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# ULAM STABILITY AND DATA DEPENDENCE FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH CAPUTO DERIVATIVE

JinRong Wang[*,1], Linli Lv[1] Yong Zhou[2]

1. Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China
2. Department of Mathematics, Xiangtan University, Xiangtan, Hunan 411105, P.R. China

**ABSTRACT.** In this paper, Ulam stability and data dependence for fractional differential equations with Caputo fractional derivative of order $\alpha$ are studied. We present four types of Ulam stability results for the fractional differential equation in the case of $0 < \alpha < 1$ and $b = +\infty$ by virtue of the Henry-Gronwall inequality. Meanwhile, we give an interesting data dependence results for the fractional differential equation in the case of $1 < \alpha < 2$ and $b < +\infty$ by virtue of a generalized Henry-Gronwall inequality with mixed integral term. Finally, examples are given to illustrate our theory results.

**Keywords.** Fractional differential equations; Caputo derivative; Ulam stability; Data dependence; Gronwall inequality.

## 1. Introduction

Fractional differential equations have been proved to be strong tools in the modelling of many physical phenomena. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. There has been a significant development in fractional ordinary differential equations and partial differential equations. For more details on fractional calculus theory, one can see the monographs of Kilbas et al. [17], Miller and Ross [20], Podlubny [23], Tarasov [26] and the papers of Agarwal et al. [1, 2], Ahmad and Nieto [3], Balachandran et al. [5], Bai [6], Benchohra et al. [7], Henderson and Ouahab [13], Li et al. [18, 19], Mophou and N'Guérékata [21], Wang et al. [27, 28, 29, 30, 31], Zhang [34] and Zhou et al. [35, 36].

On the other hand, numerous monographs have discussed the data dependence in the theory of ordinary differential equations (see for example [4, 9, 10, 14, 22, 24]). Meanwhile, there are some special data dependence in the theory of functional equations such as Ulam-Hyers, Ulam-Hyers-Rassias and Ulam-Hyers-Bourgin. The stability properties of all kinds of equations have attracted the attention of many mathematicians. Particularly, the Ulam-Hyers-Rassias stability was taken up by a number of mathematicians and the study of this area has the grown to be one of the central subjects in the mathematical analysis area. For more information, we can see the monographs Cadariu [8], Hyers [15] and Jung [16].

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*Corresponding author.
Email addresses: wjr9668@126.com (J. Wang); lvlinli2008@126.com (L. Lv); yzhou@xtu.edu.cn (Y. Zhou).

Although, there are some work on the local stability and Mittag-Leffler stability for fractional differential equations (see [11, 18, 19]), to the best of my knowledge, there are very rare works on the Ulam stability for fractional differential equations. Motivated by [1, 25, 32], we will study the Ulam stability for the following fractional differential equation

$$^{c}D^{\alpha}x(t) = f(t, x(t)),\ t \in [a, b),\ b = +\infty,$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$ and the function $f$ satisfies some conditions will be specified later. Meanwhile, we will study the data dependence for the following fractional differential equation

$$^{c}D^{\alpha}x(t) = f(t, x(t)),\ t \in [a, b),\ b < +\infty,$$

where the Caputo fractional derivative of order $\alpha \in (1, 2)$.

In the present paper, we introduce four types of Ulam stability definitions for fractional differential equations: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability. We present the four types of Ulam stability results for a fractional differential equation in the case $0 < \alpha < 1$ and $b = +\infty$ by virtue of a Henry-Gronwall inequality. Meanwhile, we give data dependence results for a fractional differential equation in the case $1 < \alpha < 2$ and $b < +\infty$ by virtue of Henry-Gronwall inequality with mixed integral term. Finally, examples are given to illustrate our theory results.

## 2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. We denote $(\mathbb{B}, |\cdot|)$ be a Banach space. Let $a \in \mathbb{R}, b \in \mathbb{R}, a < b \le +\infty$, Let $C([a, b), \mathbb{B})$ be the Banach space of all continuous functions from $[a, b)$ into $\mathbb{B}$ with the norm $|y|_C = \sup\{|y(t)| : t \in [a, b)\}$. If $\mathbb{B} := \mathbb{R}$, we simply denote $C([a, b), \mathbb{R})$ by $C[a, b)$.

We need some basic definitions and properties of the fractional calculus theory which are used further in this paper. For more details, see [17].

**Definition 2.1.** The fractional integral of order $\gamma$ with the lower limit zero for a function $f$ is defined as

$$I^{\gamma}f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \frac{f(s)}{(t-s)^{1-\gamma}}ds,\ t > 0,\ \gamma > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

**Definition 2.2.** The Riemann-Liouville derivative of order $\gamma$ with the lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$$^{L}D^{\gamma}f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}}ds,\ t > 0,\ n-1 < \gamma < n.$$

**Definition 2.3.** The Caputo derivative of order $\gamma$ for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$$^{c}D^{\gamma}f(t) = {}^{L}D^{\gamma}\left(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right),\ t > 0,\ n-1 < \gamma < n.$$

Let $\epsilon$ be a positive real number, $f : [a, b) \times \mathbb{B} \to \mathbb{B}$ be a continuous operator and $\varphi : [a, b) \to \mathbb{R}_+$ be a continuous function. We consider the following differential equation

$$^{c}D^{\alpha}x(t) = f(t, x(t)),\ \alpha \in (0, 1)\ (\text{or } (1, 2)),\ t \in [a, b), \tag{2.1}$$

and the following inequalities

$$|{}^cD^\alpha y(t) - f(t, y(t))| \le \epsilon,\ \ t \in [a, b), \tag{2.2}$$

$$|{}^cD^\alpha y(t) - f(t, y(t))| \le \varphi(t),\ \ t \in [a, b), \tag{2.3}$$

$$|{}^cD^\alpha y(t) - f(t, y(t))| \le \epsilon\varphi(t),\ \ t \in [a, b). \tag{2.4}$$

**Definition 2.4.** The equation (2.1) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $y \in C^1([a, b), \mathbb{B})(or\ C^2([a, b), \mathbb{B}))$ of the inequality (2.2) there exists a solution $x \in C^1([a, b), \mathbb{B})(or\ C^2([a, b), \mathbb{B}))$ of the equation (2.1) with

$$|y(t) - x(t)| \le c_f \epsilon,\ \ t \in [a, b).$$

**Definition 2.5.** The equation (2.1) is generalized Ulam-Hyers stable if there exists $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_f(0) = 0$ such that for each solution $y \in C^1([a, b), \mathbb{B})(or\ C^2([a, b), \mathbb{B}))$ of the inequality (2.2) there exists a solution $x \in C^1([a, b), \mathbb{B})(or\ C^2([a, b), \mathbb{B})$ of the equation (2.1) with

$$|y(t) - x(t)| \le \theta_f(\epsilon),\ \ t \in [a, b).$$

**Definition 2.6.** The equation (2.1) is Ulam-Hyers-Rassias stable with respect to $\varphi$ if there exists $c_{f,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in C^1([a, b), \mathbb{B})(or\ C^2([a, b), \mathbb{B}))$ of the inequality (2.4) there exists a solution $x \in C^1([a, b), \mathbb{B})(or\ C^2([a, b), \mathbb{B}))$ of the equation (2.1) with

$$|y(t) - x(t)| \le c_{f,\varphi}\epsilon\varphi(t),\ \ t \in [a, b).$$

**Definition 2.7.** The equation (2.1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi$ if there exists $c_{f,\varphi} > 0$ such that for each solution $y \in C^1([a, b), \mathbb{B})(or\ C^2([a, b), \mathbb{B}))$ of the inequality (2.3) there exists a solution $x \in C^1([a, b), \mathbb{B})(or\ C^2([a, b), \mathbb{B}))$ of the equation (2.1) with

$$|y(t) - x(t)| \le c_{f,\varphi}\varphi(t),\ \ t \in [a, b).$$

**Remark 2.8.** It is clear that: (i) Definition 2.4 $\Longrightarrow$ Definition 2.5; (ii) Definition 2.6 $\Longrightarrow$ Definition 2.7; (iii) Definition 2.6 $\Longrightarrow$ Definition 2.4.

**Remark 2.9.** A function $y \in C^1([a, b), \mathbb{B})(or\ C^2([a, b), \mathbb{B}))$ is a solution of the inequality (2.2) if and only if there exists a function $g \in C^1([a, b), \mathbb{B})(or\ C^2([a, b), \mathbb{B}))$ (which depend on $y$) such that
(i) $|g(t)| \le \epsilon,\ \ t \in [a, b)$;
(ii) ${}^cD^\alpha y(t) = f(t, y(t)) + g(t),\ \ t \in [a, b)$.

One can have similar remarks for the inequations (2.3) and (2.4).

So, the Ulam stabilities of the fractional differential equations are some special types of data dependence of the solutions of fractional differential equations.

**Remark 2.10.** Let $0 < \alpha < 1$, if $y \in C^1([a, b), \mathbb{B})$ is a solution of the inequality (2.2) then $y$ is a solution of the following integral inequality

$$\left| y(t) - y(a) - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, y(s)) ds \right| \le \frac{(t-a)^\alpha}{\Gamma(\alpha+1)}\epsilon,\ \ t \in [a, b).$$

Indeed, by Remark 2.9 we have that

$$^cD^\alpha y(t) = f(t, y(t)) + g(t), \forall\ t \in [a, b).$$

Then

$$y(t) - y(a) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, y(s))ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)ds,\ t \in [a, b).$$

This implies that

$$y(t) = y(a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, y(s))ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)ds,\ t \in [a, b).$$

From this it follows that

$$\begin{aligned} \left| y(t) - y(a) - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, y(s))ds \right| &= \left| \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)ds \right| \\ &\le \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |g(s)|ds \\ &\le \frac{\epsilon}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} ds \\ &\le \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \epsilon. \end{aligned}$$

We have similar remarks for the solutions of the inequations (2.3) and (2.4).

In what follows, we collect the Henry-Gronwall inequality (see Lemma 7.1.1, [12]), which can be used in fractional differential equations with initial value conditions.

**Lemma 2.11.** *Let $z$, $\omega : [0, T) \to [0, +\infty)$ be continuous functions where $T \le \infty$. If $\omega$ is nondecreasing and there are constants $\kappa \ge 0$ and $q > 0$ such that*

$$z(t) \le \omega(t) + \kappa \int_0^t (t-s)^{q-1} z(s)ds,\ t \in [0, T),$$

*then*

$$z(t) \le \omega(t) + \int_0^t \left[ \sum_{n=1}^{\infty} \frac{(\kappa\Gamma(q))^n}{\Gamma(nq)} (t-s)^{nq-1} \omega(s) \right] ds,\ t \in [0, T).$$

*If $\omega(t) = \bar{a}$, constant on $0 \le t < T$, then the above inequality is reduce to*

$$z(t) \le \bar{a} E_q(\kappa\Gamma(q)t^q),\ 0 \le t < T,$$

*where $E_q$ is the Mittag-Leffler function* [17] *defined by*

$$E_\beta(y) := \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(k\beta+1)},\ y \in \mathbb{C},\ \mathfrak{Re}(\beta) > 0.$$

**Remark 2.12.** (i) There exists a constant $M_\kappa^* > 0$ independent of $\bar{a}$ such that

$$z(t) \le M_\kappa^* \bar{a} \text{ for all } 0 \le t < T.$$

(ii) For more generalized Henry-Gronwall inequalities see Ye et al. [33].

To end this section, we collect a generalized Henry-Gronwall inequality with mixed integral term, which can be used in boundary value problems for fractional differential equations.

**Lemma 2.13.** *Let $b < +\infty$ and $y \in C([0,b],\mathbb{B})$ satisfy the following inequality:*

$$|y(t)| \leq a_1 + b_1 \int_0^t (t-s)^{\alpha-1}|y(s)|^\lambda ds + c_1 \int_0^b (b-s)^{\alpha-1}|y(s)|^\lambda ds, \tag{2.5}$$

*where $\alpha \in (1,2)$, $\lambda \in [0, 1-\frac{1}{p}]$ for some $1 < p < +\infty$, $a_1, b_1, c_1 \geq 0$ are constants. Then there exists a constant $M := (b_1 + c_1)\left[\frac{b^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]^{\frac{1}{p}} > 0$ such that*

$$|y(t)| \leq (a_1+1)e^{Mb}.$$

*Proof.* Similar to the proof of Lemma 3.2 in our previous work [32], one can obtain the result immediately. □

## 3. Ulam stability results

Let $0 < \alpha < 1$. Without loss of generality, we consider the equation (2.1) and the inequality (2.3) in the case $b = +\infty$.

We suppose that:

$(H_1)$ $f \in C([a,+\infty) \times \mathbb{B}, \mathbb{B})$;

$(H_2)$ There exists $m_f > 0$ such that

$$|f(t,u_1) - f(t,u_2)| \leq m_f |u_1 - u_2|, \text{ for each } t \in [a,+\infty), \text{ and all } u_1, u_2 \in \mathbb{B};$$

$(H_3)$ Let $\varphi \in C([a,+\infty), \mathbb{R}_+)$ be an increasing function. There exists $\lambda_\varphi > 0$ such that

$$\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\varphi(s)ds \leq \lambda_\varphi \varphi(t), \text{ for each } t \in [a,+\infty).$$

We obtain the following generalized Ulam-Hyers-Rassias stable results.

**Theorem 3.1.** *In the conditions $(H_1)$, $(H_2)$ and $(H_3)$ the equation (2.1) $(b = +\infty)$ is generalized Ulam-Hyers-Rassias stable.*

*Proof.* Let $y \in C^1([a,+\infty), \mathbb{B})$ be a solution of the inequality (2.3) $(b = +\infty)$. Denote by $x$ the unique solution of the Cauchy problem

$$\begin{cases} {}^cD^\alpha x(t) = f(t,x(t)), \ 0 < \alpha < 1, \ t \in [a,+\infty), \\ x(a) = y(a). \end{cases} \tag{3.1}$$

Then we have

$$x(t) = y(a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s,x(s))ds, \ t \in [a,+\infty).$$

By differential inequality (2.3), we have

$$\begin{aligned} & \left| y(t) - y(a) - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s,y(s))ds \right| \\ \leq \ & \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\varphi(s)ds \\ \leq \ & \lambda_\varphi \varphi(t), \ t \in [a,+\infty). \end{aligned}$$

From these relation it follows

$$\begin{aligned}
&|y(t)-x(t)| \\
\leq\ & \left|y(t)-y(a)-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,x(s))ds\right| \\
\leq\ & \left|y(t)-y(a)-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,y(s))ds\right. \\
& \left.+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,y(s))ds-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,x(s))ds\right| \\
\leq\ & \left|y(t)-y(a)-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,y(s))ds\right| \\
& +\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|f(s,y(s))-f(s,x(s))|ds \\
\leq\ & \lambda_\varphi\varphi(t)+\frac{m_f}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|y(s)-x(s)|ds.
\end{aligned}$$

By Lemma 2.11 and Remark 2.12(i), there exists a constant $M_f^*>0$ independent of $\lambda_\varphi\varphi(t)$ such that

$$|y(t)-x(t)|\leq M_f^*\lambda_\varphi\varphi(t):=c_{f,\varphi}\varphi(t),\ t\in[a,+\infty).$$

Thus, the equation (2.1) ($b=+\infty$) is generalized Ulam-Hyers-Rassias stable. □

**Corollary 3.2.** *(i) Under the assumptions of Theorem 3.1, we consider the equation (2.1)* ($b=+\infty$) *and the inequality (2.4). One can repeat the same process to verify that the equation (2.1)* ($b=+\infty$) *is Ulam-Hyers-Rassias stable.*
*(ii) Under the assumptions ($H_1$) and ($H_2$), we consider the equation (2.1)* ($b=+\infty$) *and the inequality (2.2). One can repeat the same process to verify that the equation (2.1)* ($b=+\infty$) *is Ulam-Hyers stable.*

## 4. Data Dependence

Let $1<\alpha<2$, we reconsider the equation (2.1) ($b<+\infty$) and the inequality (2.2).

We suppose that:

($H_4$) $f\in C([a,b]\times\mathbb{B})$.

($H_5$) There exist $m_f>0$ and $\lambda\in[0,1-\frac{1}{p}]$ for some $1<p<\infty$ such that

$$|f(t,u_1)-f(t,u_2)|\leq m_f|u_1-u_2|^\lambda,\ \text{for each } t\in[a,b],\ \text{and all } u_1,u_2\in\mathbb{B}.$$

The following result is interesting although the proof is not very difficult.

**Theorem 4.1.** *Assumptions ($H_4$) and ($H_5$) hold. Let $y\in C^2[a,b]$ be a solution of the inequality (2.2). Denote by $x$ the solution of the following fractional boundary value problem*

$$\left\{\begin{array}{l} {}^cD^\alpha x(t)=f(t,x(t)),\ 1<\alpha<2,\ t\in[a,b], \\ x(a)=y(a),\ x(b)=y(b). \end{array}\right. \tag{4.1}$$

*Then the following relation holds:*

$$|y(t)-x(t)|\leq c_f(\epsilon+1),\ t\in[a,b], \tag{4.2}$$

*where*

$$c_f:=e^{Mb}\max\left\{\frac{(b-a)^\alpha}{\Gamma(\alpha+1)},1\right\}>0 \text{ and } M:=\frac{2m_f}{\Gamma(\alpha)}\left[\frac{b^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]^{\frac{1}{p}}.$$

*Proof.* By Lemma 3.17 of [1], it is clear that the solution of the fractional boundary value problem (4.1) given by

$$
\begin{aligned}
x(t) &= \frac{b-t}{b-a}y(a)+\frac{t-a}{b-a}y(b)+\frac{a-t}{b-a}\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}f(s,x(s))ds \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,x(s))ds.
\end{aligned}
$$

By differential inequality (2.2), we have

$$
\begin{aligned}
&\left|y(t)-\frac{b-t}{b-a}y(a)-\frac{t-a}{b-a}y(b)-\frac{a-t}{b-a}\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}f(s,y(s))ds\right. \\
&\left.-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,y(s))ds\right| \\
\leq\ & \frac{\epsilon}{\Gamma(\alpha)}\int_a^b (t-s)^{\alpha-1}ds \\
\leq\ & \frac{(b-a)^\alpha\epsilon}{\Gamma(\alpha+1)}.
\end{aligned}
$$

From these relation it follows

$$
\begin{aligned}
& |y(t)-x(t)| \\
\leq\ & \left|y(t)-\frac{b-t}{b-a}y(a)-\frac{t-a}{b-a}y(b)-\frac{a-t}{b-a}\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}f(s,x(s))ds\right. \\
&\left.-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,x(s))ds\right| \\
\leq\ & \left|y(t)-\frac{b-t}{b-a}y(a)-\frac{t-a}{b-a}y(b)-\frac{a-t}{b-a}\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}f(s,y(s))ds\right. \\
&\left.-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,y(s))ds\right| \\
&+\left|\frac{a-t}{b-a}\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}f(s,y(s))ds-\frac{a-t}{b-a}\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}f(s,x(s))ds\right. \\
&\left.+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,y(s))ds-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,x(s))ds\right| \\
\leq\ & \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\epsilon+\frac{|a-t|}{b-a}\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|f(s,y(s))-f(s,x(s))|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|f(s,y(s))-f(s,x(s))|ds \\
\leq\ & \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\epsilon+\frac{m_f}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|y(s)-x(s)|^\lambda ds \\
&+\frac{m_f}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|y(s)-x(s)|^\lambda ds.
\end{aligned}
$$

Applying Lemma 2.13 to the above inequality and yields the aim inequality (4.2). □

## 5. Example

In this section, some examples are given to illustrate our theory results.

Let $0<\alpha<1$. We consider in the case $\mathbb{B}:=\mathbb{R}$ the equation

$$
{}^cD^\alpha x(t)=0,\ t\in[a,b), \tag{5.1}
$$

and the inequation

$$|^{c}D^{\alpha}y(t)| \le \epsilon,\ t \in [a,b). \tag{5.2}$$

Let $y \in C^{1}[a,b)$ be a solution of the inequation (5.2). Then there exists $g \in C[a,b)$ such that:

$$(i)\ |g(t)| \le \epsilon,\ t \in [a,b),$$

$$(ii)\ {}^{c}D^{\alpha}y(t) = g(t),\ t \in [a,b). \tag{5.3}$$

Integrating (5.3) from $a$ to $b$ by virtue of Definition 2.4, we have

$$y(t) = y(a) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}g(s)ds,\ t \in [a,b).$$

We have, for all $c \in \mathbb{R}$,

$$\begin{aligned} |y(t)-c| &= \left| y(a) - c + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}g(s)ds \right| \\ &\le |y(a)-c| + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|g(s)|ds \\ &\le |y(a)-c| + \frac{\epsilon}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}ds \\ &\le |y(a)-c| + \frac{(t-a)^{\alpha}\epsilon}{\Gamma(\alpha+1)},\ t \in [a,b). \end{aligned}$$

If we take $c := y(a)$, then

$$|y(t)-y(a)| \le \frac{(t-a)^{\alpha}\epsilon}{\Gamma(\alpha+1)},\ t \in [a,b).$$

If $b < +\infty$, then

$$|y(t)-y(a)| \le \frac{(b-a)^{\alpha}\epsilon}{\Gamma(\alpha+1)},\ t \in [a,b).$$

So, the equation (5.1) is Ulam-Hyers stable.

Let $b = +\infty$. The function

$$y(t) = \frac{(t-a)^{\alpha}\epsilon}{\Gamma(\alpha+1)}$$

is a solution of the inequality (5.2) and

$$|y(t)-c| = \left| \frac{(t-a)^{\alpha}\epsilon}{\Gamma(\alpha+1)} - c \right| \to +\infty,\ \text{as } t \to +\infty.$$

So, the equation (5.1) is not Ulam-Hyers stable on the interval $[a,+\infty)$.

Let us consider the inequation

$$|^{c}D^{\alpha}y(t)| \le \varphi(t),\ t \in [a,+\infty). \tag{5.4}$$

Let $y$ be a solution of (5.4) and $x(t) = y(a)$, $t \in [a,+\infty)$ a solution of the equation (5.1). We have that

$$|y(t)-x(t)| = |y(t)-y(a)| \le \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\varphi(s)ds,\ t \in [a,+\infty)$$

If there exists $c_{\varphi} > 0$ such that

$$\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\varphi(s)ds \le c_{\varphi}\varphi(t),\ t \in [a,+\infty),$$

then the equation (5.1) is generalized Ulam-Hyers-Rassias stable on $[a,+\infty)$ with respect to $\varphi$.

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