Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **96**, 1-11; **http://www.math.u-szeged.hu/ejqtde/**

Three symmetric positive solutions of fourth-order nonlocal boundary value problems

Fuyi Xu

School of Science, Shandong University of Technology, Zibo, 255049, Shandong, China
School of Mathematics and Systems Science, Beihang University, Beijing 100191, China

Abstract. In this paper, we study the existence of three positive solutions for fourth-order singular nonlocal boundary value problems. We show that there exist triple symmetric positive solutions by using Leggett-Williams fixed-point theorem. The conclusions in this paper essentially extend and improve some known results.

MSC: 34B16.

Keywords: nonlocal boundary condition, symmetric positive solutions, singular, Green's function, cone.

1 Introduction

Boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics, the existence of positive solutions for such problems has become an important area of investigation in recent years. To identify a few, we refer the reader to [1-3,6,10,11,13,17,18] and references therein.

At the same time, a class of boundary value problems with nonlocal boundary conditions appeared in heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics. Such problems include two-point, three-point, multi-point boundary value problems as special cases and have attracted the attention of Gallardo [1], Karakostas and Tsamatos [2], Lomtatidze and Malaguti [3] (and see the references therein). For more information about the general theory of integral equations and their relation to boundary value problems, see for example, [4,5].

Motivated by the works mentioned above, in this paper, we study the existence of three symmetric positive solutions for the following fourth-order singular nonlocal boundary value problem(NBVP):

$$\begin{cases} u''''(t)(t) = g(t)f(t,u), \qquad 0<t<1, \\ u(0) = u(1) = \displaystyle\int_0^1 a(s)u(s)ds, \\ u''(0) = u''(1) = \displaystyle\int_0^1 b(s)u''(s)ds, \end{cases} \tag{1.1}$$

where $a, b \in L^1[0,1]$, $g : (0,1) \to [0,\infty)$ is continuous, symmetric on $(0,1)$ and may be singular at $t = 0$ and $t = 1$, $f : [0,1] \times [0,\infty) \longrightarrow [0,\infty)$ is continuous and $f(\cdot, x)$

[1]E-mail addresses: zbxufuyi@163.com(F.Xu)

[2]This work was supported financially by the National Natural Science Foundation of China (11171034).

is symmetric on $[0,1]$ for all $x \in [0,+\infty)$. We show that there exist triple symmetric positive solutions by using Leggett-Williams fixed-point theorem.

2 Preliminaries and Lemmas

In this section, we present some definitions and lemmas that are important to prove our main results.

Definition 2.1. Let E be a real Banach space over R. A nonempty closed set $P \subset E$ is said to be a cone provided that
(i) $u \in P$, $a \geq 0$ implies $au \in P$; and
(ii) $u,\ -u \in P$ implies $u = 0$.

Definition 2.2. Given a cone P in a real Banach space E, a functional $\psi : P \to P$ is said to be increasing on P provided $\psi(x) \leq \psi(y)$, for all $x, y \in P$ with $x \leq y$.

Definition 2.3. Given a nonnegative continuous functional γ on P in a real Banach space E, we define for each $d > 0$ the following set

$$P(\gamma, d) = \{x \in P | \gamma(x) < d\}.$$

Definition 2.4. The function w is said to be symmetric on $[0,1]$, if

$$w(t) = w(1-t), t \in [0,1].$$

Definition 2.5. A function u^* is called a symmetric positive solution of the NBVP(1.1) if u^* is symmetric and positive on $[0,1]$, and satisfies the differential equation and the boundary value conditions in NBVP(1.1) .

Definition 2.6. Given a cone P in a real Banach space E, a functional $\alpha : P \to [0,\infty)$ is said to be nonnegative continuous concave on P provided $\alpha(tx+(1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$, for all $x,\ y \in P$ with $t \in [0,1]$.

Let $a,\ b,\ r > 0$ be constants with P and α as defined above, we note

$$P_r = \{y \in P|\ \ \|y\| < r\}, \quad P\{\alpha, a, b\} = \{y \in P|\ \ \alpha(y) \geq a, \ \ \|y\| \leq b\}.$$

The main tool of this paper is the following well-known Leggett-Williams fixed-point theorem.

Theorem 2.1.[15-16] *Assume E be a real Banach space, $P \subset E$ be a cone. Let $T : \overline{P}_c \to \overline{P}_c$ be completely continuous and α be a nonnegative continuous concave functional on P such that $\alpha(y) \leq \|y\|$, for $y \in \overline{P}_c$. Suppose that there exist $0 < a < b < d \leq c$ such that*
(i) $\{y \in P(\alpha,\ b,\ d)|\ \ \alpha(y) > b\} \neq \emptyset$ and $\alpha(Ty) > b$, for all $y \in P(\alpha,\ b,\ d)$;
(ii) $\|Ty\| < a$, for all $\|y\| \leq a$;
(iii) $\alpha(Ty) > b$ for all $y \in P(\alpha,\ b,\ c)$ with $\|Ty\| > d$.
Then T has at least three fixed points $y_1,\ y_2,\ y_3$ satisfying

$$\|y_1\| < a,\ \ b < \alpha(y_2),$$

and

$$\|y_3\| > a,\ \ \alpha(y_3) < b.$$

Lemma 2.1. [14] *Suppose that* $d := \int_0^1 m(s)ds \neq 1$, $m \in L^1[0,1], y \in C[0,1]$, *then BVP*

$$u''(t) + y(t) = 0, \qquad 0 < t < 1, \tag{2.1}$$

$$u(0) = u(1) = \int_0^1 m(s)u(s)ds, \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 H(t,s)y(s)ds, \tag{2.3}$$

where

$$H(t,s) = G(t,s) + \frac{1}{1-d}\int_0^1 G(s,x)m(x)dx, \;\; G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Proof. Integrating both sides of (2.1) on $[0,t]$, we have

$$u'(t) = -\int_0^t y(s)ds + B. \tag{2.4}$$

Again integrating (2.4) from 0 to t, we get

$$u(t) = -\int_0^t (t-s)y(s)ds + Bt + A. \tag{2.5}$$

In particular,

$$u(1) = -\int_0^1 (1-s)y(s)ds + B + A, \qquad u(0) = A.$$

By the boundary value conditions (2.2) we get

$$B = \int_0^1 (1-s)y(s)ds. \tag{2.6}$$

By $G(s,x) = G(x,s)$ and (2.5), we can obtain

$$\begin{aligned} A &= u(0) = \int_0^1 m(x)u(x)dx = \int_0^1 m(x)\left(-\int_0^x (x-s)y(s)ds + Bx + A\right)dx \\ &= \int_0^1 m(x)\left(-\int_0^x (x-s)y(s)ds + x\int_0^1 (1-s)y(s)ds\right)dx + A\int_0^1 m(x)dx \\ &= \int_0^1 m(x)\left(\int_0^x s(1-x)y(s)ds + \int_x^1 x(1-s)y(s)ds\right)dx + Ad \\ &= \int_0^1 m(x)\left(\int_0^1 G(s,x)y(s)ds\right)dx + Ad \\ &= \int_0^1 \left(\int_0^1 G(s,x)m(x)dx\right)y(s)ds + Ad. \end{aligned}$$

So, we have

$$A = \frac{1}{1-d}\int_0^1 \left(\int_0^1 G(s,x)m(x)dx\right) y(s)ds. \tag{2.7}$$

By (2.5), (2.6) and (2.7), we obtain

$$\begin{aligned}
u(t) &= -\int_0^t (t-s)y(s)ds + Bt + A \\
&= -\int_0^t (t-s)y(s)ds + t\int_0^1 (1-s)y(s)ds + \frac{1}{1-d}\int_0^1 \left(\int_0^1 G(s,x)m(x)dx\right) y(s)ds \\
&= \int_0^t s(1-t)y(s)ds + \int_t^1 t(1-s)y(s)ds + \frac{1}{1-d}\int_0^1 \left(\int_0^1 G(s,x)m(x)dx\right) y(s)ds \\
&= \int_0^1 G(t,s)y(s)ds + \frac{1}{1-d}\int_0^1 \left(\int_0^1 G(s,x)m(x)dx\right) y(s)ds \\
&= \int_0^1 H(t,s)y(s)ds.
\end{aligned}$$

This completes the proof of Lemma 2.1.

It is easy to verify the following properties of $H(t,s)$ and $G(t,s)$.

Lemma 2.2. *If* $m(t) > 0$, *and* $d := \int_0^1 m(s)ds \in (0,1)$, *then*

(1) $H(t,s) \geq 0, t,s \in [0,1]$, $H(t,s) > 0, t,s \in (0,1)$;

(2) $G(1-t,1-s) = G(t,s), G(t,t) \leq G(t,s) \leq G(s,s), t,s \in [0,1]$;

(3) $\gamma H(s,s) \leq H(t,s) \leq H(s,s)$, *where* $\gamma = \dfrac{\eta}{1-d+\eta} \in (0,1), \eta = \int_0^1 G(x,x)m(x)dx.$

So we may denote Green's functions of the following boundary value problems

$$\begin{cases} -u''(t) = 0, & 0 < t < 1, \\ u(0) = u(1) = \displaystyle\int_0^1 a(s)u(s)ds \end{cases}$$

and

$$\begin{cases} -u''(t) = 0, & 0 < t < 1, \\ u(0) = u(1) = \displaystyle\int_0^1 b(s)u(s)ds, \end{cases}$$

by $H_1(t,s)$ and $H_2(t,s)$, respectively. By Lemma 2.1, we know that $H_1(t,s)$ and $H_2(t,s)$ can be written by

$$H_1(t,s) = G(t,s) + \frac{1}{1-\displaystyle\int_0^1 a(s)ds}\int_0^1 G(s,x)a(x)dx,$$

and

$$H_2(t,s) = G(t,s) + \frac{1}{1-\displaystyle\int_0^1 b(s)ds}\int_0^1 G(s,x)b(x)dx.$$

Obviously, $H_1(t,s)$ and $H_2(t,s)$ have the same properties with $H(t,s)$ in Lemma 2.2.

Remark 2.1. For notational convenience, we introduce the following constants

$$\alpha=\int_0^1 a(s)ds,\ \beta=\int_0^1 b(s)ds,$$

$$\gamma_1=\frac{\eta_1}{1-\alpha+\eta_1},\gamma_2=\frac{\eta_2}{1-\beta+\eta_2}\in(0,1),$$

$$\eta_1=\int_0^1 G(x,x)a(x)dx,\eta_2=\int_0^1 G(x,x)b(x)dx.$$

Lemma 2.3. *Assume that $\alpha,\beta\neq 1, h\in C[0,1]$, then NBVP*

$$\begin{cases} u''''(t)=h(t), \qquad 0<t<1,\\ u(0)=u(1)=\int_0^1 a(s)u(s)ds,\\ u''(0)=u''(1)=\int_0^1 b(s)u''(s)ds \end{cases} \tag{2.8}$$

has a unique solution

$$u(t)=\int_0^1\int_0^1 H_1(t,\tau)H_2(\tau,s)h(s)dsd\tau. \tag{2.9}$$

Lemma 2.4. *Assume that $\alpha,\beta\neq 1, h\in C[0,1]$ is symmetric, then the solution $u(t)$ of NBVP (2.8) is symmetric on $[0,1]$.*

Proof. For notational convenience, we set

$$E_1(\tau)=\frac{1}{1-\displaystyle\int_0^1 a(s)ds}\int_0^1 G(\tau,x)a(x)dx, E_2(s)=\frac{1}{1-\displaystyle\int_0^1 b(s)ds}\int_0^1 G(s,x)b(x)dx.$$

For $\forall t, s \in [0,1]$, by (2.9) and Lemma 2.2 we have

$$\begin{aligned}
u(1-t) &= \int_0^1 \int_0^1 H_1(1-t,\tau)H_2(\tau,s)h(s)dsd\tau \\
&= \int_0^1 \int_0^1 [G(1-t,\tau)+E_1(\tau)][G(\tau,s)+E_2(s)]h(s)dsd\tau \\
&= \int_0^1 \int_0^1 G(1-t,\tau)G(\tau,s)h(s)dsd\tau + \int_0^1 \int_0^1 G(1-t,\tau)E_2(s)h(s)dsd\tau \\
&+ \int_0^1 \int_0^1 E_1(\tau)[G(\tau,s)+E_2(s)]h(s)dsd\tau \\
&= \int_1^0 \int_1^0 G(1-t,1-\tau)G(1-\tau,1-s)h(1-s)d(1-s)d(1-\tau) \\
&+ \int_1^0 \int_0^1 G(1-t,1-\tau)E_2(s)h(s)dsd(1-\tau) + \int_0^1 \int_0^1 E_1(\tau)[G(\tau,s)+E_2(s)]h(s)dsd\tau \\
&= \int_0^1 \int_0^1 G(t,\tau)G(\tau,s)h(s)dsd\tau + \int_0^1 \int_0^1 G(t,\tau)E_2(s)h(s)dsd\tau \\
&+ \int_0^1 \int_0^1 E_1(\tau)[G(\tau,s)+E_2(s)]h(s)dsd\tau \\
&= \int_0^1 \int_0^1 H_1(t,\tau)H_2(\tau,s)h(s)dsd\tau \\
&= u(t).
\end{aligned}$$

Therefore, the solution $u(t)$ of NBVP (2.8) is symmetric on $[0,1]$.

Lemma 2.5. *Assume that $a(t) \geq 0, b(t) \geq 0,$ and $\alpha, \beta \in (0,1), h \in C^+[0,1]$, then the solution $u(t)$ of NBVP (2.8) is positive on $[0,1]$.*

Proof. Set $v(t) = -u''(t)$. By $v''(t) = -h(t) \leq 0, t \in [0,1]$, we know that $v(t)$ is a concave function on $[0,1]$. Thus, by (2.3) we have

$$v(1) = v(0) = \frac{1}{1 - \int_0^1 b(s)ds} \int_0^1 \left(\int_0^1 G(s,x)b(x)dx \right) h(s)ds \geq 0.$$

On the other hand, due to $u''(t) = -v(t) \leq 0, t \in [0,1]$, we deduce that $u(t)$ is a concave function on $[0,1]$. It follows that by (2.3)

$$u(1) = u(0) = \frac{1}{1 - \int_0^1 a(s)ds} \int_0^1 \left(\int_0^1 G(s,x)a(x)dx \right) h(s)ds \geq 0,$$

which implies that the solution $u(t) \geq 0$.

Lemma 2.6. *Assume that $a(t) \geq 0, b(t) \geq 0,$ and $\alpha, \beta \in (0,1), h \in C^+[0,1]$, then the solution $u(t)$ of NBVP (2.8) satisfies*

$$\min_{t \in [0,1]} u(t) \geq \gamma \|u\|, \tag{2.10}$$

where $\gamma = \gamma_1\gamma_2$, $\|\cdot\|$ *is the supremum norm on* $C^+[0,1]$.

Proof. By Lemma 2.2 and (2.3), we obtain

$$u(t) = \int_0^1 \int_0^1 H_1(t,\tau)H_2(\tau,s)h(s)dsd\tau \leq \int_0^1 \int_0^1 H_1(\tau,\tau)H_2(s,s)h(s)dsd\tau.$$

So,

$$\|u\| \leq \int_0^1 \int_0^1 H_1(\tau,\tau)H_2(s,s)h(s)dsd\tau. \tag{2.11}$$

On the other hand, by Lemma 2.2 and (2.3) we have

$$\begin{aligned} u(t) &= \int_0^1 \int_0^1 H_1(t,\tau)H_2(\tau,s)h(s)dsd\tau \\ &\geq \gamma_1\gamma_2 \int_0^1 \int_0^1 H_1(\tau,\tau)H_2(s,s)h(s)dsd\tau. \\ &= \gamma \int_0^1 \int_0^1 H_1(\tau,\tau)H_2(s,s)h(s)dsd\tau. \end{aligned} \tag{2.12}$$

Combined (2.11) with (2.12), we deduce inequality (2.10).

Now we define an integral operator $T: C[0,1] \to C[0,1]$ by

$$(Tu)(t) = \int_0^1 \int_0^1 H_1(t,\tau)H_2(\tau,s)g(s)f(s,u(s))dsd\tau.$$

Define a set P by

$$P = \left\{ u \in C^+[0,1] : u(t) \text{ is a symmetric and concave function on } [0,1], \min_{t\in[0,1]} x(t) \geq \gamma\|u\| \right\},$$

$\|\cdot\|$ is the supremum norm on $C^+[0,1]$. It is easy to see that P is a cone in $C[0,1]$. Clearly, u is a solution of the NBVP (1.1) if and only if u is a fixed point of the operator T.

In the rest of the paper, we make the following assumptions:

(B_1) $a, b \in L^1[0,1], a(t), b(t) \geq 0, \alpha, \beta \in (0,1)$;

(B_2) $g : (0,1) \to [0,+\infty)$ is continuous, symmetric, and $0 < \int_0^1 H_2(s,s)g(s)ds < +\infty$;

(B_3) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, and $f(\cdot,x)$ is symmetric on $[0,1]$ for all $x \in [0,+\infty)$.

Remark 2.2. (B_2) implies that $g(t)$ may be singular at $t = 0$ and $t = 1$.

Remark 2.3. If (B_1) holds, then for all $t, s \in [0,1]$, we have

$$H_1(1-t,1-s) = H_1(t,s), \quad H_2(1-t,1-s) = H_2(t,s).$$

Lemma 2.7. *Assume that conditions* (B_1), (B_2) *and* (B_3) *hold. Then* $T: P \to P$ *is a completely continuous operator.*

Proof From Lemma 2.4, Lemma 2.5 and Lemma 2.6, we know that $T(P) \subset P$. Now we prove that operator T is completely continuous. For $n \geq 2$ define g_n by

$$g_n(t) = \begin{cases} \inf\{g(t), g(\frac{1}{n})\}, & 0 < t \leq \frac{1}{n}, \\ g(t), & \frac{1}{n} \leq t \leq 1 - \frac{1}{n}, \\ \inf\{g(t), g(1-\frac{1}{n})\}, & 1 - \frac{1}{n} \leq t < 1. \end{cases}$$

Then, $g_n : [0,1] \to [0,+\infty)$ is continuous and $g_n(t) \le g(t), t \in (0,1)$. And $T_n : P \to P$ by

$$(T_n u)(t) = \int_0^1 \int_0^1 H_1(t,\tau)H_2(\tau,s)g_n(s)f(s,u(s))dsd\tau.$$

Obviously, T_n is compact on P for any $n \ge 2$ by an application of the Ascoli- Arzela Theorem. Let $B_R = \{u \in P : \|u\| \le R\}$. We claim that T_n converges uniformly to T as $n \to \infty$ on B_R. In fact, let $M_R = \max\{f(s,x) : (s,x) \in [0,1] \times [0,R]\}, M = \max\{H_1(\tau,t) : \tau \in [0,1]\}$, then $M_R, M < \infty$. Since $0 < \int_0^1 H_2(s,s)g(s) < \infty$, by the absolute continuity of integral, we have

$$\lim_{n\to\infty} \int_{e(\frac{1}{n})} H_2(s,s)g(s)ds = 0,$$

where $e(\frac{1}{n}) = [0,\frac{1}{n}] \cup [1-\frac{1}{n},1]$. So, for any $t \in [0,1]$, fixed $R > 0$ and $u \in B_R$, we have

$$\begin{aligned}
|(T_n u)(t) - (Tu)(t)| &= \left| \int_0^1 \int_0^1 H_1(t,\tau)H_2(\tau,s)(g_n(s) - g(s))f(s,u(s))dsd\tau \right| \\
&\le MM_R \int_0^1 H_2(s,s)|g_n(s) - g(s)|ds \\
&\le MM_R \int_{e(\frac{1}{n})} H_2(s,s)g(s)ds \\
&\to 0 \ (n \to \infty),
\end{aligned}$$

where we have used assumptions (B_1)-(B_3) and the fact that $H_2(t,s) \le H_2(s,s)$ for $t,s \in [0,1]$. Hence the completely continuous operator T_n converges uniformly to T as $n \to \infty$ on any bounded subset of P, and therefore T is completely continuous.

3 The Main Results

We first define the nonnegative, continuous concave functional $\varphi : P \to [0,\infty)$ by

$$\varphi(u) = \min_{t\in[0,1]} u(t).$$

Obviously, for every $u \in P$ we have

$$\varphi(u) \le \|u\|.$$

We shall use the following notation:

$$\Lambda = \frac{1}{\int_0^1 \int_0^1 H_1(\tau,\tau)H_2(s,s)g(s)dsd\tau}.$$

Our main result is the following theorem.

Theorem 3.1. *Suppose conditions* (B_1), (B_2) *and* (B_3) *hold, and there exist positive constants* a, b *and* c *with* $0 < a < b < \gamma c$ *such that*

(A_1) $f(t,u) < \Lambda c$, for $t \in [0,1], 0 \le u \le c$;
(A_2) $f(t,u) \ge \dfrac{\Lambda b}{\gamma}$, for $t \in [0,1], b \le u \le \dfrac{b}{\gamma}$;
(A_3) $f(t,u) \le \Lambda a$, for $t \in [0,1], 0 \le u \le a$.
Then the NBVP(1.1) has at least three symmetric positive solutions u_1, u_2 and u_3 such that

$$\|u_1\| < a, b < \varphi(u_2), \text{ and } \|u_3\| > a \text{ with } \varphi(u_3) < b.$$

Proof. we show that all the conditions of Theorem 2.1 are satisfied. We first assert that there exists a positive number c such that $T(\overline{P}_c) \subset \overline{P}_c$. By (A_1) we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]}(Tu)(t) \\
&= \max_{t\in[0,1]} \int_0^1 \int_0^1 H_1(t,\tau)H_2(\tau,s)g(s)f(s,u(s))dsd\tau \\
&\le \Lambda c \int_0^1 \int_0^1 H_1(t,\tau)H_2(\tau,s)g(s)dsd\tau \\
&\le \Lambda c \int_0^1 \int_0^1 H_1(\tau,\tau)H_2(s,s)g(s)dsd\tau \\
&= c.
\end{aligned}$$

Therefore, we have $T(\overline{P}_c) \subset \overline{P}_c$. Especially, if $u \in \overline{P}_a$, then assumption (A_3) yields $T : \overline{P}_a \to P_a$.

We now show that condition (i) of Theorem 2.1 is satisfied. Clearly, $\{u \in P(\varphi,\ b,\ \frac{b}{\gamma})|\ \ \varphi(u) > b\} \ne \emptyset$. Moreover, if $u \in P(\varphi,\ b,\ \frac{b}{\gamma})$, then $\varphi(u) \ge b$, so $b \le \|u\| \le \frac{b}{\gamma}$. By the definition of φ and (A_2), we obtain

$$\begin{aligned}
\varphi(Tu) &= \min_{t\in[0,1]}(Tu)(t) \\
&= \min_{t\in[0,1]} \int_0^1 \int_0^1 H_1(t,\tau)H_2(\tau,s))g(s)f(s,u(s))dsd\tau \\
&\ge \int_0^1 \int_0^1 \gamma H_1(\tau,\tau)H_2(s,s)g(s)f(s,u(s))dsd\tau \\
&\ge \int_0^1 \int_0^1 \gamma H_1(\tau,\tau)H_2(s,s)g(s)\gamma^{-1}\Lambda b dsd\tau \\
&= b.
\end{aligned}$$

Therefore, condition (i) of Theorem 2.1 is satisfied.

Finally, we address condition (iii) of Theorem 2.1. For this we choose $u \in P(\varphi,\ b,\ c)$ with $\|Tu\| > \frac{b}{\gamma}$. Then from Lemma 2.6, we deduce

$$\varphi(Tu) = \min_{t\in[0,1]}(Tu)(t) \ge \gamma\|Tu\| > b.$$

Hence, condition (iii) of Theorem 2.1 holds. By Theorem 2.1, we obtain the NBVP(1.1) has at least three symmetric positive solutions u_1, u_2 and u_3 such that

$$\|u_1\| < a, b < \varphi(u_2), \text{ and } \|u_3\| > a \text{ with } \varphi(u_3) < b.$$

4 Examples

In the section, we present a simple example to explain our results.
Example 4.1. Consider the following fourth-order singular nonlocal boundary value problems (NBVP)

$$\begin{cases} u''''(t) = \dfrac{1}{6t(1-t)}(\dfrac{1}{2}|1-2t| + 2\min\{u^2, \sqrt{2u}\}), \qquad 0 < t < 1, \\ u(0) = u(1) = \dfrac{48}{25}\displaystyle\int_0^1 su(s)ds, \\ u''(0) = u''(1) = \dfrac{48}{25}\displaystyle\int_0^1 su''(s)ds, , \end{cases} \tag{4.1}$$

where $g(t) = \frac{1}{6t(1-t)}$, $a(t) = b(t) = \frac{48}{25}t$, and $f(t,u) = \frac{1}{2}|1-2t| + 2\min\{u^2, \sqrt{2u}\}$, then $g(t), a(t), b(t)$ and $f(t,u)$ satisfy the assumptions (B_1)-(B_3). A direct computation shows

$$\alpha = \beta = \frac{24}{25}, \eta_1 = \eta_2 = \frac{4}{25}, \gamma = \frac{16}{25}, \Lambda = \frac{12}{5}.$$

We choose $a = \frac{1}{2},\ b = \frac{8}{15},\ c = 8$. Obviously, $a < b < \gamma c$. Moreover,
(i) for $(t,x) \in [0,1] \times [0,c]$, we have
$f(t,u) \le f(1,c) = \frac{1}{2} + 4\sqrt{2} < \frac{12}{5} \times 8 = \Lambda c$;
(ii) for $(t,x) \in [0,1] \times [b, \gamma^{-1}b]$, we have
$f(t,u) \ge f(\frac{1}{2}, b) = \frac{8}{15}\sqrt{15} > 2 = \gamma^{-1}b\Lambda$;
(iii) for $(t,x) \in [0,1] \times [0,a]$, we have
$f(t,u) \le f(1,a) \le \frac{1}{2} + \frac{2}{4} < \frac{12}{5} \times \frac{1}{2} = \Lambda a.$

By Theorem 3.1, we know the NBVP (4.1) has at least three positive solutions.

Acknowledgments The authors are grateful to the referees for their valuable suggestions and comments.

References

[1] J.M. Gallardo, Second order differential operators with integral boundary conditions and generation of semigroups, Rocky Mountain J. Math. 30 (2000) 1265-1292.

[2] G.L. Karakostas and P.Ch. Tsamatos, Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary value problems, Electron. J. Diff. Equ. 30 (2002) 1-17.

[3] A. Lomtatidze and L. Malaguti, On a nonlocal boundary-value problems for second order nonlinear singular differential equations, Georgian Math. J. 7 (2000) 133-154.

[4] C. Corduneanu, Integral Equations and Applications, Cambridge University Press, Cambridge, UK, (1991).

[5] R.P. Agarwal and D. O'Regan, Infinite Interval Problems for Differential, Difference and Integral Equations, Kluwer Academic Publishers, Dordrecht, The Netherlands, (2001).

[6] X.G. Zhang, L.S. Liu and H.C. Zou, Positive solutions of fourth-order singular three point eigenvalue problems, Appl. Math. Comput. 189 (2007) 1359-1367.

[7] Z.C. Hao, L.S. Liu and L. Debnath, A necessary and sufficient condition for the existence of positive solutions of fourth-order singular boundary value problems, Appl. Math. Lett. 16(3)(2003) 279-285.

[8] S. Timoshenko, Theory of Elastic Stability, Mc. Graw, New York, 1961.

[9] M.A. Krasnoselskii, Positive Solution of Operator Equations, Noordhoof, Gronignen, 1964.

[10] J.R. Graef, C. Qian and B. Yang, A three point boundary value problem for nonlinear fourth order equations, J. Math. Anal. Appl. 287 (2003) 217-233.

[11] F.Y. Xu, H. Su and X.Y. Zhang, Positive solutions of fourth-order nonlinear singular boundary value problems, Nonlinear. Anal. 68(5)(2008) 1284-1297.

[12] D.J. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cone, Academic Press, San Diego, 1988.

[13] X.G. Zhang, L.S. Liu and H.C. Zou, Eigenvalues of fourth-order singular Sturm-Liouville boundary value problems. Nonlinear. Anal. 68(2)(2008) 384-392.

[14] F.Y. Xu and J. Liu, Symmetric positive solutions for nonlinear singular fourth-order eigenvalue problems with nonlocal boundary condition, Discrete Dynamics in Nature and Society, Volume 2010, Article ID 187827, 16 pages.

[15] J. Henderson and H.B. Thompson, Multiple symmetric positive solutions for second order boundary value problems, Proc. Amer. Math. Soc. 128 (2000) 2373-2379.

[16] R.W. Legget and R.L. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, Indiana. Univ. Math. J. 28 (1979) 673-688.

[17] Y.P. Sun, Positive solutions of singular third-order three-point boundary value problem, J. Math. Anal. Appl. 306 (2005) 589-603.

[18] Y.P. Sun, Existence and multiplicity of symmetric positive solutions for three-point boundary value problem, J. Math. Anal. Appl. 329 (2007) 998-1009.

(Received August 8, 2011)