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Blowup solutions and their blowup rates for parabolic equations with non-standard growth conditions

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Abstract. This paper concerns classical solutions for homogeneous Dirichlet problem of parabolic equations coupled via exponential sources involving variable exponents. We first establish blowup criteria for positive solutions. And then, for radial solutions, we obtain optimal classification for simultaneous and non-simultaneous blowup, which is represented by using the maxima of the involved variable exponents. At last, all kinds of blowup rates are determined for both simultaneous and non-simultaneous blowup solutions.
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1 Introduction

In this paper, we consider the homogeneous Dirichlet problem of parabolic equations, involving non-standard growth conditions,

$$u_t = \Delta u + \mathrm{e}^{m(x)u+p(x)v}, \quad v_t = \Delta v + \mathrm{e}^{q(x)u+n(x)v}, \quad (x,t) \in \Omega \times (0,T) \tag{1.1}$$

with initial data $u(x,0) = u_0(x) \geq 0$ and $v(x,0) = v_0(x) \geq 0$ in Ω, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $\partial\Omega \in C^\infty$; variable exponents $m(x)$, $n(x)$, $p(x)$, $q(x)$ are positive continuous functions on $\bar{\Omega}$; initial data u_0, v_0 are smooth on $\bar{\Omega}$ and satisfy the compatibility conditions on $\partial\Omega$. The existence and uniqueness of local classical solutions to (1.1) is well-known (see, for example, [1]), and T represents the maximal existence time of classical solutions. The nonlinear parabolic problems like (1.1) come from several branches of applied mathematics and physics, such as, flows of electrorheological or thermo-rheological fluids [2, 3, 4], and the processing of digital images [5, 6, 7]. For more detail information, the interested readers can refer to books [8, 9].

Zheng, Zhao, and Chen [10] discussed the parabolic equations

$$u_t = \Delta u + \mathrm{e}^{mu+pv}, \quad v_t = \Delta v + \mathrm{e}^{qu+nv}, \quad (x,t) \in \Omega \times (0,T), \tag{1.2}$$

with homogeneous Dirichlet boundary conditions, where $\Omega = B_R = \{x \in R^N : |x| < R\}$; constants m, n, p, q satisfy $0 \leq m < q$ and $0 \leq n < p$. The simultaneous blowup rate for radial blowup solutions was obtained as

$$\mathrm{e}^{u(0,t)} = O\Big((T-t)^{-\frac{p-n}{pq-mn}}\Big), \quad \mathrm{e}^{v(0,t)} = O\Big((T-t)^{-\frac{q-m}{pq-mn}}\Big)$$

under suitable assumptions on initial data. The other known results for the special cases of system (1.2) were studied in [11, 12, 13, 14, 15], etc., where critical blowup exponent, blowup rate, and even

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blowup profile were considered. For the introduced nonlinear term e^{mu} (or e^{nv}), the component u (or v) of the classical solution for (1.2) blows up by itself for $m > 0$ (or $n > 0$) for large initial data. So the non-simultaneous blowup may happen, which is defined as, for example,

$$\limsup_{t\to T} \|u(\cdot,t)\|_\infty = +\infty \quad \text{and} \quad \|v(\cdot,t)\|_\infty < +\infty, \quad t \in [0,T).$$

The simultaneous and non-simultaneous blowup for (1.2) have been studied in [16]. The results are (i) There exist initial data such that non-simultaneous blowup occurs if and only if $m > q$ or $n > p$. (ii) Any blowup is simultaneous if and only if $m \le q$ and $n \le p$. (iii) Both simultaneous and non-simultaneous blowup may occur if and only if $m > q$ and $n > p$. (iv) Any blowup is non-simultaneous if and only if $m > q$ and $n \le p$, or $m \le q$ and $n > p$. Moreover, blowup rates are obtained.

For the parabolic problem with non-standard growth conditions, Pinasco [17] in 2009 considered the following parabolic problems

$$u_t = \Delta u + a(x)u^{p(x)} \quad \text{or} \quad a(x)\int_\Omega u^{p(x)}(x,t)\mathrm{d}x, \quad (x,t) \in \Omega \times (0,T),$$

subject to homogeneous Dirichlet boundary conditions, where $\Omega \subset \mathbf{R}^N$ is a bounded domain with $1 < p_- \le p(x) \le p_+ < +\infty$ and $0 < c_- \le a(x) \le c_+ < +\infty$. Here, notations $p_+ = \sup_{x\in\Omega} p(x)$ and $p_- = \inf_{x\in\Omega} p(x)$. They obtained the solution u blows up in finite time in sense of $\lim_{t\to T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = +\infty$ for large initial data.

Antontsev and Shmarev [18] discussed the evolution $p(x)$-Laplace parabolic equation

$$u_t = \mathrm{div}(a(x,t)|\nabla u|^{p(x)-2}\nabla u) + b(x,t)|u|^{\sigma(x,t)-2}u, \quad (x,t) \in \Omega \times (0,T),$$

subject to null Dirichlet boundary condition, with variable functions $p(x)$, $\sigma(x,t) \in (1,+\infty)$. If $p(x) \equiv 2$, $a(x,t) \equiv 1$, and $b(x,t) \ge b^- > 0$ (i.e. the semilinear equation), blow-up happens if the initial data are sufficiently large and either $\min_{x\in\Omega} \sigma(x,t) = \sigma^-(t) > 2$ for all $t > 0$, or $\sigma^-(t) \ge 2$, $\sigma^-(t) \searrow 2$ as $t \to \infty$ and $\int_1^\infty e^{s(2-\sigma^-(s))}\mathrm{d}s < \infty$. For the Laplace equation with the exponents $p(x)$ and $\sigma(x)$, they proved that every solution, corresponding to sufficiently large initial data, exhibits blow-up if $b(x,t) \ge b^- > 0$, $a_t(x,t) \le 0$, $b_t(x,t) \ge 0$, $\min_{x\in\Omega} \sigma(x) > 2$, $\max_{x\in\Omega} p(x) \le \min_{x\in\Omega} \sigma(x)$.

In work [19], Ferreira, Pablo, Pérez-Llanos, and Rossi discussed the homogeneous Dirichlet problem of $u_t = \Delta u + u^{p(x)}$ and also its corresponding Cauchy problem in $\mathbf{R}^N$. They obtained some interesting results for nonnegative $p(x)$ as follows, for $\Omega = \mathbf{R}^n$ or bounded Ω, if $p_+ > 1$, there exist blow-up solutions, while if $p_+ \le 1$, then every solution is global. For the Cauchy problem, if $p_- > 1 + 2/N$, there exist global nontrivial solutions; If $1 < p_- < p_+ \le 1 + 2/N$, all solutions blow up; If $p_- < 1 + 2/N < p_+$, there are functions $p(x)$ such that the problem possesses global nontrivial solutions and functions $p(x)$ such that all solutions blow up. Two more results of global solutions were obtained: If $\Omega \subset B_r(x_0)$ for some $x_0 \in \mathbf{R}^N$ and $r < \sqrt{2N}$, then the problem possesses global nontrivial solutions, regardless of the exponent $p(x)$; If $p_- > 1$, then there are global solutions, regardless of the size of Ω. The authors of [19] found out some new phenomena in bounded domains, which are quite different from the corresponding parabolic problems without variable exponents: There are functions $p(x)$ and bounded domains Ω such that positive solutions blow up in finite time for any initial data.

The first results for the homogeneous Dirichlet problem of parabolic equations

$$u_t = \Delta u + v^{p(x)}, \quad v_t = \Delta v + u^{q(x)}, \quad (x,t) \in \Omega \times (0,T),$$

have been obtained by Bai and Zheng [20]. Some criteria are established for distinguishing global and non-global solutions of the problem, depending or independent on initial data. Especially, some Fujita-type result is obtained: there exist suitable domain Ω and variable exponents such that any solution blows up in finite time, just as that of [19].

How to use the four variable exponents to describe blowup classifications of solutions and their blowup rates is worth to be studied for system (1.1). Up to now, few research works discussed simultaneous and non-simultaneous blowup solutions for parabolic systems with non-standard growth conditions. Motivated by this, in the present paper, we discuss the complete and optimal classification for blowup solutions of (1.1) and also consider the blowup rate estimates.

This paper is arranged as follows: in the next section, we give the main results of the present paper; in Section 3, critical blowup criteria are obtained for classical solutions; in Section 4, critical exponents for simultaneous and non-simultaneous blowup are proved; the final section deals with all kinds of blowup rates for solutions.

2 Main Results

The first result concerns the blowup criteria for classical solutions of (1.1). In what follows, we still use the notation of [17] e.g., p_+ to represent $\sup\{p(x), x \in \Omega\}$.

Theorem 2.1 *For problem (1.1), if $m_+ > 0$ or $n_+ > 0$ or $p_+q_+ > 0$, classical solutions of (1.1) blow up in finite time for large initial data. Conversely, if $m_+ = n_+ = p_+q_+ = 0$, every classical solution remains global.*

In order to obtain the simultaneous and non-simultaneous blowup estimates, we assume

(H_1) $\Omega = B_R = \{x \in R^N : |x| < R\}$; both initial data and variable exponents are radially non-increasing in B_R, satisfying $\Delta m, \Delta n, \Delta p, \Delta q \le 0$.

(H_2) $\Delta u_0 + (1-\varepsilon\varphi)\mathrm{e}^{m_+u_0+p_+v_0}$, $\Delta v_0 + (1-\varepsilon\varphi)\mathrm{e}^{q_+u_0+n_+v_0} \ge 0$ in B_R, where $\varepsilon > 0$ is a small constant and φ is the first eigenfunction of the Dirichlet Laplacian in B_R normalized by $\displaystyle\int_{B_R} \varphi(x)\mathrm{d}x = 1$.

It is easy to check that the radial solution (u, v) satisfies $u_t,\ v_t \ge 0$ and $\nabla u, \nabla v \le 0$ in B_R by the comparison principle.

Theorem 2.2 (i) *There exist initial data such that u (or v) blows up alone if and only if $m_+ > q_+$ (or $n_+ > p_+$).*

(ii) *Any blowup is simultaneous if and only if $m_+ \le q_+$ and $n_+ \le p_+$.*

(iii) *Both simultaneous and non-simultaneous blowup may occur if and only if $m_+ > q_+$ and $n_+ > p_+$.*

(iv) *Any blowup is non-simultaneous if and only if $m_+ > q_+$ and $n_+ \le p_+$ (for u blowing up alone), or $m_+ \le q_+$ and $n_+ > p_+$ (for v blowing up alone).*

The following theorem shows blowup rate estimates for solutions of (1.1).

Theorem 2.3 (i) *If $m_+ < q_+$ and $n_+ < p_+$, or $m_+ > q_+$, $n_+ > p_+$ and assume that simultaneous blow-up happens, then*

$$\mathrm{e}^{u(0,t)} = O\left((T-t)^{-\frac{p_+-n_+}{p_+q_+-n_+m_+}}\right), \quad \mathrm{e}^{v(0,t)} = O\left((T-t)^{-\frac{q_+-m_+}{p_+q_+-n_+m_+}}\right).$$

(ii) *If $m_+ < q_+$ and $n_+ = p_+$, then*

$$\mathrm{e}^{(q_+-m_+)u(0,t)} = O\big(|\log(T-t)|\big), \quad \mathrm{e}^{n_+v(0,t)} v^{\frac{q_+}{q_+-m_+}}(0,t) = O\big((T-t)^{-1}\big).$$

(iii) *If $m_+ = q_+$ and $n_+ < p_+$, then*

$$\mathrm{e}^{m_+u(0,t)} u^{\frac{p_+}{p_+-n_+}}(0,t) = O\big((T-t)^{-1}\big), \quad \mathrm{e}^{(p_+-n_+)v(0,t)} = O\big(|\log(T-t)|\big).$$

(iv) *If* $m_+ = q_+$ *and* $n_+ = p_+$, *then*

$$u(0,t) = O\big(|\log(T-t)|\big), \quad v(0,t) = O\big(|\log(T-t)|\big).$$

(v) *If* u *(or* v*) blows up while* v *(or* u*) remains bounded up to blowup time* T, *then*

$$\mathrm{e}^{u(0,t)} = O\left((T-t)^{-\frac{1}{m_+}}\right) \quad \left(\text{or } \mathrm{e}^{v(0,t)} = O\left((T-t)^{-\frac{1}{n_+}}\right)\right).$$

3 Proof of Theorem 2.1

We show the proof of blowup criteria for classical solutions of (1.1).

Proof of Theorem 2.1. Let $m_+ > 0$. Due to the continuity of $m(x)$, there exists a ball $B \subset \Omega$, in which $m_+ \geq m(x) \geq \delta > 0$. Introduce a function

$$\eta(t) = \int_B \varphi_1(x)u(x,t)\mathrm{d}x,$$

where φ_1 and λ_1 are the first eigenfunction and the first eigenvalue of the Dirichlet Laplacian in B respectively with $\int_B \varphi_1(x)\mathrm{d}x = 1$. It is easy to see that

$$\eta(t) \leq \|u(\cdot,t)\|_{L^\infty(B)} \leq \|u(\cdot,t)\|_{L^\infty(\Omega)}.$$

We only need to prove that $\eta(t)$ blows up in finite time. One can obtain

$$\begin{aligned} \eta'(t) &\geq -\lambda_1\eta(t) + \int_B \varphi_1(x)\mathrm{e}^{\delta u(x,t)}\mathrm{d}x \\ &\geq -\lambda_1\eta(t) + \left(\frac{\delta}{\delta+1}\right)^{\delta+1}\eta^{\delta+1}(t), \end{aligned} \tag{3.1}$$

if $\eta(0) = \int_B \varphi_1(x)u_0(x)\mathrm{d}x \geq \max\left\{1, \lambda_1^{\frac{1}{\delta}}\left[(\delta+1)/\delta\right]^{\frac{\delta+1}{\delta}}\right\}$. Hence there exists constant $c > 0$ such that $\eta'(t) \geq c\eta^{\delta+1}(t)$. By integration from 0 to t, one can obtain that

$$\eta(t) \geq \left(\eta^{-\delta}(0) - \frac{\delta}{c}t\right)^{-1/\delta}.$$

So $\eta(t)$ blows up in finite time for positive δ. Hence u blows up in finite time for $m_+ > 0$ and large $\int_B \varphi_1(x)u_0(x)\mathrm{d}x$.

Similarly, v blows up in finite time for $n_+ > 0$ and large initial data.

Now, let $p_+q_+ > 0$. For the continuity of positive variable exponents, there exists some positive constant $\beta = \min_{x\in\bar{\Omega}}\{p(x), q(x)\}$. Define

$$\zeta(t) = \int_\Omega \varphi(x)u(x,t)\mathrm{d}x, \quad \xi(t) = \int_\Omega \varphi(x)v(x,t)\mathrm{d}x,$$

where φ and λ are the first eigenfunction and the first eigenvalue of the Dirichlet Laplacian in Ω respectively with $\int_\Omega \varphi(x)\mathrm{d}x = 1$. As the discussion of (3.1), we have

$$\begin{aligned} \zeta'(t) &\geq -\lambda\zeta(t) + \int_B \varphi(x)\mathrm{e}^{p(x)v(x,t)}\mathrm{d}x \\ &\geq -\lambda\zeta(t) + \int_B \varphi(x)\mathrm{e}^{\beta v(x,t)}\mathrm{d}x \end{aligned}$$

$$\geq -\lambda\zeta(t) + c\xi^{\beta+1}(t), \tag{3.2}$$

similarly, there is the inequality

$$\xi'(t) \geq -\lambda\xi(t) + c\zeta^{\beta+1}(t). \tag{3.3}$$

Combining (3.2) with (3.3), we have

$$K'(t) \geq -\lambda K(t) + c(\zeta^{\beta+1}(t) + \xi^{\beta+1}(t)) \geq -\lambda K(t) + cK^{\beta+1}(t)$$

with $K(t) = \zeta(t) + \xi(t)$. Hence $K(t)$ blows up in finite time for large initial data, which deduces $(\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty)$ blows up.

If $m_+ = n_+ = p_+q_+ = 0$, then

$$\begin{cases} u_t = \Delta u + \mathrm{e}^{p(x)v} \leq \Delta u + \mathrm{e}^{p_+v}, & (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v + \mathrm{e}^{q(x)u} \leq \Delta v + \mathrm{e}^{q_+u}, & (x,t) \in \Omega \times (0,T). \end{cases}$$

It is easy to see that the classical solutions remain global for $p_+q_+ = 0$. $\Box$

4 Proof of Theorem 2.2

In order to prove Theorem 2.2, we introduce the following lemma.

Lemma 4.1 *Assume (H_1) and (H_2) hold. Then*

$$u_t(x,t) \geq \varepsilon\phi(x,t)\mathrm{e}^{m(x)u(x,t)+p(x)v(x,t)}, \quad (x,t) \in B_R \times [0,T), \tag{4.1}$$

$$v_t(x,t) \geq \varepsilon\phi(x,t)\mathrm{e}^{q(x)u(x,t)+n(x)v(x,t)}, \quad (x,t) \in B_R \times [0,T), \tag{4.2}$$

where $\phi(x,t)$ satisfies

$$\begin{cases} \phi_t(x,t) = \Delta\phi(x,t), & (x,t) \in B_R \times (0,T), \\ \phi(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ \phi(x,0) = \varphi(x), & x \in B_R, \end{cases}$$

where φ is the first eigenfunction of the Dirichlet Laplacian in B_R normalized by $\displaystyle\int_{B_R} \varphi(x)\mathrm{d}x = 1$.

Proof. Construct functions

$$J(x,t) = u_t(x,t) - \varepsilon\phi(x,t)\mathrm{e}^{m(x)u(x,t)+p(x)v(x,t)}, \quad (x,t) \in B_R \times (0,T),$$
$$K(x,t) = v_t(x,t) - \varepsilon\phi(x,t)\mathrm{e}^{q(x)u(x,t)+n(x)v(x,t)}, \quad (x,t) \in B_R \times (0,T).$$

It is easy to check that

$$J(x,t) \geq 0,\ K(x,t) \geq 0, \quad (x,t) \in \partial B_R \times (0,T);$$
$$J(x,0) = \Delta u_0(x) + (1 - \varepsilon\varphi(x))\mathrm{e}^{m_+u_0(x)+p_+v_0(x)} \geq 0, \quad x \in B_R,$$
$$K(x,0) = \Delta v_0(x) + (1 - \varepsilon\varphi(x))\mathrm{e}^{q_+u_0(x)+n_+v_0(x)} \geq 0, \quad x \in B_R.$$

By computation, we have

$$J_t - \Delta J + \varepsilon\phi\mathrm{e}^{m(x)u+p(x)v}\big[(-\Delta m(x))u + (-\Delta p(x))v\big]$$
$$\geq 2\varepsilon\nabla\phi \cdot \big[u\nabla m(x) + m(x)\nabla u + v\nabla p(x) + p(x)\nabla v\big]\mathrm{e}^{m(x)u+p(x)v}$$

$$
\begin{aligned}
&+\varepsilon\phi \mathrm{e}^{m(x)u+p(x)v}\big[u\nabla m(x)+m(x)\nabla u+v\nabla p(x)+p(x)\nabla v\big]^2\\
&+\varepsilon\phi \mathrm{e}^{m(x)u+p(x)v}\big[2\nabla m(x)\cdot\nabla u+2\nabla p(x)\cdot\nabla v+m(x)\Delta u+p(x)\Delta v\big]\\
&+u_{tt}-\Delta u_t-\varepsilon\phi \mathrm{e}^{m(x)u+p(x)v}(m(x)u_t+p(x)v_t)\\
&-\varepsilon \mathrm{e}^{m(x)u+p(x)v}(\phi_t-\Delta\phi)\\
\geq & m(x)\mathrm{e}^{m(x)u+p(x)v}J+p(x)\mathrm{e}^{m(x)u+p(x)v}K, \quad (x,t)\in B_R\times(0,T),
\end{aligned}
$$

similarly,

$$
\begin{aligned}
&K_t-\Delta K+\varepsilon\phi \mathrm{e}^{q(x)u+n(x)v}\big[(-\Delta q(x))u+(-\Delta n(x))v\big]\\
\geq & q(x)\mathrm{e}^{q(x)u+n(x)v}J+n(x)\mathrm{e}^{q(x)u+n(x)v}K, \quad (x,t)\in B_R\times(0,T).
\end{aligned}
$$

Then (4.1) and (4.2) hold by the comparison principle. $\square$

By (4.1), we obtain

$$u_t(0,t)\geq \varepsilon\phi(0,T)\mathrm{e}^{p_+v_0(0)}\mathrm{e}^{m_+u(0,t)}, \quad t\in[0,T).$$

Integrating the above inequality from t to T, we have the estimate for u,

$$u(0,t)\leq \log\left\{\big[\varepsilon m_+\phi(0,T)\mathrm{e}^{p_+v_0(0)}\big]^{-\frac{1}{m_+}}(T-t)^{-\frac{1}{m_+}}\right\}, \quad t\in[0,T). \tag{4.3}$$

By the similar method,

$$v(0,t)\leq \log\left\{\big[\varepsilon n_+\phi(0,T)\mathrm{e}^{q_+u_0(0)}\big]^{-\frac{1}{n_+}}(T-t)^{-\frac{1}{n_+}}\right\}, \quad t\in[0,T).$$

Considering the assumptions on u_0 and v_0, one can prove that $\Delta u(0,t)\leq 0$ and $\Delta v(0,t)\leq 0$. Hence we have the important inequalities

$$u_t(0,t)\leq \mathrm{e}^{m_+u(0,t)+p_+v(0,t)}, \quad t\in[0,T), \tag{4.4}$$

$$v_t(0,t)\leq \mathrm{e}^{q_+u(0,t)+n_+v(0,t)}, \quad t\in[0,T). \tag{4.5}$$

We use the following lemma to prove Theorem 2.2 (i). It is easy to see that Theorem 2.2 (ii) can be obtained directly from Theorem 2.2 (i).

Lemma 4.2 *There exist suitable initial data such that u blows up while v remains bounded if and only if $m_+>q_+$. There exist suitable initial data such that v blows up while u remains bounded if and only if $n_+>p_+$.*

Proof. Without loss of generality, we only prove the case for u blowing up while v remaining bounded.

At first, we prove the sufficiency. Let

$$\Gamma(x,y,t,\tau)=\frac{1}{[4\pi(t-\tau)]^{N/2}}\exp\left\{-\frac{|x-y|^2}{4(t-\tau)}\right\}$$

be the fundamental solution of the heat equation. Assume $(\tilde u_0,\tilde v_0)$ is a pair of initial data such that the solution of (1.1) blows up. Fix radially symmetric $v_0(\geq\tilde v_0)$ in B_R and take $M_1>v_0(0)$. Let $u_0(\geq\tilde u_0)$ be large such that T satisfies

$$M_1\geq v_0(0)+\frac{m}{m_+-q_+}\left[\varepsilon m_+\phi(0,T)\mathrm{e}^{p_+v_0(0)}\right]^{-\frac{q_+}{m_+}}T^{\frac{m_+-q_+}{m_+}}\mathrm{e}^{n_+M_1}.$$

Consider the auxiliary problem

$$\begin{cases} \bar{v}_t = \Delta \bar{v} + \left[\varepsilon m_+ \phi(0,T) \mathrm{e}^{p_+ v_0(0)}\right]^{-\frac{q_+}{m_+}} (T-t)^{-\frac{q_+}{m_+}} \mathrm{e}^{n_+ M_1}, & (x,t) \in B_R \times (0,T), \\ \bar{v}(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ \bar{v}(x,0) = v_0(x), & x \in B_R. \end{cases}$$

For $m_+ > q_+$ and by Green's identity [21], we have

$$\begin{aligned} \bar{v}(x,t) &= \int_{B_R} \Gamma(x,y,t,0) v_0(y) \mathrm{d}y + \int_0^t \int_{\partial B_R} \Gamma(x,y,t,\tau) \frac{\partial \bar{v}}{\partial \eta} \mathrm{d}S_y \mathrm{d}\tau \\ &\quad + \int_0^t \int_{B_R} \Gamma(x,y,t,\tau) \left[\varepsilon m_+ \phi(0,T) \mathrm{e}^{p_+ v_0(0)}\right]^{-\frac{q_+}{m_+}} (T-\tau)^{-\frac{q_+}{m_+}} \mathrm{e}^{n_+ M_1} \mathrm{d}y \mathrm{d}\tau \\ &\leq v_0(0) + \frac{m_+}{m_+ - q_+} \left[\varepsilon m_+ \phi(0,T) \mathrm{e}^{p_+ v_0(0)}\right]^{-\frac{q_+}{m_+}} T^{\frac{m_+ - q_+}{m_+}} \mathrm{e}^{n_+ M_1} \\ &\leq M_1. \end{aligned}$$

So $\bar{v}$ satisfies

$$\begin{cases} \bar{v}_t \geq \Delta \bar{v} + \left[\varepsilon m_+ \phi(0,T) \mathrm{e}^{p_+ v_0(0)}\right]^{-\frac{q_+}{m_+}} (T-t)^{-\frac{q_+}{m_+}} \mathrm{e}^{n_+ \bar{v}}, & (x,t) \in B_R \times (0,T), \\ \bar{v}(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ \bar{v}(x,0) = v_0(x), & x \in B_R. \end{cases}$$

It follows from (4.3) that v satisfies

$$\begin{cases} v_t \leq \Delta v + \left[\varepsilon m_+ \phi(0,T) \mathrm{e}^{p_+ v_0(0)}\right]^{-\frac{q_+}{m_+}} (T-t)^{-\frac{q_+}{m_+}} \mathrm{e}^{n_+ v}, & (x,t) \in B_R \times (0,T), \\ v(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ v(x,0) = v_0(x), & x \in B_R. \end{cases}$$

By the comparison principle, $v \leq \bar{v} \leq M_1$. Since $(u_0, v_0) \geq (\tilde{u}_0, \tilde{v}_0)$, (u,v) blows up. And hence only u blows up at finite time T.

Secondly, we prove the necessity. Assume u blows up while v remains bounded, say $v \leq C$. By (4.4), $u_t(0,t) \leq C\mathrm{e}^{m_+ u(0,t)}$ for $t \in [0,T)$. Hence, one obtains the estimate for u as

$$u(0,t) \geq \log \left[c(T-t)^{-\frac{1}{m_+}}\right], \quad t \in [0,T). \tag{4.6}$$

By using (4.2) and (4.6), we have

$$v_t(0,t) \geq c\varepsilon \phi(0,T) \mathrm{e}^{n_+ v_0(0)} (T-t)^{-\frac{q_+}{m_+}}. \tag{4.7}$$

Integrating (4.7) from 0 to t, we have

$$v(0,t) \geq c \int_0^t (T-\tau)^{-\frac{q_+}{m_+}} \mathrm{d}\tau + v(0,0).$$

The boundedness of v requires $m_+ > q_+$. $\square$

We introduce three lemmas to prove Theorem 2.2 (iii) and (iv). For fixed constant $\varepsilon \in (0,1)$, we define the set $\mathbb{V}_0$ making up of the initial data which satisfy (H_1) and (H_2).

Lemma 4.3 *The set of (u_0, v_0) in $\mathbb{V}_0$ such that u (or v) blows up while v (or u) remains bounded is open in L^∞-topology.*

Proof. Without loss of generality, we only prove the case for u blowing up with v remaining bounded. Let (u, v) be a solution of (1.1) with initial data $(u_0, v_0) \in \mathbb{V}_0$ such that u blows up while v remains bounded up to blowup time T, say $0 < 2\xi \le v(0,t) \le M$. It suffices to find an L^∞-neighborhood of (u_0, v_0) in $\mathbb{V}_0$ such that any solution $(\hat{u}, \hat{v})$ of (1.1) coming from this neighborhood maintains the property that $\hat{u}$ blows up while $\hat{v}$ remains bounded.

By Lemma 4.2, we know $m_+ > q_+$. Take $M_2 > M + \xi$. Let $(\tilde{u}, \tilde{v})$ be the solution of the problem

$$\begin{cases} \tilde{u}_t = \Delta\tilde{u} + \mathrm{e}^{m(x)\tilde{u}+p(x)\tilde{v}},\ \tilde{v}_t = \Delta\tilde{v} + \mathrm{e}^{q(x)\tilde{u}+n(x)\tilde{v}}, & (x,t) \in B_R \times (0, T_0), \\ \tilde{u}(x,t) = \tilde{v}(x,t) = 0, & (x,t) \in \partial B_R \times (0, T_0), \\ \tilde{u}(x,0) = \tilde{u}_0(x),\ \tilde{v}(x,0) = \tilde{v}_0(x), & x \in B_R, \end{cases}$$

where radially symmetric $(\tilde{u}_0, \tilde{v}_0)$ is to be determined.

Define

$$\mathbb{N}(u_0, v_0) = \Big\{(\tilde{u}_0, \tilde{v}_0) \mid \|\tilde{u}_0(x) - u(x, T-\varepsilon_0)\|_\infty < \xi,\ \|\tilde{v}_0(x) - v(x, T-\varepsilon_0)\|_\infty < \xi,$$
$$(\tilde{u}_0, \tilde{v}_0) = (\hat{u}(x, T-\varepsilon_0), \hat{v}(x, T-\varepsilon_0)),\ (\hat{u}_0, \hat{v}_0) \in \mathbb{V}_0\Big\}.$$

Since u blows up at time T, there exists constant $\varepsilon_0 > 0$ such that $(\tilde{u}, \tilde{v})$ blows up and T_0 satisfies

$$M_2 > M + \xi + \frac{m_+}{m_+ - q_+}\Big[m_+\varepsilon\phi(0, T_0)\mathrm{e}^{p_+\tilde{v}_0(0)}\Big]^{-\frac{q_+}{m_+}} T_0^{\frac{m_+-q_+}{m_+}} \mathrm{e}^{n_+ M_2},$$

provided that $(\tilde{u}_0, \tilde{v}_0) \in \mathbb{N}(u_0, v_0)$.

Consider the auxiliary system

$$\begin{cases} \bar{v}_t = \Delta\bar{v} + \big[m_+\varepsilon\phi(0,T_0)\mathrm{e}^{p_+\tilde{v}_0(0)}\big]^{-\frac{q_+}{m_+}} (T_0 - t)^{-\frac{q_+}{m_+}} \mathrm{e}^{n_+ M_2}, & (x,t) \in B_R \times (0, T_0), \\ \bar{v}(x,t) = 0, & (x,t) \in \partial B_R \times (0, T_0), \\ \bar{v}(x,0) = \tilde{v}_0(x), & x \in B_R. \end{cases}$$

By Green's identity, $\bar{v} \le M_2$. Hence

$$\bar{v}_t \ge \Delta\bar{v} + \Big[m_+\varepsilon\phi(0,T_0)\mathrm{e}^{p_+\tilde{v}_0(0)}\Big]^{-\frac{q_+}{m_+}} (T_0 - t)^{-\frac{q_+}{m_+}} \mathrm{e}^{n_+\bar{v}}, \quad (x,t) \in B_R \times (0, T_0).$$

On the other hand, by (4.3), we have

$$\tilde{v}_t \le \Delta\tilde{v} + \Big[m_+\varepsilon\phi(0,T_0)\mathrm{e}^{p_+\tilde{v}_0(0)}\Big]^{-\frac{q_+}{m_+}} (T_0 - t)^{-\frac{q_+}{m_+}} \mathrm{e}^{n_+\tilde{v}}, \quad (x,t) \in B_R \times (0, T_0).$$

By the comparison principle, $\tilde{v} \le \bar{v} \le M_2$, then $\tilde{u}$ must blow up.

According to the continuity with respect to initial data for bounded solutions, there must exist a neighborhood of (u_0, v_0) in $\mathbb{V}_0$ such that every solution $(\hat{u}, \hat{v})$ starting from this neighborhood will enter $\mathbb{N}(u_0, v_0)$ at time $T - \varepsilon_0$, and keeps the property that $\hat{u}$ blows up while $\hat{v}$ remains bounded. $\square$

Lemma 4.4 *Assume $m_+ > q_+$ and $n_+ > p_+$. Then both simultaneous and non-simultaneous blowup may occur.*

Proof. Assume the solution of (1.1) blows up with initial data $(u_0, v_0) \in \mathbb{V}_0$. Then the solution with initial data $(u_0/\lambda, v_0/(1-\lambda)) \in \mathbb{V}_0$ for $\lambda \in (0,1)$ also blows up. By Lemma 4.2, we know there exist some λ_1 near 0 such that u blows up while v remains bounded if $\lambda = \lambda_1$, and some λ_2 near 1 such that v blows up while u remains bounded if $\lambda = \lambda_2$, respectively. By Lemma 4.3, such initial data sets are open and connected. Then there must exist some $\lambda \in (\lambda_1, \lambda_2)$ such that simultaneous blowup happens. $\square$

Lemma 4.5 *If $m_+ \le q_+$ and $n_+ > p_+$, then any blowup must be v blowing up with u remaining bounded. If $m_+ > q_+$ and $n_+ \le p_+$, then any blowup must be u blowing up with v remaining bounded.*

Proof. We only prove the case for v blowing up with u remaining bounded. Considering (4.1), (4.2), (4.4), and (4.5), we have

$$\varepsilon\phi(0,T)\mathrm{e}^{(q_+-m_+)u(0,t)}u_t(0,t) \le \mathrm{e}^{(p_+-n_+)v(0,t)}v_t(0,t) \le \frac{\mathrm{e}^{(q_+-m_+)u(0,t)}u_t(0,t)}{\varepsilon\phi(0,T)}, \quad t \in [0,T). \tag{4.8}$$

By Lemma 4.2, there is not the case for u blowing up alone. We only need to prove that u and v cannot blow up simultaneously. If not, assume simultaneous blowup happens.

If $m_+ < q_+$ and $n_+ > p_+$, then by integrating the left inequality of (4.8) from 0 to t, one obtains

$$\frac{\varepsilon\phi(0,T)}{q_+-m_+}\mathrm{e}^{(q_+-m_+)u(0,t)} \le C - \frac{1}{n_+-p_+}\mathrm{e}^{-(n_+-p_+)v(0,t)}.$$

This is a contradiction to simultaneous blowup occurring.

If $m_+ = q_+$ and $n_+ > p_+$, then

$$\varepsilon\phi(0,T)u(0,t) \le C - \frac{1}{n_+-p_+}\mathrm{e}^{-(n_+-p_+)v(0,t)}.$$

It is also a contradiction. $\square$

Proofs of Theorem 2.2 (iii) and (iv). The sufficiency for the cases (iii) and (iv) can be obtained by Lemmas 4.4 and 4.5, respectively. Now, for the necessity of case (iii), we only need to prove that, if the exponents do not satisfy $m_+ > p_+$ and $q_+ > n_+$, then there is not the phenomenon of coexistence for simultaneous and non-simultaneous blowup. It can be obtained by Theorem 2.2 (ii) and Lemma 4.5, directly. Similarly, the necessary condition of case (iv) can be proved by Theorem 2.2 (ii) and Lemma 4.4. $\square$

5 Proof of Theorem 2.3

In this section, we give the estimates of blowup rates.

Proof of Theorem 2.3 (i). We only prove simultaneous blowup rate in the region $m_+ < q_+$ and $n_+ < p_+$. The case for $m_+ > q_+$ and $n_+ > p_+$ can be obtained by the similar methods. Integrating (4.8), we have the relationships between u and v as follows,

$$\mathrm{e}^{(q_+-m_+)u(0,t)} \le C\mathrm{e}^{(p_+-n_+)v(0,t)}, \quad \mathrm{e}^{(p_+-n_+)v(0,t)} \le C\mathrm{e}^{(q_+-m_+)u(0,t)}, \quad t \in [0,T). \tag{5.1}$$

Considering (4.1), (4.2), (4.4), (4.5) with (5.1), we have

$$c \le \left(\mathrm{e}^{-\frac{p_+q_+-m_+n_+}{p_+-n_+}u(0,t)}\right)_t \le C, \quad c \le \left(\mathrm{e}^{-\frac{p_+q_+-m_+n_+}{q_+-m_+}v(0,t)}\right)_t \le C. \tag{5.2}$$

Then simultaneous blowup rate follows from (5.2) immediately. $\square$

Proofs of Theorem 2.3 (ii) and (iii). We only prove case (ii), and case (iii) can be proved by the similar method. By using the similar method to establish (5.1), we have that

$$cv^{\frac{1}{q_+-m_+}}(0,t) \le \mathrm{e}^{u(0,t)} \le Cv^{\frac{1}{q_+-m_+}}(0,t), \quad t \in [0,T). \tag{5.3}$$

Then v satisfies

$$c \le \mathrm{e}^{-n_+v(0,t)}v^{-\frac{q_+}{q_+-m_+}}(0,t)v_t(0,t) \le C, \quad t \in [0,T).$$

Integrating the above inequalities from t to T, we obtain

$$c(T-t) \le \int_{v(0,t)}^{+\infty} \mathrm{e}^{-n_+ s} s^{-\frac{q_+}{q_+-m_+}} \mathrm{d}s \le C(T-t), \quad t \in [0,T).$$

Since

$$\lim_{t\to T} \frac{\displaystyle\int_{v(0,t)}^{+\infty} \mathrm{e}^{-n_+ s} s^{-\frac{q_+}{q_+-m_+}} \mathrm{d}s}{\mathrm{e}^{-n_+ v(0,t)} v^{-\frac{q_+}{q_+-m_+}}(0,t)} = \lim_{v(0,t)\to+\infty} \frac{\displaystyle\int_{v(0,t)}^{+\infty} \mathrm{e}^{-n_+ s} s^{-\frac{q_+}{q_+-m_+}} \mathrm{d}s}{\mathrm{e}^{-n_+ v(0,t)} v^{-\frac{q_+}{q_+-m_+}}(0,t)} = \frac{1}{n},$$

that is,

$$c\mathrm{e}^{-n_+ v(0,t)} v^{-\frac{q_+}{q_+-m_+}}(0,t) \le n \int_{v(0,t)}^{+\infty} \mathrm{e}^{-n_+ s} s^{-\frac{q_+}{q_+-m_+}} \mathrm{d}s \le C\mathrm{e}^{-n_+ v(0,t)} v^{-\frac{q_+}{q_+-m_+}}(0,t).$$

Then we have the blowup rate for v as

$$c(T-t)^{-1} \le \mathrm{e}^{n_+ v(0,t)} v^{\frac{q_+}{q_+-m_+}}(0,t) \le C(T-t)^{-1}, \quad t \in [0,T). \tag{5.4}$$

On the other hand, by (5.3) and (5.4),

$$c(T-t)^{-1} \le \varepsilon\phi(0,T)\mathrm{e}^{q_+ u(0,t)+n_+ v(0,t)} \le v_t(0,t), \quad t \in [0,T),$$

$$v_t(0,t) \le \mathrm{e}^{q_+ u(0,t)+n_+ v(0,t)} \le C(T-t)^{-1}, \qquad t \in [0,T).$$

Since

$$cv_t(0,t) \le \mathrm{e}^{(q_+-m_+)u(0,t)} u_t(0,t) \le Cv_t(0,t), \quad t \in [0,T),$$

we obtain

$$c(T-t)^{-1} \le \mathrm{e}^{(q_+-m_+)u(0,t)} u_t(0,t) \le C(T-t)^{-1}, \quad t \in [0,T),$$

hence blowup rate for u is followed. $\square$

Proof of Theorem 2.3 (iv). By a method used to establish inequalities (4.8), we obtain

$$\varepsilon\phi(0,T)u_t(0,t) \le v_t(0,t) \le \frac{1}{\varepsilon\phi(0,T)} u_t(0,t).$$

It is easy to prove that

$$\varepsilon\phi(0,T)u(0,t) \le v(0,t) + C, \quad t \in [0,T),$$

$$v(0,t) \le \frac{1}{\varepsilon\phi(0,T)} u(0,t) + C, \quad t \in [0,T).$$

Then

$$c\mathrm{e}^{[m_+ + \varepsilon p_+ \phi(0,T)]u(0,t)} \le u_t(0,t) \le C\mathrm{e}^{[m_+ + \frac{p_+}{\varepsilon\phi(0,T)}]u(0,t)}, \quad t \in [0,T).$$

By integration, we have

$$c|\log(T-t)| \le u(0,t) \le C|\log(T-t)|, \quad t \in [0,T).$$

Similarly, the blowup rate for v is obtained,

$$c|\log(T-t)| \le v(0,t) \le C|\log(T-t)|, \quad t \in [0,T). \qquad \square$$

The non-simultaneous blowup rate is equivalent to that of the scalar equation, which can be obtained from e.g. (4.3) and (4.6). Theorem 2.3 (v) is proved.

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