial(\Psi(u))}{\partial d}|_{d=u_h}$ . Since  $\frac{\partial^2}{\partial d^2}(\Psi^2(d))|_{d=u_h} = 2(1 + \Psi(u_h))\Psi'(u_h)^2$ , for the other parts of (3.6),

$$\begin{aligned} I & \equiv \int_{Q^t(\theta, \rho^*)} [a(x, \tau, u, \nabla u)]_h \nabla \left[ \frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h} \right] dx d\tau \\ & + \int_{Q^t(\theta, \rho^*)} [a_0(x, \tau, u, \nabla u)]_h \frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h} dx d\tau \\ & = 2 \int_{Q^t(\theta, \rho^*)} [a(x, \tau, u, \nabla u)]_h \nabla u_h (1 + \Psi(u_h))\Psi'(u_h)^2 \eta^{p_\rho^+} dx d\tau \\ & + 2 \int_{Q^t(\theta, \rho^*)} [a(x, \tau, u, \nabla u)]_h \Psi'(u_h)\Psi(u_h)\nabla \eta^{p_\rho^+} dx d\tau \\ & + 2 \int_{Q^t(\theta, \rho^*)} [a_0(x, \tau, u, \nabla u)]_h \Psi'(u_h)\Psi(u_h)\eta^{p_\rho^+} dx d\tau. \end{aligned}$$

Next, we consider the problem on the set  $\{(x, t) \in K_{\rho^*} \times (-\theta, 0) : u(x, t) < k - \omega 2^{-(m+2)}\}$ , thus  $\frac{\omega}{2} \leq |\Psi'(u)| \leq \frac{2^{(m+2)}}{\omega}$ . When  $h \rightarrow 0$ ,  $u_h \rightarrow u$  and  $(u_h - k)_- \rightarrow (u - k)_-$  a.e. in  $(x, t) \in Q(\theta, \rho^*)$ , so  $(1 + \Psi(u_h))\Psi'(u_h)^2 \rightarrow (1 + \Psi(u))\Psi'(u)^2$  a.e. in  $(x, t) \in Q(\theta, \rho^*)$ . Since

$$|(1 + \Psi(u_h))\Psi'(u_h)^2 - (1 + \Psi(u))\Psi'(u)^2|^{p(x)} \leq [2(1 + m \ln 2)(\frac{2^{m+2}}{\omega})^2]^{p^+}$$

and by Lebesgue's theorem, we get

$$(1 + \Psi(u_h))\Psi'(u_h)^2 \nabla u \rightarrow (1 + \Psi(u))\Psi'(u)^2 \nabla u$$

in  $L^{p(x)}(Q^t(\theta, \rho^*))$  for a.e.  $t \in (-\theta, 0)$ . Because  $[a(x, t, u, \nabla u)]_h \rightarrow a(x, t, u, \nabla u)$  in  $L^{p(x)}(Q^t(\theta, \rho^*))$ ,

$$\begin{aligned} & \int_{Q^t(\theta, \rho^*)} [a(x, \tau, u, \nabla u)]_h \nabla u_h (1 + \Psi(u_h))\Psi'(u_h)^2 \eta^{p^+} dx d\tau \\ & \rightarrow \int_{Q^t(\theta, \rho^*)} a(x, \tau, u, \nabla u) \nabla u (1 + \Psi(u))\Psi'(u)^2 \eta^{p^+} dx d\tau. \end{aligned}$$

In the same way

$$\begin{aligned} & \int_{Q^t(\theta, \rho^*)} [a(x, \tau, u, \nabla u)]_h \Psi'(u_h)\Psi(u_h)\nabla \eta^{p^+} dx d\tau \\ & \rightarrow \int_{Q^t(\theta, \rho^*)} a(x, \tau, u, \nabla u)\Psi'(u)\Psi(u)\nabla \eta^{p^+} dx d\tau, \end{aligned}$$

and

$$\begin{aligned} & \int_{Q^t(\theta, \rho^*)} [a_0(x, \tau, u, \nabla u)]_h \Psi'(u_h)\Psi(u_h)\eta^{p^+} dx d\tau \\ & \rightarrow \int_{Q^t(\theta, \rho^*)} a_0(x, \tau, u, \nabla u)\Psi'(u)\Psi(u)\eta^{p^+} dx d\tau, \end{aligned}$$

are both valid.

Combining these estimates, we have

$$\begin{aligned} \lim_{h \rightarrow 0} I &= 2 \int_{Q^t(\theta, \rho^*)} a(x, \tau, u, \nabla u) \nabla u (1 + \Psi(u))\Psi'(u)^2 \eta^{p^+} dx d\tau \\ &+ 2 \int_{Q^t(\theta, \rho^*)} a(x, \tau, u, \nabla u)\Psi'(u)\Psi(u)\nabla \eta^{p^+} dx d\tau \\ &+ 2 \int_{Q^t(\theta, \rho^*)} a_0(x, \tau, u, \nabla u)\Psi'(u)\Psi(u)\eta^{p^+} dx d\tau. \end{aligned} \quad (3.8)$$

With (1.4)-(1.5), (1.7)-(1.8), we can get

$$\begin{aligned} \lim_{h \rightarrow 0} I &\geq 2\beta \int_{Q^t(\theta, \rho^*)} (|\nabla u|^{p(x)} + |u|^{p(x)})(1 + \Psi(u))|\Psi'(u)|^2 \eta^{p^+} dx d\tau \\ &- 2\alpha \int_{Q^t(\theta, \rho^*)} (|\nabla u|^{p(x)-1} + |u|^{p(x)-1})|\Psi'(u)|\Psi(u)\nabla \eta^{p^+} dx d\tau \\ &- 2\alpha \int_{Q^t(\theta, \rho^*)} (|\nabla u|^{p(x)-1} + |u|^{p(x)-1})|\Psi'(u)|\Psi(u)\eta^{p^+} dx d\tau. \end{aligned} \quad (3.9)$$

Since  $\frac{(p^+ - 1)p(x)}{p(x) - 1} > \frac{p^+(p(x) - 1)}{p(x) - 1}$ , by Young's inequality,

$$\begin{aligned} & \int_{Q^t(\theta, \rho^*)} |\nabla u|^{p(x)-1} |\Psi'(u)|\Psi(u)\eta^{p^+ - 1} |\nabla \eta| dx d\tau \\ & \leq \varepsilon \int_{Q^t(\theta, \rho^*)} |\nabla u|^{p(x)} (\Psi'(u))^2 (\Psi(u) + 1) \eta^{p^+} dx d\tau \\ & + C(\varepsilon) \int_{Q^t(\theta, \rho^*)} (\Psi'(u))^{2-p(x)} \Psi(u) |\nabla \eta|^{p(x)} dx d\tau. \end{aligned} \quad (3.10)$$

In the same way, we have

$$\begin{aligned}
 & \int_{Q^t(\theta, \rho^*)} |u|^{p(x)-1} \Psi'(u) \Psi(u) \eta^{p_\rho^+ - 1} |\nabla \eta| dx d\tau \\
 \leq & \varepsilon \int_{Q^t(\theta, \rho^*)} |u|^{p(x)} (\Psi'(u))^2 (\Psi(u) + 1) \eta^{p_\rho^+} dx d\tau \\
 & + C(\varepsilon) \int_{Q^t(\theta, \rho^*)} (\Psi'(u))^{2-p(x)} \Psi(u) |\nabla \eta|^{p(x)} dx d\tau, \\
 & \int_{Q^t(\theta, \rho^*)} |\nabla u|^{p(x)-1} \Psi'(u) \Psi(u) \eta^{p_\rho^+} dx d\tau \\
 \leq & \varepsilon \int_{Q^t(\theta, \rho^*)} |\nabla u|^{p(x)} (\Psi'(u))^2 (\Psi(u) + 1) \eta^{p_\rho^+} dx d\tau \\
 & + C(\varepsilon) \int_{Q^t(\theta, \rho^*)} (\Psi'(u))^{2-p(x)} \Psi(u) \eta^{p_\rho^+} dx d\tau,
 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
 & \int_{Q^t(\theta, \rho^*)} |u|^{p(x)-1} \Psi'(u) \Psi(u) \eta^{p_\rho^+} dx d\tau \\
 \leq & \varepsilon \int_{Q^t(\theta, \rho^*)} |u|^{p(x)} (\Psi'(u))^2 (\Psi(u) + 1) \eta^{p_\rho^+} dx d\tau \\
 & + C(\varepsilon) \int_{Q^t(\theta, \rho^*)} (\Psi'(u))^{2-p(x)} \Psi(u) \eta^{p_\rho^+} dx d\tau.
 \end{aligned}$$

Combining (3.8)-(3.11),

$$\begin{aligned}
 \lim_{h \rightarrow 0} I \geq & (2\beta - 4\alpha p^+ \varepsilon) \int_{Q^t(\theta, \rho^*)} (|\nabla u|^{p(x)} + |u|^{p(x)})(1 + \Psi(u)) \Psi'(u)^2 \eta^{p_\rho^+} dx d\tau \\
 & - C(\varepsilon) \int_{Q^t(\theta, \rho^*)} (\Psi'(u))^{2-p(x)} \Psi(u) (\eta^{p_\rho^+} + |\nabla \eta|^{p(x)}) dx d\tau.
 \end{aligned}$$

Take  $4\alpha p^+ \varepsilon = \beta$ , then

$$\begin{aligned}
 \lim_{h \rightarrow 0} I \geq & \beta \int_{Q^t(\theta, \rho^*)} (|\nabla u|^{p(x)} + |u|^{p(x)})(1 + \Psi(u)) \Psi'(u)^2 \eta^{p_\rho^+} dx d\tau \\
 & - C(p^+) \int_{Q^t(\theta, \rho^*)} (\Psi'(u))^{2-p(x)} \Psi(u) (\eta^{p_\rho^+} + |\nabla \eta|^{p(x)}) dx d\tau.
 \end{aligned} \tag{3.12}$$

In view of (3.7) and (3.12),

$$\int_{K_{\rho^*}} \Psi^2(u(x, t)) \eta^{p_\rho^+} dx \leq C \int_{Q^t(\theta, \rho^*)} (\Psi'(u))^{2-p(x)} \Psi(u) (\eta^{p_\rho^+} + |\nabla \eta|^{p(x)}) dx d\tau.$$

By  $\Psi(u) \leq m \ln 2$ ,  $|\Psi'(u)|^{-1} \leq \frac{\omega}{2}$ ,  $|\nabla \eta| \leq \frac{4}{\rho}$ ,  $|\Psi'(u)| \leq \frac{2^{m+2}}{\omega}$ , we can get

$$\int_{K_{\rho^*}} \Psi^2(u(x, t)) \eta^{p_\rho^+} dx \leq Cm |K_{\rho^*}|. \tag{3.13}$$

$\forall t \in (-\theta, 0)$ , for such a set  $\{(x, t) \in K_{2\rho^*} : u(x, t) < \mu^- + \frac{\omega}{2^{m+2}}\}$  we have

$$\Psi^2(u) \geq \ln^2 \frac{H_k^-}{H_k^- - \frac{\omega}{4} + \frac{\omega}{2^{m+1}}}.$$

Since  $-\frac{\omega}{4} + \frac{\omega}{2^{m+1}} < 0$ , we obtain  $\ln^2 \frac{H_k^-}{H_k^- - \frac{\omega}{4} + \frac{\omega}{2^{m+1}}}$  is decreasing about  $H_k^-$  and  $H_k^- \leq \frac{\omega}{4}$ , thus

$$\Psi^2(u) \geq \ln^2 \frac{H_k^-}{H_k^- - \frac{\omega}{4} + \frac{\omega}{2^{m+1}}} \geq \ln^2 \frac{\frac{\omega}{4}}{\frac{\omega}{4} - \frac{\omega}{4} + \frac{\omega}{2^{m+1}}} = [(m-1) \ln 2]^2.$$

Because  $\eta = 1$  in  $K_{\frac{\rho^*}{2}}$ , by (3.13)

$$|x \in K_{\frac{\rho^*}{2}} : u(x, t) < \mu^- + \frac{\omega}{2^{m+2}}| \leq C \frac{m}{(m-1)^2} |K_{\frac{\rho^*}{2}}|,$$

where  $C = C(M, p^+)$ . To prove the lemma we have only to choose  $m$  sufficiently large and  $s = m + 2$ .  $\square$

**Lemma 3.4** *Let (3.3)-(3.4) hold, then there exist  $\sigma_1 \in (0, 1)$  and an integer  $s > 1$  independent of  $\omega$  and  $\rho$ , so that  $u(x, t) > \mu^- + \frac{\omega}{2^{s+1}}$ , a.e.  $(x, t) \in Q(\theta, \frac{\rho^*}{4})$ .*

*Proof:* Let  $\rho_m^* = \frac{\rho^*}{4} + \frac{\rho^*}{2^{m+2}}$ ,  $k_m = \mu^- + \frac{\omega}{2^{s+1}} + \frac{\omega}{2^{s+m+1}}$ ,  $m = 0, 1, 2, \dots$ , and  $s > 1$  is to be chosen later. By lemma 3.2, for a.e.  $x \in K_{\rho_m^*}$  we have  $u(x, -\theta) > \mu^- + \frac{\omega}{4} \geq k_m$ , thus  $(u - k_m)_-(x, -\theta) = 0$ . Let  $\eta_m(x)$  be a smooth cutoff function in  $K_{\rho_m^*}$  satisfying  $\eta_m \equiv 1$  in  $K_{\rho_{m+1}^*}$ ,  $|\nabla \eta_m| \leq \frac{2^{m+4}}{\rho}$ , and  $\eta_m = 0$  outside  $K_{\rho_m^*}$ .

We take  $\varphi = -(u - k_m)_- \eta_m^{p_\rho^+}$  as the testing function in (1.11), by the fact that

$$\|u\|_{L_{loc}^\infty(Q_{\rho_m^*})} \leq M, \quad \|(u - k_m)_-\|_{L_{loc}^\infty(Q_{\rho_m^*})} \leq \|(k_m - u)\|_{L_{loc}^\infty(Q_{\rho_m^*})} \leq \frac{\omega}{2^s},$$

similar to lemma 3.2, we have

$$\begin{aligned} & \sup_{-\theta < t < 0} \int_{K_{\rho_m^*}} (u - k_m)_-^2 \eta_m^{p_\rho^+} dx + \int_{Q(\theta, \rho_m^*)} |\nabla(u - k_m)_-|^{p(x)} \eta_m^{p_\rho^+} dx dt \\ & \leq C 2^{mp_\rho^+} \rho^{-p_\rho^+} \int_{Q(\theta, \rho_m^*)} \chi[(u - k_m)_- > 0] dx dt. \end{aligned} \tag{3.14}$$

On the other hand, we have

$$\int_{K_{\rho_m^*}} (u - k_m)_-^2 \eta_m^{p_\rho^+} dx \geq \left(\frac{\omega}{2^s}\right)^{2-p_\rho^-} \int_{K_{\rho_m^*}} (u - k_m)_-^{p_\rho^-} \eta_m^{p_\rho^+} dx \geq \frac{\theta}{\rho^*} \int_{K_{\rho_m^*}} (u - k_m)_-^{p_\rho^-} \eta_m^{p_\rho^+} dx$$

and

$$\begin{aligned} & \int_{Q(\theta, \rho_m^*)} |\nabla(u - k_m)_-|^{p_\rho^-} \eta_m^{p_\rho^+} dx dt \\ & \leq \int_{Q(\theta, \rho_m^*)} |\nabla(u - k_m)_-|^{p(x)} \eta_m^{p_\rho^+} dx dt + \int_{Q(\theta, \rho_m^*)} \chi[(u - k_m)_- > 0] \eta_m^{p_\rho^+} dx dt, \end{aligned}$$

where  $s$  is chosen so large as to satisfy the conclusion of lemma 3.3.

Combining the above two inequalities with (3.14), we get

$$\begin{aligned} & \sup_{-\theta < t < 0} \int_{K_{\rho_m^*}} (u - k_m)_-^{p_\rho^-} \eta_m^{p_\rho^+} dx + \frac{(\rho^*)^{p_\rho^+}}{\theta} \int_{Q(\theta, \rho_m^*)} |\nabla(u - k_m)_-|^{p_\rho^-} \eta_m^{p_\rho^+} dx dt \\ & \leq C 2^{mp_\rho^+} \rho^{-p_\rho^+} \frac{(\rho^*)^{p_\rho^+}}{\theta} \int_{Q(\theta, \rho_m^*)} \chi[(u - k_m)_- > 0] dx dt. \end{aligned}$$

We introduce the change of variable  $z = t(\rho^*)^{p_\rho^+} \theta^{-1}$ , which maps  $Q(\theta, \rho_m^*)$  into  $Q_m = K_{\rho_m^*} \times (-(\rho^*)^{p_\rho^+}, 0)$ . Let  $v(x, t) = u(x, \theta z(\rho^*)^{-p_\rho^+})$ ,  $\tilde{\eta}_m(x, z) =$



$\eta_m(x, \theta z(\rho^*)^{-p_\rho^+})$ , and denote  $|A_m| = \text{meas}\{(x, z) \in Q : v(x, z) < k_m\}$ , then

$$\begin{aligned}
 & \| (v - k_m)_- \tilde{\eta}_m^{p_\rho^+} \|_{V^{p_\rho^-, p_\rho^-}(Q_m)}^{p_\rho^-} \\
 = & \text{ess sup}_{-(\rho^*)^{p_\rho^+} < z < 0} \int_{K_{\rho_m^*}} (v - k_m)_-^{p_\rho^-} (\tilde{\eta}_m^{p_\rho^+})^{p_\rho^-} dx + \int_{Q_m} |\nabla[(v - k_m)_- \tilde{\eta}_m^{p_\rho^+}]|^{p_\rho^-} dx dz \\
 \leq & C \left( \sup_{-(\rho^*)^{p_\rho^+} < z < 0} \int_{K_{\rho_m^*}} (v - k_m)_-^{p_\rho^-} \tilde{\eta}_m^{p_\rho^+} dx \right. \\
 & \left. + \int_{Q_m} |\nabla(v - k_m)_-|^{p_\rho^-} \tilde{\eta}_m^{p_\rho^+} dx dz + \int_{Q_m} |(v - k_m)_- \nabla \tilde{\eta}_m^{p_\rho^+}|^{p_\rho^-} dx dz \right) \\
 \leq & C 2^{mp_\rho^+} \rho^{-p_\rho^+} |A_m|,
 \end{aligned} \tag{3.15}$$

by lemma 2.6 and (3.15),

$$\begin{aligned}
 \frac{1}{2^{p_\rho^- (m+2)}} \left(\frac{\omega}{2^s}\right)^{p_\rho^-} |A_{m+1}| & = |k_m - k_{m+1}|^{p_\rho^-} |A_{m+1}| \\
 & \leq \| (v - k_m)_- \|_{L^{p_\rho^-}(Q_{m+1})}^{p_\rho^-} \leq \| (v - k_m)_- \tilde{\eta}_m^{p_\rho^+} \|_{L^{p_\rho^-}(Q_m)}^{p_\rho^-} \\
 & \leq \| (v - k_m)_- \tilde{\eta}_m^{p_\rho^+} \|_{V^{p_\rho^-, p_\rho^-}(Q_m)}^{p_\rho^-} |A_m|^{\frac{p_\rho^-}{p_\rho^- + N}} \\
 & \leq C 2^{mp_\rho^+} \rho^{-p_\rho^+} |A_m|^{1 + \frac{p_\rho^-}{p_\rho^- + N}}.
 \end{aligned} \tag{3.16}$$

We take  $A > 2^s$ , by (3.3), we get  $(\frac{\omega}{2^s})^{p_\rho^- - 2} \geq \rho^\varepsilon \geq \rho^{p_\rho^- - 2}$ , therefore  $\frac{\omega}{2^s} \geq \rho$ . Thus we obtain

$$\left(\frac{\omega}{2^s}\right)^{-p_\rho^+} \leq \rho^{-p_\rho^+}.$$

Denote  $Z_m = \frac{|A_m|}{|Q_m|}$ . By (3.16) and (1.10),

$$Z_{m+1} \leq C 4^{mp_\rho^+} Z_m^{1 + \frac{p_\rho^-}{p_\rho^- + N}} \leq C 4^{mp_\rho^+} Z_m^{1 + \frac{p_\rho^-}{p_\rho^- + N}},$$

where  $C = C(M, p^+)$ . Since

$$Z_0 = \frac{|A_0|}{|Q_0|} = \frac{|\{(x, t) \in Q(\theta, \frac{\theta^*}{2}) : u(x, t) < \mu^- + \frac{\omega}{2^s}\}|}{|Q(\theta, \frac{\theta^*}{2})|},$$

by lemma 3.3 there exists  $s$  such that  $Z_0 < \sigma_1$  where  $\sigma_1 \equiv C^{-\frac{N+p_\rho^-}{p_\rho^-}} 4^{-p^+(\frac{N+p_\rho^-}{p_\rho^-})^2}$ . Then by lemma 2.6 it follows that  $Z_m \rightarrow 0$  as  $m \rightarrow \infty$ . So we can get

$$u(x, t) > \mu^- + \frac{\omega}{2^{s+1}}, \quad a.e. \quad (x, t) \in Q(\theta, \frac{\rho^*}{4}).$$

**Proposition 3.1** *There exist  $\sigma \in (0, 1)$ ,  $\nu_1 \in (0, 1)$  and  $A_1 \gg 1$  independent of  $\omega$  and  $\rho$ , such that if for some cylinder of the type  $[(0, t^*) + Q(l\rho^{p_\rho^+}, \rho)]$ ,*

$$|(x, t) \in [(0, t^*) + Q(l\rho^{p_\rho^+}, \rho)] : u < \mu^- + \frac{\omega}{2}| \leq \sigma |Q(l\rho^{p_\rho^+}, \rho)|,$$

then either

$$\omega \leq A_1 \rho^{\frac{\varepsilon}{p_\rho^- - 2}} \quad (3.17)$$

or

$$\operatorname{ess\,osc}_{Q(l(\frac{\rho}{8})^{p_\rho^+}, \frac{\rho}{8})} u \leq \nu_1 \omega. \quad (3.18)$$

Proof: Assume (3.17) is violated. By lemma 3.4, we can determine a positive integer number  $s$  such that  $\operatorname{ess\,inf}_{Q(l(\frac{\rho}{8})^{p_\rho^+}, \frac{\rho}{8})} u \geq \mu^- + \frac{\omega}{2^{s+1}}$ , this gives

$$- \operatorname{ess\,inf}_{Q(l(\frac{\rho}{8})^{p_\rho^+}, \frac{\rho}{8})} u \leq -\mu^- - \frac{\omega}{2^{s+1}}, \quad (3.19)$$

and further

$$\operatorname{ess\,osc}_{Q(\theta, \frac{\rho}{8})} u \leq (1 - \frac{1}{2^{s+1}})\omega.$$

therefore the proposition follows with  $\nu_1 = (1 - \frac{1}{2^{s+1}})$ , since  $Q(l(\frac{\rho}{8})^{p_\rho^+}, \frac{\rho}{8}) \subset Q(\theta, \frac{\rho}{8})$ .  $\square$

Next assume that the condition of proposition 3.1 is violated, i.e. for every cylinder  $[(0, t^*) + Q(l\rho^{p_\rho^+}, \rho)] \subset Q(a\rho^{p_\rho^+}, \rho)$ , where  $\frac{1}{a} = (\frac{\omega}{A})^{p_\rho^- - 2}$ ,

$$|(x, t) \in [(0, t^*) + Q(l\rho^{p_\rho^+}, \rho)] : u < \mu^- + \frac{\omega}{2}| > \sigma |Q(l\rho^{p_\rho^+}, \rho)|.$$

Since  $\mu^- + \frac{\omega}{2} \leq \mu^+ - \frac{\omega}{2}$ , we can get

$$|(x, t) \in [(0, t^*) + Q(l\rho^{p_\rho^+}, \rho)] : u > \mu^+ - \frac{\omega}{2}| \leq (1 - \sigma) |Q(l\rho^{p_\rho^+}, \rho)|. \quad (3.20)$$

**Lemma 3.5** *Let (3.20) hold, then there exists a  $\bar{t} \in [t^* - l\rho^{p_\rho^+}, t^* - \frac{\sigma}{2}l\rho^{p_\rho^+}]$  such that*

$$|\{x \in K_\rho : u(x, \bar{t}) > \mu^+ - \frac{\omega}{2}\}| \leq \frac{1 - \sigma}{1 - \frac{\sigma}{2}} |K_\rho|.$$

Proof: If not, for all  $t \in [t^* - l\rho^{p_\rho^+}, t^* - \frac{\sigma}{2}l\rho^{p_\rho^+}]$ ,

$$|\{x \in K_\rho : u(x, t) > \mu^+ - \frac{\omega}{2}\}| > \frac{1 - \sigma}{1 - \frac{\sigma}{2}} |K_\rho|$$

and

$$\begin{aligned} & |(x, t) \in [(0, t^*) + Q(l\rho^{p_\rho^+}, \rho)] : u > \mu^+ - \frac{\omega}{2}| \\ & \geq \int_{t^* - l\rho^{p_\rho^+}}^{t^* - \frac{\sigma}{2}l\rho^{p_\rho^+}} |\{x \in K_\rho : u(x, t) > \mu^+ - \frac{\omega}{2}\}| dt \\ & > (1 - \frac{\sigma}{2}) l\rho^{p_\rho^+} (1 - \sigma) (1 - \frac{\sigma}{2})^{-1} |K_\rho| = (1 - \sigma) |Q(l\rho^{p_\rho^+}, \rho)|, \end{aligned}$$

contradicting (3.20).  $\square$

**Lemma 3.6** *Let (3.20) hold, then there exists a positive integer  $\bar{s} > 2$ , such that*

$$|\{x \in K_\rho : u(x, t) > \mu^+ - \frac{\omega}{2^{\bar{s}}}\}| \leq (1 - (\frac{\sigma}{2})^2) |K_\rho|, \quad \forall t \in [t^* - \frac{\sigma}{2}l\rho^{p_\rho^+}, t^*].$$

Proof: Let  $k = \mu^+ - \frac{\omega}{2}$ ,  $Q_\rho = K_\rho \times (\bar{t}, t^*)$ . Similar to lemma 3.3, we take  $\varphi = \frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h}$  as the testing function in (3.2), where the cutoff function  $\eta$  independent of  $t$  is taken so that  $\eta \equiv 1$  in the cube  $K_{(1-\alpha)\rho}$ ,  $\alpha \in (0, 1)$ , and  $|\nabla\eta| \leq \frac{1}{\alpha\rho}$ ,  $0 < \alpha < 1$ . We take  $H_k^+ = \operatorname{ess\,sup}_{[(0,t^*)+Q(\theta,\rho^*)]}(u-k)_+$ , and consider

$$\Psi(u) = \max\left\{0, \ln \frac{H_k^+}{H_k^+ - (u-k)_+ + \omega 2^{-(m+2)}}\right\} = \ln^+ \frac{H_k^+}{H_k^+ - (u-k)_+ + \omega 2^{-(m+2)}},$$

then

$$\begin{aligned} \int_{K_{(1-\alpha)\rho}} \Psi^2(u(x,t))dx &\leq \int_{K_\rho} \Psi^2(u(x,t))\eta^{p_\rho^+} dx \leq \int_{K_\rho} \Psi^2(u(x,\bar{t}))dx \\ &\quad + C \int_{\bar{t}}^{t^*} \int_{K_\rho} (\Psi'(u))^{2-p(x)} \Psi(u)(\eta^{p_\rho^+} + |\nabla\eta|^{p(x)})dx, \end{aligned}$$

where  $|t^* - \bar{t}| \leq l\rho^{p_\rho^+}$ ,  $l = (\frac{\omega}{2})^{2-p_\rho^-}$ ,  $C = C(M, p^+)$ .

When  $u(x,\bar{t}) > k + \frac{\omega}{2^{m+1}} > \mu^+ - \frac{\omega}{2}$ ,  $\Psi^2(u(x,\bar{t})) \neq 0$ , by lemma 3.5,

$$\begin{aligned} \int_{K_\rho} \Psi^2(u(x,\bar{t}))dx &= \int_{\{x \in K_\rho : u(x,\bar{t}) > \mu^+ - \frac{\omega}{2} + \frac{\omega}{2^{m+1}}\}} \Psi^2(u(x,\bar{t}))dx \\ &\leq \int_{\{x \in K_\rho : u(x,\bar{t}) > \mu^+ - \frac{\omega}{2}\}} \Psi^2(u(x,\bar{t}))dx \leq (m \ln 2)^2 (1-\sigma)(1-\frac{\sigma}{2})^{-1} |K_\rho|, \end{aligned}$$

so we have

$$\int_{K_{(1-\alpha)\rho}} \Psi^2(u(x,t))dx \leq C[m^2(1-\sigma)(1-\frac{\sigma}{2})^{-1} + m\alpha^{-p_\rho^+}]|K_\rho|.$$

$\forall t \in (\bar{t}, t^*)$ , in  $\{x \in K_{(1-\alpha)\rho} : u(x,t) > \mu^+ - \frac{\omega}{2^{m+1}}\}$  we can get

$$\Psi^2(u) \geq \ln^2 \frac{H_k^+}{H_k^+ - \frac{\omega}{4} + \omega 2^{-(m+1)}} \geq \ln^2 \frac{\omega 2^{-2}}{\omega 2^{-(m+1)}} = (m-1)^2 \ln^2 2,$$

so  $\forall t \in (\bar{t}, t^*)$ ,

$$\begin{aligned} &|x \in K_{(1-\alpha)\rho} : u(x,t) > \mu^+ - \omega 2^{-(m+2)}| \\ &\leq C[(\frac{m}{m+1})^2(1-\sigma)(1-\frac{\sigma}{2})^{-1} + \frac{1}{m}\alpha^{-p_\rho^+}]|K_\rho|. \end{aligned}$$

On the other hand,  $\forall t \in (\bar{t}, t^*)$ ,

$$\begin{aligned} |x \in K_\rho : u(x,t) > \mu^+ - \omega 2^{-(m+2)}| &\leq |x \in K_{(1-\alpha)\rho} : u(x,t) > \mu^+ - \omega 2^{-(m+1)}| \\ &\quad + |K_\rho \setminus K_{(1-\alpha)\rho}| \leq |x \in K_{(1-\alpha)\rho} : u(x,t) > \mu^+ - \omega 2^{-(m+1)}| + \alpha N |K_\rho|, \end{aligned}$$

so  $\forall t \in (\bar{t}, t^*)$ ,

$$\begin{aligned} &|x \in K_{(1-\alpha)\rho} : u(x,t) > \mu^+ - \omega 2^{-(m+1)}| \\ &\leq C(\frac{m}{m-1})^2[(1-\sigma)(1-\frac{\sigma}{2})^{-1} + \frac{1}{m}\alpha^{-p_\rho^+} + N\alpha]|K_\rho|. \end{aligned}$$

Choose  $\alpha$  so small and then  $m$  so large that  $C(\frac{m}{m-1})^2 \leq (1+\sigma)(1-\frac{\sigma}{2})$ ,  $\frac{C}{m}\alpha^{-p^+} \leq \frac{3}{8}\sigma^2$  and  $C\alpha N \leq \frac{3}{8}\sigma^2$ . Then for such a choice of  $m$  the lemma follows with  $\bar{s} = m + 1$ .  $\square$

Since (3.20) holds for all  $[(0, t^*) + Q(l\rho^{p^+}, \rho)]$ , the conclusion of lemma 3.6 holds for all time levels satisfying  $t \geq -(a-l)\rho^{p^+} = -(1 - (\frac{2}{A})^{p^- - 2})a\rho^{p^+}$ . If the number  $A$  is chosen sufficiently large such that  $1 - (\frac{2}{A})^{p^- - 2} > \frac{1}{2}$ , we deduce the following corollary.

**Corollary 3.1** *Let (3.20) hold, then for all  $t \in (-\frac{a}{2}\rho^{p^+}, 0)$ ,*

$$|\{x \in K_\rho : u(x, t) > \mu^+ - \omega 2^{-\bar{s}}\}| \leq (1 - (\frac{\sigma}{2})^2)|K_\rho|.$$

**Lemma 3.7** *Let (3.20) hold, then for every  $\bar{\sigma} \in (0, 1)$ , there exists positive integer  $s^* > \bar{s}$ , such that*

$$|\{x \in K_\rho : u(x, t) > \mu^+ - \frac{\omega}{2^{s^*}}\}| \leq \bar{\sigma}|Q(2^{-1}a\rho^{p^+}, \rho)|, \quad \forall t \in (-\frac{a}{2}\rho^{p^+}, 0).$$

Proof: Consider the problem in  $Q(a\rho^{p^+}, 2\rho)$ . Let  $k = \mu^+ - \frac{\omega}{2^{\bar{s}}}$ , where  $\bar{s} \leq s \leq s^*$ . Take  $\varphi = (u_n - k)_+ \zeta^{p^+}$  as the testing function in (1.9), where  $\zeta$  is a cutoff function that equals one on  $Q(\frac{a}{2}\rho^{p^+}, \rho)$ , vanishes on the parabolic boundary of  $Q(a\rho^{p^+}, 2\rho)$  and such that  $|\nabla\zeta| \leq \frac{1}{\rho}$ ,  $0 \leq \zeta_t \leq \frac{2}{a\rho^{p^+}}$ . Similar to lemma 3.2, we get

$$\begin{aligned} \int_{A_s} |\nabla u|^{p^-} dxdt &\leq \int_{Q(\frac{a}{2}\rho^{p^+}, \rho)} |\nabla(u - k)_+|^{p(x)} dxdt + |A_s| \\ &\leq C\rho^{-p^+} |Q(\frac{a}{2}\rho^{p^+}, \rho)|, \end{aligned}$$

where  $C = C(p^+)$  and

$$A_s = \{(x, t) \in Q(\frac{a}{2}\rho^{p^+}, \rho) : u(x, t) > \mu^+ - \frac{\omega}{2^s}\},$$

$$A_s(t) = \{x \in K_\rho : u(x, t) > \mu^+ - \frac{\omega}{2^s}\}.$$

By corollary 3.1,  $\forall t \in (-\frac{a}{2}\rho^{p^+}, 0)$ ,

$$|\{x \in K_\rho : u(x, t) < \mu^+ - \frac{\omega}{2^s}\}| = |K_\rho| - |A_s(t)| \geq (\frac{\sigma}{2})^2 |K_\rho|. \quad (3.21)$$

In lemma 2.8, take  $k = \mu^+ - \frac{\omega}{2^s}$ ,  $h = \mu^+ - \frac{\omega}{2^{s+1}}$ , then  $\forall t \in [-\frac{a}{2}\rho^{p^+}, 0]$ , by (3.21), we get

$$\frac{\omega}{2^{s+1}} |A_{s+1}(t)| \leq \frac{C}{\sigma^2} \frac{\rho^{N+1}}{|K_\rho|} \int_{A_s(t) \setminus A_{s+1}(t)} |\nabla u| dx. \quad (3.22)$$

Take  $A > 2^s$ , there exists  $C = C(M, p^+, p^-)$  such that  $(\frac{\omega}{2})^{p_\rho^+ - p_\rho^-} \leq C$  and  $(\frac{\omega}{2})^{-p_\rho^-} \leq \rho^{-p_\rho^-}$  hold. Integrating on  $(-a\rho^{p_\rho^+}, 0)$ , from (3.22) we get

$$\begin{aligned} & (\frac{\omega}{2^s})^{-p_\rho^-} \frac{\omega}{2^{s+1}} |A_{s+1}| \leq (\frac{\omega}{2^s})^{-p_\rho^-} \frac{C\rho}{\sigma^2} \int_{A_s \setminus A_{s+1}} |\nabla u| dx dt \\ & \leq (\frac{\omega}{2^s})^{-p_\rho^-} \frac{C\rho}{\sigma^2} (\int_{A_s} |\nabla u|^{p_\rho^-} dx dt)^{\frac{1}{p_\rho^-}} |A_s \setminus A_{s+1}|^{\frac{p_\rho^- - 1}{p_\rho^-}} \\ & \leq \frac{C}{\sigma^2} |Q(\frac{a}{2}\rho^{p_\rho^+}, \rho)|^{\frac{1}{p_\rho^-}} |A_s \setminus A_{s+1}|^{\frac{p_\rho^- - 1}{p_\rho^-}}, \end{aligned} \quad (3.23)$$

If  $s$  is large enough so that  $(\frac{\omega}{2^s})^{p_\rho^-} \frac{2^{s+1}}{\omega} < 1$ , from (3.23) we get

$$|A_{s+1}|^{\frac{p_\rho^-}{p_\rho^- - 1}} \leq C\sigma^{-2\frac{p_\rho^-}{p_\rho^- - 1}} |Q(\frac{a}{2}\rho^{p_\rho^+}, \rho)|^{\frac{1}{p_\rho^- - 1}} |A_s \setminus A_{s+1}|, \quad (3.24)$$

for all  $\bar{s} \leq s \leq s^*$ . We add them for  $s = \bar{s}, \bar{s} + 1, \bar{s} + 2, \dots, s^* - 1$ , then

$$(s^* - \bar{s}) |A_{s^*}|^{\frac{p_\rho^-}{p_\rho^- - 1}} \leq C\sigma^{-2\frac{p_\rho^-}{p_\rho^- - 1}} |Q(\frac{a}{2}\rho^{p_\rho^+}, \rho)|^{\frac{p_\rho^-}{p_\rho^- - 1}}.$$

After taking  $s^*$  so large that  $C(s^* - \bar{s})^{\frac{1-p_\rho^-}{p_\rho^-}} \leq \sigma^2 \bar{\sigma}$ , we conclude the lemma.  $\square$

**Lemma 3.8** *Let (3.20) hold, then there exists  $\bar{\sigma} \in (0, 1)$  so that*

$$u(x, t) \leq \mu^+ - \frac{\omega}{2^{s^*+1}}, \quad a.e. \quad Q(\frac{a}{2}(\frac{\rho}{2})^{p_\rho^+}, \frac{\rho}{2}),$$

where  $\frac{1}{a} = (\frac{\omega}{A})^{p_\rho^- - 2}$ ,  $A = 2^{s^*}$ .

Proof: We will consider the problem over the boxes  $Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m)$ . Let  $\rho_m = \frac{\rho}{2} + \frac{\rho}{2^{m+1}}$ ,  $k_m = \mu^+ - \frac{\omega}{2^{s^*+1}} - \frac{\omega}{2^{s^*+m+1}}$ .  $\zeta_m$  is a cutoff function with  $0 \leq \zeta_m \leq 1$  in  $Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m)$ ,  $\zeta_m \equiv 1$  in  $Q(\frac{a}{2}\rho_{m+1}^{p_\rho^+}, \rho_{m+1})$ ,  $\zeta_m \equiv 0$  on the parabolic boundary of  $Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m)$ ,  $|\nabla \zeta_m| \leq \frac{2^{m+2}}{\rho}$ ,  $0 \leq \frac{\partial \zeta_m}{\partial t} \leq \frac{2}{a}(\frac{2^{m+2}}{\rho})^{p_\rho^+}$ . Take  $(u_m - k_m)_+ \zeta_m^{p_\rho^+}$  as the testing function in (1.11), by  $\|u\|_{L_{loc}^\infty(Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m))} \leq M$  and  $\|(u - k_m)_+\|_{L_{loc}^\infty(Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m))} \leq \|(u - k_m)\|_{L_{loc}^\infty(Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m))} \leq \frac{\omega}{2^{s^*}}$ , similar to lemma 3.2, we obtain

$$\begin{aligned} & \sup_{-\frac{a}{2}\rho_m^{p_\rho^+} < t < 0} \int_{K_{\rho_m}} (u - k_m)_+^2 \zeta_m^{p_\rho^+} dx + \int_{Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m)} |\nabla (u - k_m)_+|^{p(x)} \zeta_m^{p_\rho^+} dx dt \\ & \leq C 2^{mp_\rho^+} \rho^{-p_\rho^+} \int_{Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m)} \chi[(u - k_m)_+ > 0] dx dt. \end{aligned} \quad (3.25)$$

On the other hand, we have

$$\int_{K_{\rho_m}} (u - k_m)_+^{p_\rho^-} \zeta_m^{p_\rho^+} dx \leq (\frac{\omega}{2^{s^*}})^{p_\rho^- - 2} \int_{K_{\rho_m}} (u - k_m)_+^2 \zeta_m^{p_\rho^+} dx$$

and

$$\begin{aligned} \int_{Q(\frac{a}{2}\rho_m^+, \rho_m)} |\nabla(u - k_m)_+|^{p_\rho^-} \zeta_m^{p_\rho^+} dxdt \leq & \int_{Q(\frac{a}{2}\rho_m^+, \rho_m)} |\nabla(u - k_m)_+|^{p(x)} \zeta_m^{p_\rho^+} dxdt \\ & + \int_{Q(\frac{a}{2}\rho_m^+, \rho_m)} \chi[(u - k_m)_+ > 0] \zeta_m^{p_\rho^+} dxdt \end{aligned}$$

then by (3.25),

$$\begin{aligned} & \sup_{-\frac{a}{2}\rho_m^+ < t < 0} \int_{K_{\rho_m}} (u - k_m)_+^{p_\rho^-} \zeta_m^{p_\rho^+} dx + \frac{1}{a} \int_{Q(\frac{a}{2}\rho_m^+, \rho_m)} |\nabla(u - k_m)_+|^{p_\rho^-} \zeta_m^{p_\rho^+} dxdt \\ \leq & C2^{mp_\rho^+} \rho^{-p_\rho^+} \frac{1}{a} \int_{Q(\frac{a}{2}\rho_m^+, \rho_m)} \chi[(u - k_m)_+ > 0] dxdt. \end{aligned}$$

Next, we introduce the change of time-variable  $z = 2l^{-1}t$  which transforms  $Q(2^{-1}a\rho_m^+, \rho_m)$  into  $Q_m = K_{\rho_m} \times (-\rho_m^+, 0)$ . Setting  $v(x, t) = u(x, 2^{-1}az)$ ,  $\tilde{\zeta}_m(x, z) = \zeta_m(x, 2^{-1}az)$ ,  $|A_m| = \text{meas}\{(x, z) \in Q_m : v(x, z) > k_m\}$ , then

$$\begin{aligned} & \|(v - k_m)_+ \tilde{\zeta}_m^{p_\rho^+}\|_{V^{p_\rho^-, p_\rho^-}(Q_m)}^{p_\rho^-} \\ \leq & C \left( \sup_{-\rho_m^+ < z < 0} \int_{K_{\rho_m}} (v - k_m)_+^{p_\rho^-} \tilde{\zeta}_m^{p_\rho^+} dx + \int_{Q_m} |\nabla(v - k_m)_+|^{p_\rho^-} \tilde{\zeta}_m^{p_\rho^+} dx dz \right. \\ & \left. + \int_{Q_m} |(v - k_m)_+ \nabla \tilde{\zeta}_m^{p_\rho^+}|^{p_\rho^-} dx dz \right) \leq C2^{mp_\rho^+} \rho^{-p_\rho^+} |A_m|. \end{aligned} \tag{3.26}$$

By lemma 2.8 and (3.26),

$$\begin{aligned} \frac{1}{2^{p_\rho^-(m+2)}} \left(\frac{\omega}{2^{s^*}}\right)^{p_\rho^-} |A_{m+1}| &= |k_m - k_{m+1}|^{p_\rho^-} |A_{m+1}| \\ &\leq \|(v - k_m)_+\|_{L^{p_\rho^-}(Q_{m+1})}^{p_\rho^-} \leq \|(v - k_m)_+ \tilde{\zeta}_m^{p_\rho^+}\|_{L^{p_\rho^-}(Q_m)}^{p_\rho^-} \\ &\leq \|(v - k_m)_+ \tilde{\zeta}_m^{p_\rho^+}\|_{V^{p_\rho^-, p_\rho^-}(Q_m)}^{p_\rho^-} |A_m|^{\frac{p_\rho^-}{p_\rho^- + N}} \\ &\leq C2^{mp_\rho^+} \rho^{-p_\rho^+} |A_m|^{1 + \frac{p_\rho^-}{p_\rho^- + N}}. \end{aligned}$$

Take  $A = 2^{s^*}$ , then  $(\frac{\omega}{2^{s^*}})^{-p_\rho^-} \leq \rho^{-p_\rho^-}$ .

Next, we obtain

$$Z_{m+1} \leq C4^{mp_\rho^+} Z_m^{1 + \frac{p_\rho^-}{p_\rho^- + N}}.$$

By lemma 2.7, when  $m \rightarrow \infty$ ,  $Z_m \rightarrow 0$  where  $Z_0 \leq C^{-\frac{N+p_\rho^-}{p_\rho^-}} 4^{-p^+(\frac{N+p_\rho^-}{p_\rho^-})^2} \equiv \bar{\sigma}$ . Thus as  $m \rightarrow \infty$ ,

$$\int_{Q_m} \chi[(v - k_m)_+ > 0] dx dz \rightarrow 0,$$

i.e.  $u(x, t) \leq \mu^+ - \frac{\omega}{2^{s^*+1}}$  a.e. in  $Q(\frac{a}{2}(\frac{\rho}{2})^{p_\rho^+}, \frac{\rho}{2})$ .  $\square$

**Proposition 3.2** *There exist  $\sigma \in (0, 1)$ ,  $\nu_2 \in (0, 1)$  and  $A_2 \gg 1$  independent of  $\omega$  and  $\rho$ , such that if for all cylinders of the type  $[(0, t^*) + Q(l\rho^{p_\rho^+}, \rho)]$ ,*

$$|(x, t) \in [(0, t^*) + Q(l\rho^{p_\rho^+}, \rho)] : u > \mu^+ - \frac{\omega}{2}| \leq (1 - \sigma)|Q(l\rho^{p_\rho^+}, \rho)|,$$

then either

$$\omega \leq A_2 \rho^{\frac{\varepsilon}{p_\rho^+} - 2} \tag{3.27}$$

or

$$\operatorname{ess\,osc}_{Q(\frac{\rho}{2}, \frac{\rho}{2})^{p_\rho^+}, \frac{\rho}{2}} u \leq \nu_2 \omega. \tag{3.28}$$

Proof: Assume (3.27) is violated. By lemma 3.8, we can determine a positive integer number  $s^*$  such that

$$\operatorname{ess\,inf}_{Q(\frac{\rho}{2}, \frac{\rho}{2})^{p_\rho^+}, \frac{\rho}{2}} u \leq \mu^+ + \frac{\omega}{2^{s^*+1}}, \tag{3.29}$$

and further

$$\operatorname{ess\,osc}_{Q(\frac{\rho}{2}, \frac{\rho}{2})^{p_\rho^+}, \frac{\rho}{2}} u \leq (1 - \frac{1}{2^{s^*+1}})\omega,$$

therefore (3.28) holds with  $\nu_2 = (1 - \frac{1}{2^{s^*+1}})$ . We get the conclusion.  $\square$

Combine proposition 1 and proposition 2, we can get

**Proposition 3.3** *There exist  $\nu = \max\{\nu_1, \nu_2\}$  and  $\bar{A} = \{A_1, A_2\}$ , such that either  $\omega \leq \bar{A} \rho^{\frac{\varepsilon}{p_\rho^+} - 2}$  or  $\operatorname{ess\,osc}_{Q(l(\frac{\rho}{8})^{p_\rho^+}, \frac{\rho}{8})} u \leq \nu \omega$ , where  $\nu_1, \nu_2, A_1, A_2$  are determined by proposition 1 and proposition 2.*

Next we assume  $\omega_1 = \max\{\nu \omega, \bar{A} \rho^{\frac{\varepsilon}{p_\rho^+} - 2}\}$  and  $\frac{1}{a_1} = (\frac{\omega_1}{\bar{A}})^{p_\rho^- - 2}$ . Since

$$l(\frac{\rho}{8})^{p_\rho^+} = (\frac{2}{\omega})^{p_\rho^- - 2} (\frac{\rho}{8})^{p_\rho^+} \geq 2^{-3p_\rho^+} \nu^{p_\rho^- - 2} (\frac{2}{A})^{p_\rho^- - 2} (\frac{A}{\omega_1})^{p_\rho^- - 2} \rho^{p_\rho^+} = a_1 \rho_1^{p_\rho^+},$$

where  $\rho_1 = C^{-1} \rho$  and  $C = 8(\frac{1}{\nu})^{\frac{p_\rho^- - 2}{p_\rho^+}} (\frac{A}{2})^{\frac{p_\rho^- - 2}{p_\rho^+}}$ , so  $Q(a_1(\rho_1)^{p_\rho^+}, \rho_1) \subset Q(l(\frac{\rho}{8})^{p_\rho^+}, \frac{\rho}{8})$ .

Then we can get  $\operatorname{ess\,osc}_{Q(a_1(\rho_1)^{p_\rho^+}, \rho_1)} u \leq \omega_1$  and  $(\frac{\omega_1}{\bar{A}})^{p_\rho^- - 2} > 8^{p_\rho^+} (\frac{\omega}{2})^{p_\rho^- - 2} > \rho^\varepsilon$ . So for

$Q(a_1(\rho_1)^{p_\rho^+}, \rho_1)$ , repeating the process above, we can get the similar result, and moreover the following proposition 3.4 can be obtained:

**Proposition 3.4** *There exist  $0 < \varepsilon_0 < 1$ ,  $\nu \in (0, 1)$ ,  $C = C(N, M, p^+, p^-) > 1$  and  $A > 1$  satisfy  $\rho_0 = \rho$ ,  $\omega_0 = \omega$ ,  $\rho_n = C^{-n} \rho$  and  $\omega_{n+1} = \max\{\nu \omega_n, C \rho_n^{\varepsilon_0}\}$ ,  $n = 1, 2, \dots$ , such that for all boxes  $Q^{(n)} = Q(a_n \rho_n^{p_\rho^+}, \rho_n)$ ,  $\frac{1}{a_n} = (\frac{\omega_n}{A})^{p_\rho^- - 2}$ ,  $n = 1, 2, \dots$ , we have*

$$Q^{(n+1)} \subset Q^{(n)}, \quad \operatorname{ess\,osc}_{Q^{(n)}} u \leq \omega_n.$$

In view of proposition 3.4, we get

**Proposition 3.5** *There exist  $\lambda \in (0, 1)$ ,  $C = C(N, M, p^+, p^-)$  and  $0 < \tilde{\rho} \leq \rho$  such that for all boxes  $Q(a\rho^{p^+}, \rho)$ ,  $\frac{1}{a} = (\frac{\omega}{A})^{p^- - 2}$ , we have*

$$\operatorname{ess\,osc}_{Q(a\tilde{\rho}^{p^+}, \tilde{\rho})} u \leq C(\omega + \rho^{\varepsilon_0})\left(\frac{\tilde{\rho}}{\rho}\right)^\lambda.$$

Proof: From the iterative construction of  $\omega_n$ , it follows that  $\omega_{n+1} \leq \nu\omega_n + C\rho_n^{\varepsilon_0}$  and by iteration

$$\omega_n \leq \nu^n \omega + C(\sum_{i=0}^{n-1} \nu^i C^{-\varepsilon_0(n-i)})\rho^{\varepsilon_0}.$$

We may assume without loss of generality that  $\varepsilon_0$  is so small that  $\nu \leq C^{-\varepsilon_0}$ , then  $\omega_n \leq \nu^n \omega + Cn(\frac{\rho}{C^n})^{\varepsilon_0}$ . Let  $0 < \tilde{\rho} \leq \rho$  be fixed, then there exists a nonnegative integer  $n$  such that

$$C^{-(n+1)}\rho \leq \tilde{\rho} \leq C^{-n}\rho,$$

which implies the inequalities

$$(n+1) \geq \ln\left(\frac{\tilde{\rho}}{\rho}\right)^{-\frac{1}{\ln C}},$$

$$\nu^n \leq \nu^{-1}\left(\frac{\tilde{\rho}}{\rho}\right)^{\lambda_1}, \quad \lambda_1 = \frac{|\ln \nu|}{\ln C},$$

$$Cn\left(\frac{\rho}{C^n}\right)^{\varepsilon_0} \leq C^{1+\varepsilon_0} \ln\left(\frac{\tilde{\rho}}{\rho}\right)^{-\frac{1}{\ln C}} \tilde{\rho}^{\varepsilon_0} \leq C(\varepsilon_0)\rho^{\frac{\varepsilon_0}{2}} \tilde{\rho}^{\frac{\varepsilon_0}{2}}.$$

Therefore

$$\omega_n \leq C(\omega + \rho^{\varepsilon_0})\left(\frac{\tilde{\rho}}{\rho}\right)^\lambda, \quad \lambda = \min\left\{\lambda_1, \frac{\varepsilon_0}{2}\right\}.$$

On the other hand, by (3.3) we get  $\omega > C\rho^{\varepsilon_0}$ . Thus by the definition of  $\omega_n$ ,  $\omega_1 = \max\{\nu\omega, C\rho^{\varepsilon_0}\} \leq \omega$  and  $\omega_2 = \max\{\nu\omega_1, C(C^{-1}\rho)^{\varepsilon_0}\} \leq \omega, \dots$ , so  $\omega_n \leq \omega$ . Since  $Q(a\tilde{\rho}^{p^+}, \tilde{\rho}) \subset Q^{(n)}$ , by proposition 3.4, we obtain  $\operatorname{ess\,osc}_{Q(a\tilde{\rho}^{p^+}, \tilde{\rho})} u \leq \omega_n$ , so we

conclude proposition 3.5.  $\square$

By proposition 3.5, we know  $u$  is Hölder continuity in  $Q(a\tilde{\rho}^{p^+}, \tilde{\rho})$ , so for every point in  $Q$  we can obtain such a cylinder as  $Q(a\tilde{\rho}^{p^+}, \tilde{\rho})$ , then by limited coverage theorem,  $u$  is local Hölder continuity in  $Q$ , thus we get theorem 1.

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