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COMPACTNESS METHODS FOR HÖLDER ESTIMATES OF CERTAIN DEGENERATE ELLIPTIC EQUATIONS

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Abstract. In this paper we obtain the interior $C^{1,\alpha}$ regularity of the quasilinear elliptic equations of divergence form. Our basic tools are the elementary local L^∞ estimates and weak Harnack inequality for second-order linear elliptic equations, and the compactness method.

1. Introduction

In this paper we consider the following nonlinear elliptic problem

$$\operatorname{div}\left(g\left(|\nabla u|^2\right)\nabla u\right)=0 \quad \text{in } \Omega. \tag{1.1}$$

Here $g \in C^1([0,\infty))$ satisfies the following ellipticity condition

$$K^{-1}(Q+s)^{\frac{p}{2}-1} \le g(Q)+2g'(Q)Q \le K(Q+s)^{\frac{p}{2}-1}, \tag{1.2}$$

for $s \ge 0$ and $1<p<\infty$. In fact, condition (1.2) implies the following condition for a possibly larger constant K

$$K^{-1}(Q+s)^{\frac{p}{2}-1} \le g(Q)+2g'(Q)Q \le K(Q+s)^{\frac{p}{2}-1} \tag{1.3}$$

$$K^{-1}(Q+s)^{\frac{p}{2}-1} \le g(Q) \le K(Q+s)^{\frac{p}{2}-1} \tag{1.4}$$

$$|g'(Q)Q| \le K(Q+s)^{\frac{p}{2}-1}. \tag{1.5}$$

Especially when $g(x)=x^{\frac{p-2}{2}}$, (1.1) is reduced to

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)=0 \quad \text{in } \Omega, \tag{1.6}$$

which can be derived from the variational problem

$$\Phi(u)=\min_{v|_{\partial\Omega}=g}\Phi(v)=:\min_{v|_{\partial\Omega}=g}\int_\Omega |\nabla v|^p\,dx.$$

As usual, the solutions of (1.1) are taken in a weak sense. We now state the definition of weak solutions.

Definition 1.1. *A function $u \in W^{1,p}_{loc}(\Omega)$ is a local weak solution of (1.1) if for any $\varphi \in W^{1,p}_0(\Omega)$ we have*

$$\int_\Omega g\left(|\nabla u|^2\right)\nabla u\cdot\nabla\varphi\,dx=0.$$

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Evans [6] have shown that ∇u is local Hölder continuous for weak solutions of (1.6) for $p \geq 2$ and then Lewis [9] extended the corresponding result to the case that $1 < p < \infty$. Moreover, Uhlenbeck [10] obtained the interior $C^{1,\alpha}$ regularity estimates for weak solutions of (1.1) with condition (1.2) and

$$|\rho'(Q_1)Q_1 - \rho'(Q_2)Q_2| \leq K\left(Q_1 + Q_2 + s\right)^{p/2-1-\beta}\left(Q_1 - Q_2\right)^{\beta}$$

for $s \geq 0$, $\beta > 0$ and $p \geq 2$, and DiBenedetto [3] considered the more general equations. Moreover, Wang [12] used compactness methods to give a quick proof of the interior $C^{1,\alpha}$ regularity for weak solutions of (1.6) for $1 < p < \infty$. Recently, Duzaar and Mingione [4,5] proved local Lipschitz regularity of the gradient for weak solutions of (1.1) for $1 < p < \infty$ and the more general equations. In this paper we will prove the interior $C^{1,\alpha}$ regularity for weak solutions of (1.1) with condition (1.2) by a compactness method, which is introduced by the authors (see [1, 11, 12, 13]). Our basic tools are the elementary local L^∞ estimates and weak Harnack inequality for second-order linear elliptic equations, and the compactness method.

The essence of $C^{1,\alpha}$ regularity of the solution is that the solution is almost a linear function. Actually, we can show that the difference between the solution and a linear function is like $|x|^{1+\alpha}$. Moreover, we can use the same method to prove $C^{k,\alpha}$ estimates for the solution if we replace the linear function by the k-th order polynomial function.

Definition 1.2. (1) *We call $u \in C_p^\alpha$ at the point $x = 0$ for $1 < p < \infty$ and $0 < \alpha < 1$ if*

$$[u]_{C_p^\alpha(0)} = \sup_{0<r\leq 1} \frac{1}{r^\alpha}\left(\fint_{B_r} |u - \overline{u}_{B_r}|^p \ dx\right)^{\frac{1}{p}} < \infty,$$

where $\overline{u}_{B_r} = \frac{1}{|B_r|}\int_{B_r} u \ dx$.

(2) *We call $u \in C_p^{1,\alpha}$ at the point $x = 0$ for $1 < p < \infty$ if there is a linear function $L(x) = Ax + B$ such that*

$$[u]_{C_p^{1,\alpha}(0)} = \sup_{0<r\leq 1} \frac{1}{r^{1+\alpha}}\left(\fint_{B_r} |u - L|^p \ dx\right)^{\frac{1}{p}} < \infty.$$

Now let us state the main result of this work.

Theorem 1.3. *If $u \in W^{1,p}_{loc}(B_1)$ is a weak solution of (1.1) with condition (1.2), then $u \in C_p^{1+\alpha}(0)$ for some $\alpha \in (0, 1)$.*

Remark 1.4. *If $u \in C_p^{1+\alpha}(0)$, then by Theorem 1.3, page 72 in [7], u is locally $C^{1,\alpha}$ in the classical sense.*

2. Compactness method

In this section we will finish the proof of Theorem 1.3 by the compactness method. We first consider the following approximation problem

$$\operatorname{div}\left(g\left(\epsilon + |\nabla u^\epsilon|^2\right)\nabla u^\epsilon\right) = 0, \qquad x \in \Omega,\ \epsilon \in (0, 1]. \tag{2.1}$$

We shall show uniform $C^{1,\alpha}$ estimates in Theorem 1.3 for u^ϵ for small $\epsilon > 0$. We will omit the index ϵ since the $C^{1,\alpha}$ estimates are uniform, and then $u^\epsilon \to u$ uniformly. Actually, from (2.1) we have

$$a_{ij}u_{ij} =: \left[g\left(\epsilon + |\nabla u|^2\right)\delta_{ij} + g'\left(\epsilon + |\nabla u|^2\right) 2u_iu_j \right] u_{ij} = 0. \tag{2.2}$$

Now we denote $\widetilde{a_{ij}}$ by

$$\widetilde{a_{ij}} = \frac{g\left(\epsilon + |\nabla u|^2\right)\delta_{ij} + g'\left(\epsilon + |\nabla u|^2\right) 2u_iu_j}{\left(s + \epsilon + |\nabla u|^2\right)^{\frac{p}{2}-1}}. \tag{2.3}$$

Then from (1.3)-(1.5) we have

$$K^{-1}|\xi|^2 \le \widetilde{a_{ij}}\xi_i\xi_j \le 3K|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n,$$

and

$$\widetilde{a_{ij}}u_{ij} = 0.$$

Lemma 2.1. *If u is a local weak solution of (2.1) in B_1, then*

$$\|\nabla u\|_{L^\infty(B_{1/2})} \le C\left(\|\nabla u\|_{L^p(B_1)} + 1\right),$$

where C is independent of ϵ.

Proof. Let $v = \left(s + \epsilon + |\nabla u|^2\right)^{p/2}$. Then we find that

$$\left(\widetilde{a_{ij}}v_j\right)_i = \left(pa_{ij}u_{kj}u_k\right)_i. \tag{2.4}$$

Moreover, differentiating (2.1) with respect to x_k, we have

$$\left(a_{ij}u_{kj}\right)_i = 0.$$

Furthermore, (2.3) and (2.4) imply that

$$\left(\widetilde{a_{ij}}v_j\right)_i = pa_{ij}u_{kj}u_{ki} \ge 0. \tag{2.5}$$

Therefore, from the maximum principle (see Lemma 1.2, Chapter 4 in [2])) we obtain

$$\|\nabla u\|^p_{L^\infty(B_{1/2})} \le \|v\|_{L^\infty(B_{1/2})} \le C\left(\|\nabla u\|^p_{L^p\left(B_{3/4}\right)} + 1\right),$$

which finishes our proof. □

From the lemma above, we may as well assume that

$$|\nabla u| \le 1.$$

Lemma 2.2. *Let u be a local weak solution of (2.1) in B_1 and $|\nabla u| \le 1$. For any $\sigma > 0$, there exists an $\eta(\sigma) > 0$ such that if*

$$|\{x \in B_1 : |\nabla u| \le 1 - \eta\}| \le \eta |B_1|,$$

then there is a harmonic function v such that

$$\int_{B_{1/2}} |u - v|^p \, dx \le \sigma.$$

Proof. We prove it by contradiction. If the result is false, then there would exist $\sigma_0 > 0$, $\{\epsilon_k\}_{k=1}^{\infty}$ and $\{u_k\}_{k=1}^{\infty}$ satisfying

$$\begin{aligned}
&\int_{B_1} g\left(\epsilon_k + |\nabla u_k|^2\right) \nabla u_k \cdot \nabla \phi \; dx = 0 \quad \text{for any} \quad \phi \in C_0^{\infty}(B_1), \\
&|\nabla u_k| \le 1, \\
&|D_k| \le \frac{1}{2^k} |B_1|, \text{ where } D_k = \left\{ x \in B_1 : |\nabla u_k| \le 1 - \frac{1}{2^k} \right\},
\end{aligned}$$

so that for any harmonic function v in $B_{1/2}$ we have

$$\int_{B_{1/2}} |u - v|^p \; dx \ge \sigma_0. \tag{2.6}$$

Hence, we may assume that

$$\begin{aligned}
&\epsilon_k \to \epsilon_0, \\
&u_k \to v \qquad \text{in } L^p(B_1), \\
&\nabla u_k \to \nabla v \quad \text{weakly in } L^p(B_1), \\
&|\nabla u_k| \to 1 \quad \text{in } B_1 \setminus D_k.
\end{aligned}$$

Since

$$\left\{ \int_{B_1 \setminus D_k} + \int_{D_k} \right\} g\left(\epsilon_k + |\nabla u_k|^2\right) \nabla u_k \cdot \nabla \phi \; dx = 0,$$

we deduce that

$$\int_{B_1} g\left(\epsilon_0 + 1\right) \nabla v \cdot \nabla \phi \; dx = 0$$

as $k \to \infty$. That is to say, v is a harmonic function, which is contradictory to (2.6). Thus, we complete the proof. □

Lemma 2.3. *Let u be a local weak solution of (2.1) in B_1 with $|\nabla u| \le 1$. If*

$$|\{x \in B_1 : |\nabla u| \le 1 - \eta\}| \ge \eta |B_1|,$$

then

$$|\nabla u| \le 1 - \eta^2/C \quad \text{in } B_{1/2},$$

where C is independent of ϵ.

Proof. Let $w = (s + \epsilon + 1)^{p/2} - \left(s + \epsilon + |\nabla u|^2\right)^{p/2} \ge 0$. Then w is a local weak solution of

$$\left(\widetilde{a_{ij}} w_j\right)_i = -p a_{ij} u_{kj} u_{ki} \le 0 \text{ in } B_1,$$

in view of (2.5). Therefore, from Theorem 8.18 in [8] we have

$$\inf_{B_{1/2}} w \ge \frac{1}{C} \int_{B_1} w \; dx,$$

which implies that

$$\inf_{B_{1/2}} \left((s + \epsilon + 1)^{p/2} - \left(s + \epsilon + |\nabla u|^2\right)^{p/2} \right)$$

$$\geq \frac{1}{C}\int_{B_1} (s+\epsilon+1)^{p/2} - \left(s+\epsilon+|\nabla u|^2\right)^{p/2}\ dx$$
$$\geq \frac{\eta}{C}\left((s+\epsilon+1)^{p/2} - \left(s+\epsilon+(1-\eta)^2\right)^{p/2}\right).$$

Thus we can easily obtain the desired result by using the elementary inequality $(1-x)^{\theta} \leq 1 - C\theta x$ for $0 < x < 1/2$ and $\theta > 0$. □

Corollary 2.4. *Let $\delta_0 = \eta^2/C$ as in the lemma above. Assume that u is a local weak solution of (2.1) in B_1 with $|\nabla u| \leq 1$. If*

$$\left|\left\{x \in B_{1/2^i} : |\nabla u| \leq (1-\eta)(1-\delta_0)^i\right\}\right| \geq \eta\left|B_{1/2^i}\right| \ \text{for } i = 0, 1, ..., k,$$

then

$$|\nabla u| \leq (1-\delta_0)^i \quad \text{in } B_{1/2^i} \text{ for } i = 1, 2, ..., k+1,$$

where C is independent of ϵ.

Proof. We can prove by induction on i. From the lemma above, it is easy to check that our conclusion is valid for $i = 0$. Assume that the conclusion is valid for some i. We denote $w_1(x)$ by

$$w_1(x) = \frac{2^i}{(1-\delta_0)^i} u\left(\frac{x}{2^i}\right).$$

Then we can obtain the result from the lemma above. □

Lemma 2.5. *Let u be a local weak solution of (2.1) in B_1 with $|\nabla u| \leq 1$, $⨍_{B_1} |u|^p\, dx \leq 1$ and*

$$|\{x \in B_1 : |\nabla u| \leq 1-\eta\}| \leq \eta\,|B_1|.$$

(1) *For any $0 < \alpha < 1$ and $\theta > 0$, there exist $\eta > 0$ and $r_0 \in (0, 1/4)$ depending on θ, α, p, and a linear function $L_1(x) = A_1 x + B_1$ such that*

$$⨍_{B_{r_0}} |u - L_1|^p\ dx \leq \theta r_0^{p(1+\alpha)}.$$

(2) *For any $0 < \alpha < 1$, there exist $\eta > 0$ and $r_0 \in (0, 1/4)$ depending on α, p, and linear functions $L_k(x) = A_k x + B_k$ for $k = 0, 1, 2, 3, ...$, with uniformly bounded coefficients such that*

$$⨍_{B_{r_0^k}} |u - L_k(x)|^p\ dx \leq r_0^{pk(1+\alpha)} \tag{2.7}$$

and

$$|A_{k+1} - A_k| \leq C r_0^{pk\alpha}, \tag{2.8}$$
$$|B_{k+1} - B_k| \leq C r_0^{pk(1+\alpha)}. \tag{2.9}$$

(3) *For any $0 < \alpha < 1$, there exist $\eta > 0$ depending on α, p, and a linear function $L(x) = Ax + B$ such that*

$$⨍_{B_r} |u - L|^p\ dx \leq C r^{p(1+\alpha)} \quad \text{for any } 0 < r \leq 1.$$

Proof. (1) For any $\sigma > 0$, from Lemma 2.2 there exists $\eta = \eta(\sigma) > 0$ such that

$$\int_{B_{1/2}} |u - v|^p \; dx \leq \sigma, \tag{2.10}$$

where v is a harmonic function in B_1. Since $u \in W^{1,p}_{loc}(B_1)$ is a weak solution of (2.1), then

$$\int_{B_{1/2}} |v|^p \; dx \leq C,$$

which implies that

$$\sup_{B_{1/4}} \left| D^2 v \right| \leq C.$$

Now, let $L_1(x) = A_1 x + B_1$ be the Taylor polynomial of v at 0. Then we have

$$\sup_{x \in B_{1/4}} |v - L_1| \leq C|x|^2.$$

Therefore, for any $0 < r < 1/4$ we have

$$\begin{aligned} ⨍_{B_r} |u - L_1|^p \; dx \quad &\leq \quad 2^{p-1} \left(⨍_{B_r} |u - v|^p \; dx + ⨍_{B_r} |v - L_1|^p \; dx \right) \\ &\leq \quad 2^{p-1} \frac{\sigma}{|B_r|} + 2^{p-1} r^{2p}, \end{aligned}$$

which implies that

$$⨍_{B_{r_0}} |u - L_1|^p \; dx \leq 2^p r^{2p},$$

by taking σ small enough such that $\sigma \leq r^{2p} |B_r|$. Finally, choosing $r = r_0$ such that $2^p r_0^{p(1-\alpha)} = \theta$, we can finish the proof.

(2) We prove it by induction. From (1) we know the result is true for $k = 0, 1$, if we take $L_0 = 0$. Let us assume it is true for k. We denote $w(x)$ by

$$w(x) = \frac{(u - L_k)\left(r_0^k x\right)}{\theta r_0^{k(\alpha+1)}}.$$

Then w satisfies

$$\widetilde{a_{ij}}(w) w_{ij} = 0, \quad x \in B_1.$$

where

$$\begin{aligned} \widetilde{a_{ij}}(w) = &\frac{g\left(\epsilon + \left|\theta r_0^{k\alpha} \nabla w + A_k\right|^2\right) \delta_{ij}}{\left(s + \epsilon + \left|\theta r_0^{k\alpha} \nabla w + A_k\right|^2\right)^{\frac{p}{2}-1}} \\ &+ \frac{g'\left(\epsilon + \left|\theta r_0^{k\alpha} \nabla w + L_k\right|^2\right) 2 \left(\theta r_0^{k\alpha} w_i + (A_k)_i\right) \left(\theta r_0^{k\alpha} w_j + (A_k)_j\right)}{\left(s + \epsilon + \left|\theta r_0^{k\alpha} \nabla w + A_k\right|^2\right)^{\frac{p}{2}-1}}. \end{aligned}$$

Let v be the solution of

$$\widetilde{a_{ij}}(v) v_{ij} = 0,$$

with $v|_{B_{1/2}} = w$, where

$$\widetilde{a_{ij}}(v) = \frac{g\left(\epsilon + |A_k|^2\right)\delta_{ij}}{\left(s+\epsilon+|A_k|^2\right)^{\frac{p}{2}-1}} + \frac{g'\left(\epsilon + |A_k|^2\right)2(A_k)_i(A_k)_j}{\left(s+\epsilon+|A_k|^2\right)^{\frac{p}{2}-1}}.$$

Since $g \in C^1$, $\|\widetilde{a_{ij}}(w) - \widetilde{a_{ij}}(v)\|_{L^\infty(B_1)}$ is small enough if we choose θ small enough. For any $\tau > 0$, from Lemma 13 in [1] we can obtain

$$\|w - v\|_{L^\infty(B_{1/2})} \le \tau,$$

by choosing θ small enough. Now, let $L^*(x) = A^*x + B^*$ be the Taylor polynomial of v at 0. Then we have

$$\sup_{x\in B_r} |v - L^*| \le Cr^2 \quad \text{for any } r \in (0, 1/4).$$

Furthermore, choosing $\tau \le r_0^{p(1+\alpha)}$, we find that

$$\fint_{B_{r_0}} |w - L^*|^p \; dx \le \tau + Cr_0^{2p} \le Cr_0^{p(1+\alpha)}.$$

Finally, from the definition of w we can obtain

$$\fint_{B_{r_0^{k+1}}} |w - L_{k+1}|^p \; dx \le Cr_0^{p(k+1)(1+\alpha)},$$

by taking $L_{k+1} = L_k - \theta r_0^{k(\alpha+1)} L^*\left(\frac{x}{r_0^k}\right)$. Thus, (2.7)-(2.9) are true.

(3) From (2) it is easy to see that A_k, B_k converge to A_∞, B_∞ as $k \to \infty$ respectively. Now let $L(x) = A_\infty x + B_\infty$. Then we have

$$\fint_{B_{r_0^k}} |u - L(x)|^p \; dx \le r_0^{pk(1+\alpha)} \quad \text{for } k = 0, 1, 2, \ldots.$$

Therefore, we have

$$\fint_{B_r} |u - L(x)|^p \; dx \le r^{p(1+\alpha)} \quad \text{for any } 0 < r \le 1,$$

which completes our proof. □

Now we are ready to prove the main result, Theorem 1.3.

Proof. We may as well assume that $u(0) = 0$ and $\fint_{B_1} |u|^p \; dx \le 1$. We denote k by

$$\left|\left\{x \in B_{1/2^i} : |\nabla u| \le (1-\eta)(1-\delta_0)^i\right\}\right| \ge \eta\left|B_{1/2^i}\right|, \quad i = 0, 1, 2, \ldots, k-1, \tag{2.11}$$

but,

$$\left|\left\{x \in B_{1/2^k} : |\nabla u| \le (1-\eta)(1-\delta_0)^k\right\}\right| \le \eta\left|B_{1/2^k}\right|. \tag{2.12}$$

We divide into two cases:

Case 1: $k = \infty$. That is to say, (2.11) is true for any i. Then, from Corollary 2.4 we find that

$$|\nabla u| \le (1-\delta_0)^i \quad \text{in } B_{1/2^i},$$

which implies that

$$|u(x)| = |u(x) - u(0)| \leq |x|(1-\delta_0)^i \leq \frac{1}{1-\delta_0}|x|^{1+\alpha_0} \quad \text{for } |x| \leq 1,$$

where $\alpha_0 = -\log_2(1-\delta_0)$. Now fix an α and then determine δ_0 and α_0. Let $\alpha_1 = \min\{\alpha_0, \alpha\}$. Therefore, we have

$$|u(x)| \leq C|x|^{1+\alpha_0} \leq C|x|^{1+\alpha_1} \quad \text{for } |x| \leq 1.$$

Case 2: $k < \infty$. Similarly, Corollary 2.4 implies that

$$|\nabla u| \leq (1-\delta_0)^i \quad \text{in } B_{1/2^i} \quad \text{for } \ 0 \leq i \leq k, \tag{2.13}$$

which implies that

$$|u(x)| \leq C|x|^{1+\alpha_1} \quad \text{in } B_{1/2^i} \quad \text{for } \ 0 \leq i \leq k.$$

Now we denote w by

$$w(x) = \frac{2^k}{(1-\delta_0)^k} u\left(\frac{x}{2^k}\right).$$

Therefore, by Lemma 2.5 (3) and the definition of α_1, there is a linear function $L(x) = Ax + B$ such that

$$\fint_{B_r} |w - L|^p \ dx \leq Cr^{p(1+\alpha)} \leq Cr^{p(1+\alpha_1)}$$

for any $0 < r \leq 1$. Recalling the definition of w, we have

$$\fint_{B_r} \left| u(x) - (1-\delta_0)^k Ax - \frac{(1-\delta_0)^k B}{2^k} \right|^p \ dx \leq Cr^{p(1+\alpha_1)} \tag{2.14}$$

for any $0 < r \leq 1/2^k$. Moreover, for any $1/2^k < r \leq 1$ we have

$$\begin{aligned} &\fint_{B_r} \left| u(x) - (1-\delta_0)^k Ax - \frac{(1-\delta_0)^k B}{2^k} \right|^p \ dx \\ &\leq C\left(\sup_{B_r} |u|^p + \left|(1-\delta_0)^k Ar\right|^p + \left|\frac{(1-\delta_0)^k B}{2^k}\right|^p\right) \\ &\leq Cr^{p(1+\alpha_1)}, \end{aligned}$$

since $(1-\delta_0)^k = 2^{-k\alpha_0} \leq r^{\alpha_0} \leq r^{\alpha_1}$. □

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