Energy, Laplacian energy of double graphs and new families of equienergetic graphs

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Abstract. For a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, the extended double cover G^* is a bipartite graph with bipartition (X, Y), $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, where two vertices x_i and y_j are adjacent if and only if $i = j$ or v_i adjacent to v_j in G. The double graph $D[G]$ of G is a graph obtained by taking two copies of G and joining each vertex in one copy with the neighbors of corresponding vertex in another copy. In this paper we study energy and Laplacian energy of the graphs G^* and $D[G]$, L-spectra of G^{k*} the k-th iterated extended double cover of G. We obtain a formula for the number of spanning trees of G^*. We also obtain some new families of equienergetic and L-equienergetic graphs.

1 Introduction

Let G be finite, undirected, simple graph with n vertices and m edges having vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Throughout this paper we denote such a

graph by $G(n, m)$. The *adjacency matrix* $A = (a_{ij})$ of G is a $(0, 1)$-square matrix of order n whose (i, j)-entry is equal to one if v_i is adjacent to v_j and equal to zero, otherwise. The spectrum of the adjacency matrix is called the A-*spectrum* of G. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the adjacency spectrum of G, the *energy* of G is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. This quantity introduced by I. Gutman [16] has noteworthy chemical applications.

Let $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix associated to G, where d_i is the degree of vertex v_i. The matrices $L(G) = D(G) - A(G)$ and $L^+(G) = D(G) + A(G)$ are called *Laplacian* and *signless Laplacian matrices* and their spectras are respectively called *Laplacian spectrum* (L-spectrum) and *signless Laplacian spectrum* (Q-spectrum) of G. Being real symmetric, positive semi-definite matrices, let $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$ and $0 \leq \mu_n^+ \leq \mu_{n-1}^+ \leq \cdots \leq \mu_1^+$ be respectively the L-spectrum and Q-spectrum of G. It is well-known that $\mu_n = 0$ with multiplicity equal to the number of connected components of G (see [11]). Fiedler [11] showed that a graph G is connected if and only if its second smallest Laplacian eigenvalue is positive and called it as the *algebraic connectivity* of the graph G. Also it is well-known [8] that for a bipartite graph the L-spectra and Q-spectra are identical. The *Laplacian energy* of a graph G as put forward by Gutman and Zhou [17] is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| .$$

This quantity, which is an extension of graph-energy concept has found remarkable chemical applications beyond the molecular orbital theory of conjugated molecules [22]. Both energy and Laplacian energy have been extensively studied in the literature (see [1, 2, 5, 9, 10, 12, 13, 14, 15, 18, 19, 20, 24, 25, 26, 27, 29, 30] and the references in them). Based on the above definition, the *signless Laplacian energy* of a graph G is defined as

$$LE^+(G) = \sum_{i=1}^{n} \left| \mu_i^+ - \frac{2m}{n} \right| ,$$

where μ_i^+ $(i = 1, 2, \ldots, n)$is the signless Laplacian spectra of G. It is easy to see that

$$tr(L(G)) = \sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n-1} \mu_i = 2m \quad \text{and} \quad tr(LE^+(G)) = \sum_{i=1}^{n} \mu_i^+ = 2m,$$

where tr is the *trace* of the matrix.

Two graphs G_1 and G_2 of same order are said to be *equienergetic* if $E(G_1) = E(G_2)$ [3, 23]. In analogy to this two graphs G_1 and G_2 of same order are said to L-*equienergetic* if $LE(G_1) = LE(G_2)$ and Q-*equienergetic* if $LE^+(G_1) = LE^+(G_2)$. Since cospectral (Laplacian cospectral) graphs are always equienergetic (L-equienergetic), the problem of constructing equienergetic (L-equienergetic) graphs is only considered for non-cospectral (non Laplacian cospectral) graphs.

The *extended double cover* [6] of the graph $G(n, m)$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is a bipartite graph G^* with bipartition (X, Y), $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, where two vertices x_i and y_j are adjacent if and only if $i = j$ or v_i adjacent v_j in G. It is easy to see that G^* is connected if and only if G is connected and a vertex v_i is of degree d_i in G if and only if it is of degree $d_i + 1$ in G^*. Also the extended double cover G^* of the graph G always contains a perfect matching. The double graph $D[G]$ of G is a graph obtained by taking two copies of G and joining each vertex in one copy with the neighbors of corresponding vertex in another copy. The k-fold graph $D^k[G]$ [21] of the graph G is obtained by taking k copies of the graph G and joining each vertex in one of the copy with the neighbors of the corresponding vertices in the other copies. If T_n is the graph obtained from the complete graph K_n by adding a loop at each of the vertex, it is easy to see that $D^k[G] = G \otimes T_k$. In this paper we study energy, Laplacian energy of the graphs G^* and $D[G]$, the L-spectra of G^{k*} the k-th iterated extended double cover of G and obtain a formula for the number of spanning trees of G^*. We also obtain some new families of the equienergetic and L-equienergetic graphs.

We denote the complement of graph G by $\overline{G}$, the complete graph on n vertices by K_n, the empty graph by $\overline{K}_n$ and the complete bipartite graph with cardinalities of partite sets q and r by $K_{q,r}$. The rest of the paper is organized as follows. In Section 2, energy of the graphs G^* and $D^k[G]$ are obtained and some new families of equienergetic graphs are given, in Section 3 L-spectra of G^{k*} and a formula for the number of spanning tress of G^* is obtained and in Section 4 Laplacian energy of the graphs G^* and $D^k[G]$ and the construction of some new families of L-equienergetic graphs by using the graphs G^{k*} and $D^k[G]$ is presented.

2 Energy of double graphs

In this section we find the energy of the graphs G^* and $D^k[G]$. We also construct some new families of equienergetic graphs based on these graphs.

For the graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$, the

Cartesian product is a graph $G = G_1 \times G_2$ with vertex set $V(G_1) \times V(G_2)$ and an edge $((u_1, v_1), (u_2, v_2))$ if and only if $u_1 = u_2$ and (v_1, v_2) is an edge of G_2 or $v_1 = v_2$ and (u_1, u_2) is an edge of G_1. The following result gives the A-spectra (L-spectra) of the Cartesian product of graphs.

Lemma 1 (Cvetkovic, Doob, Sachs, 1980 [7]) *If $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ are two graphs having A-spectra (L-spectra) respectively as, $\mu_1, \mu_2, \ldots, \mu_{n_1}$ and $\sigma_1, \sigma_2, \ldots, \sigma_{n_2}$, then the A-spectra (L-spectra) of $G = G_1 \times G_2$ is $\mu_i + \sigma_j$, where $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$.*

The *conjunction* (*Kronecker product*) of G_1 and G_2 is a graph $G = G_1 \otimes G_2$ with vertex set $V(G_1) \times V(G_2)$ and an edge $((u_1, v_1), (u_2, v_2))$ if and only if (u_1, u_2) and (v_1, v_2) are edges in G_1 and G_2, respectively. The following result gives the A-spectra (L-spectra) of the Kronecker product of graphs.

Lemma 2 (Cvetkovic, Doob, Sachs, 1980 [7]) *If $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ are two graphs having A-spectra (L-spectra) respectively as $\mu_1, \mu_2, \ldots, \mu_{n_1}$ and $\sigma_1, \sigma_2, \ldots, \sigma_{n_2}$, then the A-spectra (L-spectra) of $G = G_1 \otimes G_2$ is $\mu_i \sigma_j$, where $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$.*

The *join product* of G_1 and G_2 is a graph $G = G_1 \vee G_2$ with vertex set $V(G_1) \cup V(G_2)$ and an edge set consisting of all the edges of G_1 and G_2 together with the edges joining each vertex of G_1 with every vertex of G_2. The L-spectra of join product of graphs is given by the following result.

Lemma 3 (Cvetkovic, Doob, Sachs, 1980 [7]) *If $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ are two graphs having L-spectra respectively as $\mu_1, \mu_2, \ldots, \mu_{n_1-1}, \mu_{n_1} = 0$ and $\sigma_1, \sigma_2, \ldots, \sigma_{n_2-1}, \sigma_{n_2} = 0$, then the L-spectra of $G = G_1 \vee G_2$ is $n_1 + n_2, n_1 + \sigma_1, n_1 + \sigma_2, \ldots, n_1 + \sigma_{n_2-1}, n_2 + \mu_1, n_2 + \mu_2, \ldots, n_2 + \mu_{n_1-1}, 0.$*

The following result gives the A-spectra of G^*, the extended double cover of the graph G.

Theorem 4 (Cvetkovic, Doob, Sachs, 1980 [7], Chen, 2004 [6]) *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the A-spectra of a graph G, then the A-spectra of the graph G^* is $\pm(\lambda_1 + 1), \pm(\lambda_2 + 1), \ldots, \pm(\lambda_n + 1)$.*

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the A-spectra of the graph G, then by Lemma 1, the A-spectra of the graph $G \times K_2$ is $\lambda_i + 1, \lambda_i - 1$ for $1 \leq i \leq n$. It is clear from Theorem 4, that the graphs $G \times K_2$ and G^* are cospectral if and only if G is bipartite [6]. If $D^k[G]$ is the k-fold graph of the graph G, the A-spectra of $D^k[G]$ is given by the following result.

Theorem 5 (Cvetkovic, Doob, Sachs, 1980 [7], Marino, Salvi, 2007 [21]) *If* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *is the* A*-spectra of a graph* G*, then the* A*-spectra of the graph* $D^k[G]$ *is* $k\lambda_1, k\lambda_2, \ldots, k\lambda_n, 0$ $((k-1)n \text{ times})$.

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the A-spectra of the graph G, then by Theorem 4, the A-spectra of the graph G^* is $\pm(\lambda_1 + 1), \pm(\lambda_2 + 1), \ldots, \pm(\lambda_n + 1)$ and by Theorem 5, the A-spectra of $D^k[G]$ is $k\lambda_1, k\lambda_2, \ldots, k\lambda_n, 0$ $((k-1)n$ times). Therefore,

$$E(G^*) = \sum_{i=1}^{n} |\lambda_i + 1| + \sum_{i=1}^{n} |-\lambda_i - 1| = 2\sum_{i=1}^{n} |\lambda_i + 1|,$$

and

$$E(D^k[G]) = \sum_{i=1}^{n} |2\lambda_i| = 2\sum_{i=1}^{n} |\lambda_| = kE(G).$$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the A-spectra of a graph G, then the A-spectra of the graph $(G \otimes K_2) \times K_2$ is $\lambda_i + 1, \lambda_i - 1, -\lambda_i + 1, -\lambda_i - 1, 1 \leq i \leq n$. Therefore,

$$E((G \otimes K_2) \times K_2) = 2\sum_{i=1}^{n} |\lambda_i + 1| + 2\sum_{i=1}^{n} |\lambda_i - 1| = 2\left(\sum_{i=1}^{n} |\lambda_i + 1| + \sum_{i=1}^{n} |\lambda_i - 1|\right)$$
$$= 2E(G \times K_2) = E(2(G \times K_2)) = E((G \times K_2) \cup (G \times K_2)).$$

From the above discussion, we observe that the graphs $(G \otimes K_2) \times K_2$ and $(G \times K_2) \cup (G \times K_2)$ are equienergetic. Moreover, if the graph G is a bipartite graph then the graphs $(G \otimes K_2) \times K_2$ and $G^* \cup G^*$ are also equienergetic graphs.

As seen above $E(D^k[G]) = k\sum_{i=1}^{n} |\lambda_i| = kE(G) = E(kG) = E(G \cup G \cup \cdots \cup G)$ (G is repeated k times). This shows that the graphs $D^k[G]$ and $(G \cup G \cup \cdots \cup G)$ (G is repeated k times) are non-cospectral equienergetic. However, we show for any graph G the graphs D[G] and $G \otimes K_2$ are always equienergetic non-cospectral graphs.

Theorem 6 *If* $D[G]$ *is the double graph of the graph* G*, then the graphs* $G \otimes K_2$ *and* $D[G]$ *are non-cospectral equienergetic graphs.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the graph G, then by Lemma 2, the eigenvalues of the graph $G \otimes K_2$ are $\lambda_i, -\lambda_i$ for $1 \leq i \leq n$ and by Theorem 5 (for $k = 2$), the eigenvalues of the graph $D[G]$ are $2\lambda_i, \quad 0$ (n times) for $1 \leq i \leq n$. Therefore,

$$E(G \otimes K_2) = \sum_{i=1}^{n} |\lambda_i| + \sum_{i=1}^{n} |-\lambda_i| = 2\sum_{i=1}^{n} |\lambda_i|.$$

Also,

$$E(D[G]) = \sum_{i=1}^{n} |2\lambda_i| = 2 \sum_{i=1}^{n} |\lambda_i|.$$

Clearly these graphs are non-cospectral, so the result follows. □

In general, if $D^k[G]$ be the k-fold graph of the graph G, we have the following observation.

Theorem 7 *If $D^k[G]$ is the k-fold graph of the graph G, then the graphs $D^k[G]$ and $G \otimes sK_2$ are non-cospectral equienergetic graphs if and only if $k = 2^s$.*

Proof. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the graph, then by Lemma 2, the eigenvalues of the graph $G \otimes sK_2$ are λ_i (2^{s-1} times), $-\lambda_i$ (2^{s-1} times) for $1 \leq i \leq n$ and by Theorem 5, the eigenvalues of the graph $D^k[G]$ are $k\lambda_i$, 0 ($(k-1)n$ times) for $1 \leq i \leq n$. Therefore,

$$E(G \otimes sK_2) = 2^{s-1} \sum_{i=1}^{n} |\lambda_i| + 2^{s-1} \sum_{i=1}^{n} |-\lambda_i| = 2^s \sum_{i=1}^{n} |\lambda_i|. \tag{1}$$

Also,

$$E(D^k[G]) = \sum_{i=1}^{n} |k\lambda_i| = k \sum_{i=1}^{n} |\lambda_i|. \tag{2}$$

From (1) and (2) it is clear that $E(G \otimes sK_2) = E(D^k[G])$ if and only if $k = 2^s$. □

Let G^{**} be the extended double cover of the graph G^*. We have the following result.

Theorem 8 *If G is an n-vertex graph, then $E(G^* \otimes K_2) = E(G^{**})$, if $|\lambda_i| \geq 2$, for all non-zero eigenvalues of G. Moreover these graphs are non-cospectral with equal number of vertices.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the graph G. By Theorem 4, the eigenvalues of the graph G^* are $\lambda_i + 1, -(\lambda_i + 1)$ for $1 \leq i \leq n$ and so of G^{**} are $\lambda_i + 2, \lambda_i, -(\lambda_i + 2), -\lambda_i$ for $1 \leq i \leq n$. Also by Lemma 2, the eigenvalues of the graph $G^* \otimes K_2$ are $\lambda_i + 1, -(\lambda_i + 1), \lambda_i + 1, -(\lambda_i + 1)$ for $1 \leq i \leq n$. Assume that $|\lambda_i| \geq 2$. Then

$$|\lambda_i + 1| = \begin{cases} |\lambda_i| + 1, & \text{if } \lambda_i \geq 0 \\ |\lambda_i| - 1, & \text{if } \lambda_i < 0 \end{cases}, \quad |\lambda_i + 2| = \begin{cases} |\lambda_i| + 2, & \text{if } \lambda_i \geq 0 \\ |\lambda_i| - 2, & \text{if } \lambda_i < 0. \end{cases}$$

Therefore,

$$
E(G^{**}) = 2\sum_{i=1}^{n}|\lambda_i + 2| + 2\sum_{i=1}^{n}|\lambda_i| = 2\left(\sum_{\lambda_i \geq 0}|\lambda_i + 2| + \sum_{\lambda_i < 0}|\lambda_i + 2| + \sum_{i=1}^{n}|\lambda_i|\right)
$$

$$
= 2\left(\sum_{\lambda_i \geq 0}|\lambda_i| + 2 + \sum_{\lambda_i < 0}|\lambda_i| - 2 + \sum_{i=1}^{n}|\lambda_i|\right)
$$

$$
= 2\left(\sum_{i=1}^{n}|\lambda_i| + \sum_{i=1}^{n}|\lambda_i| + 2(\sum_{\lambda_i \geq 0}1 - \sum_{\lambda_i < 0}1)\right)
$$

$$
= 4\sum_{i=1}^{n}|\lambda_i| + 4\theta,
$$

where θ is the difference between the number of nonnegative and negative eigenvalues of G and

$$
E(G^* \otimes K_2) = 2\left(\sum_{i=1}^{n}|\lambda_i + 1| + \sum_{i=1}^{n}|-(\lambda_i + 1)|\right) = 4\sum_{i=1}^{n}|\lambda_i + 1|
$$

$$
= 4\left(\sum_{\lambda_i \geq 0}|\lambda_i + 1| + \sum_{\lambda_i < 0}|\lambda_i + 1|\right) = 4\left(\sum_{\lambda_i \geq 0}|\lambda_i| + 1 + \sum_{\lambda_i < 0}|\lambda_i| - 1|\right)
$$

$$
= 4\sum_{i=1}^{n}|\lambda_i| + 4\left(\sum_{\lambda_i \geq 0}1 - \sum_{\lambda_i < 0}1\right) = 4\sum_{i=1}^{n}|\lambda_i| + 4\theta.
$$

Clearly these graphs are noncospectral with same number of vertices. $\square$

Let G be a bipartite graph. It is well-known that the spectra of G is symmetric about the origin, so half of the nonzero eigenvalues of G lies to the left and half lies to the right of origin. Therefore if G is a bipartite graph having all its eigenvalues nonzero, the number of positive and negative eigenvalues of G are same. Keeping this in mind we have the following result.

Theorem 9 *If G^* is the extended double cover of the bipartite graph G, then the graphs G^* and $D[G]$ are noncospectral equienergetic if and only if $|\lambda_i| \geq 1$ for all $1 \leq i \leq n$.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the graph G. By Theorem 4, the eigenvalues of the graph G^* are $\lambda_i + 1, -\lambda_i - 1$ for $1 \leq i \leq n$ and by Theorem 5,

the eigenvalues of the graph $D[G]$ are $2\lambda_i$, 0 (n times) for $1 \leq i \leq n$. Suppose that $|\lambda_i| \geq 1$ for $i = 1, 2, \ldots, n$, then

$$|\lambda_i + 1| = \begin{cases} |\lambda_i| + 1, & \text{if } \lambda_i > 0 \\ |\lambda_i| - 1, & \text{if } \lambda_i < 0. \end{cases}$$

Therefore,

$$E(G^*) = \sum_{i=1}^{n} |\lambda_i + 1| + \sum_{i=1}^{n} |-\lambda_i - 1| = 2 \sum_{i-1}^{n} |\lambda_i + 1|$$

$$= 2 \left(\sum_{\lambda_i > 0} |\lambda_i + 1| + \sum_{\lambda_i < 0} |\lambda_i + 1| \right) = 2 \left(\sum_{\lambda_i > 0} (|\lambda_i| + 1) + \sum_{\lambda_i < 0} (|\lambda_i| - 1) \right)$$

$$= 2 \left((\sum_{\lambda_i > 0} |\lambda_i| + \sum_{\lambda < 0} |\lambda_i|) + (\sum_{\lambda_i > 0} 1 - \sum_{\lambda < 0} 1) \right) = 2 \sum_{i=1}^{n} |\lambda_i| = E(D[G]).$$

Clearly these graphs are noncospectral with same number of vertices.

Conversely, suppose that the graphs G^* and $D[G]$ are noncospectral equienergetic. We will show that $|\lambda_i| \geq 1$ for all $1 \leq i \leq n$.

Assume to the contrary that $|\lambda_i| < 1$ for some i. Then for this i, $|\lambda_i + 1| = \lambda_i + 1$. Without loss of generality, suppose that the eigenvalues of G satisfy $|\lambda_i| \geq 1$, for $i = 1, 2, \ldots, k$ and $|\lambda_i| < 1$, for $i = k+1, k+2, \ldots, n$, since the eigenvalues are real and reordering does not effect the argument. We have the following cases to consider.

Case i. If $\lambda_i > 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \geq 0$ for $i = k+1, k+2, \ldots, n$, then

$$E(G^*) = 2 \left(\sum_{i=1}^{k} |\lambda_i + 1| + \sum_{i=k+1}^{n} |\lambda_i + 1| \right) = 2 \left(\sum_{i=1}^{n} |\lambda_i| + n \right).$$

Case ii. If $\lambda_i > 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \leq 0$ for $i = k+1, k+2, \ldots, n$, then if θ_0 is the number of zero eigenvalues of G, we have

$$E(G^*) = 2 \left(\sum_{i=1}^{k} |\lambda_i + 1| + \sum_{i=k+1}^{n} |\lambda_i + 1| \right) = 2 \left(\sum_{i=1}^{k} (|\lambda_i| + 1) + \sum_{i=k+1}^{n} (\lambda_i + 1) \right)$$

$$> 2 \left(\sum_{i=1}^{k} (|\lambda_i| + 1) + \sum_{i=k+1}^{n} (|\lambda_i| - 1) \right) = 2 \left(\sum_{i=1}^{n} |\lambda_i| - \theta_0 \right).$$

Case iii. If $\lambda_i < 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \geq 0$ for $i = k+1, k+2, \ldots, n$, then

$$E(G^*) = 2\left(\sum_{i=1}^{k} |\lambda_i + 1| + \sum_{i=k+1}^{n} |\lambda_i + 1|\right) = 2\left(\sum_{i=1}^{k} (|\lambda_i| - 1) + \sum_{i=k+1}^{n} (|\lambda_i| + 1)\right)$$

$$= 2\left(\sum_{i=1}^{n} |\lambda_i| + \theta_0\right).$$

Case iv. If $\lambda_i < 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \leq 0$ for $i = k+1, k+2, \ldots, n$, then

$$E(G^*) = 2\left(\sum_{i=1}^{k} |\lambda_i + 1| + \sum_{i=k+1}^{n} |\lambda_i + 1|\right) = 2\left(\sum_{i=1}^{k} (|\lambda_i| - 1) + \sum_{i=k+1}^{n} (\lambda_i + 1)\right)$$

$$> 2\left(\sum_{i=1}^{k} (|\lambda_i| - 1) + \sum_{i=k+1}^{n} (|\lambda_i| - 1)\right) = 2\left(\sum_{i=1}^{n} |\lambda_i| - n\right).$$

Clearly in all these cases, we obtain $E(G^*) \neq E(D[G])$, a contradiction. Therefore the result follows. $\square$

We can also prove Theorem 9 by using Theorem 6, the fact that the graphs G^* and $G \times K_2$ are cospectral if G is bipartite [6, Theorem 2] and the graphs $G \times K_2$ and $G \otimes K_2$ are equienergetic if an only if $|\lambda_i| \geq 1$ [4, Theorem 8].

3 The Laplacian spectra of G^{k*}

Let G^* be the extended double cover of the graph G, define $G^{**} = (G^*)^*$, and in general $G^{k*} = (G^{(k-1)*})^*$, $k \geq 1$, called the k-qtextitth iterated double cover graph of G. The A-spectra of G^{k*} is given in [6]. Here we obtain the L-spectra of the k-th iterated extended double cover G^{k*} of the graph G. Since the graph G^{k*} is always bipartite for $k \geq 1$, therefore its Laplacian (L-spectra) and signless Laplacian (Q-spectra) spectra are same.

For any complex square matrices A and B of same order, the following observation can be seen in ([28, page 41]).

Theorem 10 *If A and B are complex square matrices of same order, then*

$$\begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A + B||A - B|,$$

where the symbol $|\,|$ denotes the determinant of a matrix.

We first obtain the L-spectra of G^*, the extended double cover of the graph G, in the following result.

Theorem 11 *Let* $G(n, m)$ *be an* n-vertex graph having Laplacian and signless Laplacian spectra, respectively as $0 = \mu_n < \mu_{n-1} \leq \ldots \leq \mu_1$ and $0 < \mu_n^+ < \mu_{n-1}^+ \leq \ldots \leq \mu_1^+$. Then the Laplacian spectra of G^* is $\mu_1, \mu_2, \ldots, \mu_n,\;\; \mu_1^+ + 2, \mu_2^+ + 2, \ldots, \mu_n^+ + 2$.

Proof. Let $A(G)$ be the adjacency matrix of the graph G. By a suitable relabelling of vertices it can be seen that the adjacency matrix $A(G^*)$ of the graph G^* is

$$A(G^*) = \begin{pmatrix} 0 & A(G) + I_n \\ A(G) + I_n & 0 \end{pmatrix}.$$

Let $D(G)$ and $D(G^*)$ be respectively the degree matrices of the graphs G and G^*. It is easy to see that

$$D(G^*) = \begin{pmatrix} D(G) + I_n & 0 \\ 0 & D(G) + I_n \end{pmatrix}.$$

Therefore, Laplacian matrix $L(G^*)$ of G^* is

$$L(G^*) = D(G^*) - A(G^*) = \begin{pmatrix} D(G) + I_n & -(A(G) + I_n) \\ -(A(G) + I_n) & D(G) + I_n \end{pmatrix}.$$

So the Laplacian characteristic polynomial of G^* is

$$C_{G^*}(\lambda) = |\lambda I_{2n} - L(G^*)| = \begin{vmatrix} (\lambda - 1)I_n - D(G) & A(G) + I_n \\ A(G) + I_n & (\lambda - 1)I_n - D(G) \end{vmatrix}$$

$$= |((\lambda - 1)I_n - D(G)) - (A(G) + I_n)| \, |((\lambda - 1)I_n - D(G)) + (A(G) + I_n)|$$

$$= |(\lambda - 2)I_n - (D(G) + A(G))| \, |\lambda I_n - (D(G) - A(G))|$$

$$= Q_G(\lambda - 2)C_G(\lambda).$$

From this the result follows.   □

We now obtain the L-spectra of G^{k*} as follows.

Theorem 12 *Let* $G(n, m)$ *be a graph having* L-*spectra* μ_i, *and* Q-*spectra* μ_i^+, $1 \leq i \leq n$. *The* L-*spectra of the graph* G^{k*} *is* $\mu_i \left(\binom{k}{0} \text{ times} \right)$, $\mu_i + 2 \left(\binom{k-1}{1} \text{ times} \right)$, $\mu_i^+ + 2 \left(\binom{k-1}{0} \text{ times} \right)$, $\mu_i + 4 \left(\binom{k-1}{2} \text{ times} \right)$, $\mu_i^+ + 4 \left(\binom{k-1}{1} \times \right)$, $\ldots, \mu_i + 2(k-2) \left(\binom{k-1}{k-2} \text{ times} \right)$, $\mu_i^+ + 2(k-2) \left(\binom{k-1}{k-3} \text{ times} \right)$, $\mu_i + 2(k-1) \left(\binom{k-1}{k-1} \text{ times} \right)$, $\mu_i^+ + 2(k-1) \left(\binom{k-1}{k-2} \text{ times} \right)$, $\mu_i^+ + 2k \left(\binom{k}{k} \text{ times} \right)$, *where* $1 \leq i \leq n$.

Proof. We prove this result by induction and we use induction on k. For $k = 1$, the result follows by Theorem 11. For $k = 2$, we have $G^{2*} = G^{**}$. Let $A(G^*)$ and $A(G^{**})$ be the adjacency matrices respectively of the graphs G^* and G^{**}. It is not difficult to see that

$$A(G^{**}) = \begin{pmatrix} 0 & A(G^*) + I_{2n} \\ A(G^*) + I_{2n} & 0 \end{pmatrix}.$$

Let $D(G^*)$ and $D(G^{**})$ be respectively the degree matrices of G^* and G^{**}. It can be seen that

$$D(G^{**}) = \begin{pmatrix} D(G^*) + I_{2n} & 0 \\ 0 & D(G^*) + I_{2n} \end{pmatrix}.$$

Therefore the Laplacian matrix of G^{**} is

$$L(G^{**}) = D(G^{**}) - A(G^{**}) =$$

$$\begin{pmatrix} D(G^*) + I_{2n} & -(A(G^*) + I_{2n}) \\ -(A(G^*) + I_{2n}) & D(G^*) + I_{2n} \end{pmatrix}.$$

So the Laplacian characteristic polynomial of G^{**} is

$$C_{G^{**}}(\lambda) = |\lambda I_{4n} - L(G^{**})| = \begin{vmatrix} (\lambda-1)I_{2n} - D(G^*) & A(G^*) + I_{2n} \\ A(G^*) + I_{2n} & (\lambda-1)I_{2n} - D(G^*) \end{vmatrix}$$

$$= |((\lambda-1)I_{2n} - D(G^*)) - (A(G^*) + I_{2n})| \, |((\lambda-1)I_{2n} - D(G^*)) + (A(G^*) + I_{2n})|$$

$$= |(\lambda-2)I_{2n} - (D(G^*) + A(G^*))| \, |\lambda I_{2n} - (D(G^*) - A(G^*))|$$

$$= Q_{G^*}(\lambda - 2) C_{G^*}(\lambda).$$

From this it is clear that the L-spectra of G^{**} is $\mu_i, \ \mu_i + 2, \ \mu_i^+ + 2, \mu_i^+ + 4$, for $1 \leq i \leq n$, that is L-spectra of G^{**} is $\mu_i \left(\binom{2}{0} \text{ times} \right), \ \mu_i + 2 \left(\binom{1}{1} \text{ times} \right),$ $\mu_i^+ + 2 \left(\binom{1}{0} \text{ times} \right)$, and $\mu_i^+ + 4 \left(\binom{2}{2} \text{ times} \right)$. Therefore the result is true in this case. Assume that the result is true for $k = s - 1$. Then by induction hypothesis the L-spectra of $G^{(s-1)*}$ is $\mu_i \left(\binom{s-1}{0} \text{ times} \right), \ \mu_i + 2 \left(\binom{s-2}{1} \text{ times} \right), \ \mu_i^+ +$ $2 \left(\binom{s-2}{0} \text{ times} \right), \ \ldots, \ \mu_i + 2(s-2) \left(\binom{s-2}{s-2} \text{ times} \right), \ \mu_i^+ + 2(s-2) \left(\binom{s-2}{s-3} \text{ times} \right),$ $\mu_i^+ + 2(s-1) \left(\binom{s-1}{s-1} \text{ times} \right)$. Now for $k = s$, it can be seen by proceeding as in the case $k = 2$ the Laplacian matrix $L(G^{s*})$ of the graph G^{s*} is

$$L(G^{s*}) = D(G^{s*}) - A(G^{s*}) = \begin{pmatrix} D(G^{(s-1)*}) + I_{2^{s-1}n} & -(A(G^{(s-1)*}) + I_{2^{s-1}n}) \\ -(A(G^{(s-1)*}) + I_{2^{s-1}n}) & D(G^{(s-1)*}) + I_{2^{s-1}n} \end{pmatrix}.$$

Therefore, the Laplacian characteristic polynomial of G^{s*} is

$$C_{G^{s*}}(\lambda) = |\lambda I_{2^s n} - L(G^{s*})$$

$$= \begin{vmatrix} (\lambda - 1)I_{2^{s-1}n} - D(G^{(s-1)*}) & A(G^{(s-1)*}) + I_{2^{s-1}n} \\ A(G^{(s-1)*}) + I_{2^{s-1}n} & (\lambda - 1)I_{2^{s-1}n} - D(G^{(s-1)*}) \end{vmatrix}$$

$$= \left| ((\lambda - 1)I_{2^{s-1}n} - D(G^{(s-1)*})) - (A(G^{(s-1)*}) + I_{2^{s-1}n}) \right| \times$$

$$\left| ((\lambda - 1)I_{2^{s-1}n} - D(G^{(s-1)*})) + (A(G^{(s-1)*}) + I_{2^{s-1}n}) \right|$$

$$= \left| (\lambda - 2)I_{2^{s-1}n} - (D(G^{(s-1)*}) + A(G^{(s-1)*})) \right|$$

$$\times \left| \lambda I_{2^{s-1}n} - (D(G^{(s-1)*}) - A(G^{(s-1)*})) \right|$$

$$= Q_{G^{(s-1)*}}(\lambda - 2)C_{G^{(s-1)*}}(\lambda).$$

Therefore, it follows that the L-spectra of the graph G^{s*} is $\mu_i \left(\binom{s-1}{0} \text{ times} \right)$, $\mu_i + 2 \left(\binom{s-2}{1} \text{ times} \right)$, $\mu_i^+ + 2 \left(\binom{s-2}{0} \text{ times} \right)$, ..., $\mu_i + 2(s-2) \left(\binom{s-2}{s-2} \text{ times} \right)$, $\mu_i^+ + 2(s-2) \left(\binom{s-2}{s-3} \text{ times} \right)$, $\mu_i^+ + 2(s-1) \left(\binom{s-1}{s-1} \text{ times} \right)$, $\mu_i + 2 \left(\binom{s-1}{0} \text{ times} \right)$, $\mu_i + 4(\binom{s-2}{1} \text{ times})$, $\mu_i^+ + 4(\binom{s-2}{0} \text{ times})$, ..., $\mu_i + 2(s-1)(\binom{s-2}{s-2} \text{ times})$, $\mu_i^+ + 2(s-1)(\binom{s-2}{s-3} \text{ times})$, $\mu_i^+ + 2s(\binom{s-1}{s-1} \text{ times})$.

Using $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$, $0 \leq r \leq k$ and $\binom{s-1}{0} = \binom{s}{0} = \binom{s-1}{s-1} = \binom{s-2}{s-2} = 1$, we see that the L-spectra of G^{s*} is $\mu_i(\binom{s}{0} \text{ times})$, $\mu_i + 2(\binom{s-1}{1} \text{ times})$, $\mu_i^+ + 2(\binom{s-1}{0}$ times), $\mu_i + 4(\binom{s-1}{2} \text{ times})$, $\mu_i^+ + 4(\binom{s-1}{1} \text{ times})$, ..., $\mu_i + 2(s-2)(\binom{s-1}{s-2} \text{ times})$, $\mu_i^+ + 2(s-2)(\binom{s-1}{s-3}$ times), $\mu_i + 2(s-1)(\binom{s-1}{s-1} \text{ times})$, $\mu_i^+ + 2(s-1)(\binom{s-1}{s-2}$ times), $\mu_i^+ + 2s(\binom{s}{s} \text{ times})$. Thus the result is true in this case as well hence by induction the result follows. $\square$

If G is a bipartite graph, it is easy to see that under elementary transformation the Laplacian characteristic polynomial of G coincides with the signless Laplacian characteristic polynomial of G. Therefore the Laplacian and signless Laplacian spectra of G are same. We have the following observation.

Corollary 13 *If $G(n, m)$ is a bipartite graph having L-spectra μ_i, $1 \leq i \leq n$, then the L-spectra of k-th iterated double cover G^{k*} of G is $\mu_i \left(\binom{k}{0} \text{ times} \right)$, $\mu_i + 2 \left(\binom{k}{1} \text{ times} \right)$, ..., $\mu_i + 2(k-2) \left(\binom{k}{k-2} \text{ times} \right)$, $\mu_i + 2(k-1) \left(\binom{k}{k-1} \text{ times} \right)$, $\mu_i + 2k \left(\binom{k}{k} \text{ times} \right)$, where $1 \leq i \leq n$.*

Proof. Since for a bipartite graph G the Laplacian and the signless Laplacian spectra are same, we have $\mu_i = \mu_i^+$ for all $1 \le i \le n$. Using this in Theorem 21, we obtain the L-spectra of G^{k*} as $\mu_i \left(\binom{k}{0} \text{ times} \right)$, $\mu_i + 2 \left(\binom{k-1}{1} \text{ times} \right)$, $\mu_i + 2 \left(\binom{k-1}{0} \text{ times} \right)$, $\mu_i + 4 \left(\binom{k-1}{2} \text{ times} \right)$, $\mu_i + 4 \left(\binom{k-1}{1} \text{ times} \right)$, ..., $\mu_i + 2(k-2) \left(\binom{k-1}{k-2} \text{ times} \right)$, $\mu_i + 2(k-2) \left(\binom{k-1}{k-3} \text{ times} \right)$, $\mu_i + 2(k-1) \left(\binom{k-1}{k-1} \text{ times} \right)$, $\mu_i + 2(k-1) \left(\binom{k-1}{k-2} \text{ times} \right)$, $\mu_i + 2k \left(\binom{k}{k} \text{ times} \right)$. Now using the fact $\binom{t}{r} + \binom{t}{r-1} = \binom{t+1}{r}$, $0 \le r \le t$, the result follows. $\square$

In [6] three formulae are given for the number of spanning trees of G^* in terms of A-spectra of the corresponding graph G. We now obtain a formula for the number of spanning trees in terms of the L and Q-spectra of G^*.

Theorem 14 *The number of spanning trees $\tau(G^*)$ of the graph G^* is*

$$\tau(G^*) = \frac{1}{2}\tau(G) \prod_{i=1}^{n}(\mu_i^+ + 2).$$

Proof. Let $0 = \mu_n < \mu_{n-1} \le \cdots \le \mu_1$ and $0 < \mu_n^+ < \mu_{n-1}^+ \le \cdots \le \mu_1^+$ be respectively the L-spectra and the Q-spectra of the graph G. By Theorem 3.2, the L-spectra of the graph G^* is $\mu_i, \mu_i^+ + 2$ for $i = 1, 2, \ldots, n$. By using the fact that the number of spanning trees of a graph of order n is $\frac{1}{n}$ times the product of $(n-1)$ largest Laplacian eigenvalues of the graph, we have

$$\tau(G^*) = \frac{1}{2n} \prod_{i=1}^{n-1}\mu_i \prod_{i=1}^{n}(\mu_i^+ + 2) = \frac{1}{2}\tau(G) \prod_{i=1}^{n}(\mu_i^+ + 2).$$

$\square$

In case G is bipartite, $\mu_i = \mu_i^+$, so we have

$$\tau(G^*) = \frac{1}{2n} \prod_{i=1}^{n-1}\mu_i \prod_{i=1}^{n}(\mu_i + 2) = \tau(G) \prod_{i=1}^{n-1}(\mu_i + 2).$$

In [6] it is shown that the graphs G^* and $G \times K_2$ are A-cospectral if and only if $G = K_1$ or G is bipartite. An analogous result holds for the L-spectra and is given below.

Theorem 15 *The graphs G^* and $G \times K_2$ are L-cospectral if and only if $G = K_1$ or G is bipartite.*

Proof. If $G = K_1$, the graphs G^* and $G \times K_2$ are both isomorphic to K_1, so are L-cospectral. Now if $G \neq K_1$, assume that G is bipartite. Then $\mu_i = \mu_i^+$ and so the L-spectra of G^* is $\mu_i, \mu_i + 2$ for $1 \leq i \leq n$ which is same as the L-spectra of $G \times K_2$. Conversely, suppose that the graphs G^* and $G \times K_2$ are L-cospectral. Then $\mu_i = \mu_i^+$, which is only possible if G is bipartite. Hence the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

An *integral graph* is a graph all of whose eigenvalues are integers. Following observation is a consequence of Theorem 12.

Theorem 16 *A graph G is Laplacian integral if and only if the graph G^{k*} is Laplacian integral graph.*

It is clear from Theorem 16, that given a Laplacian integral G it is always possible to construct an infinite sequence of Laplacian integral graphs. Indeed the graph G^{k*} is Laplacian integral for all $k \geq 1$.

Two graphs G_1 and G_2 are said to be co-spectral, if they are non-isomorphic and have the same spectra. We have the following result, which follows by Theorem 12.

Theorem 17 *Two graphs G_1 and G_2 are Laplacian cospectral if and only if the graphs G_1^{k*} and G_2^{k*} are Laplacian cospectral.*

Thus given two Laplacian co-spectral graphs G_1 and G_2, it is always possible to construct an infinite sequence of Laplacian co-spectral graphs. Indeed the graphs G_1^{k*} and Since the extended double cover G^* of the graph G is always bipartite, it follows by Theorem 6, the graphs G^{**} and $G^* \times K_2$ are L-cospectral and in general the graphs G^{s*} and $G^{(s-1)*} \times K_2$ are L-cospectral. Also it is easy to see that the graphs $(G \times K_2)^*$ and $G^* \times K_2$ are L-cospectral and in general the graphs $(G \times K_2)^{s*}$ and $G^{s*} \times K_2$ are both L-cospectral as well as Q-cospectral. Moreover, if G is bipartite then as seen in Theorem 6, the graphs G^* and $G \times K_2$ are L-cospectral. Using the same argument it can be seen that the graphs G^{**} and $G \times K_2 \times K_2$ are L-cospectral if and only if G is bipartite. A repeated use of the argument as used in Theorem 6, gives the graphs G^{s*} and $G \times K_2 \times K_2 \times \cdots \times K_2 = G \times sK_2 = G \times Q_s$ (where K_2 is repeated s times) are L-cospectral if and only if G is bipartite. From this discussion it follows that the graphs G^{s*}, $G^{(s-1)*} \times K_2$, $(G \times K_2)^{(s-1)*}$ and $G \times Q_{s-1}$ are mutually non-isomorphic L-cospectral graphs if and only G is bipartite, where Q_n is the hypercube.

4 Laplacian energy of double graphs

In this section, we study the Laplacian energy of the graphs $D[G]$, $D^k[G]$ and G^*. Using these graphs we obtain some new families of non Laplacian cospectral L-equienergetic graphs. Let $D[G]$ and G^* be respectively the double graph and the extended double cover of the graph G. Then the Laplacian spectra of the graph G^* is given by Lemma 2, and the Laplacian spectra of $D^k[G]$ is given by the following result.

Theorem 18 (Marino, Salvi, 2007 [21]) *Let G be a graph with n vertices having degrees $d_1, d_2, \ldots, d_n$ and let $\mu_1, \mu_2, \ldots, \mu_n$ be its Laplacian spectra. Then the Laplacian spectra of $D^k[G]$ is $k\mu_i, kd_i$ $((k-1)n$ times) for $1 \le i \le n$.*

Let μ_i for $1 \le i \le n$ be the L-spectra of the graph G. Then by Theorem 11, the L-spectra of the extended double cover G^* of the graph G is $\mu_i, \mu_i^+ + 2$ for $1 \le i \le n$. Also the average vertex degree of G^* is $\frac{2m}{n} + 1$. Therefore,

$$LE(G^*) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n} - 1| + \sum_{i=1}^{n} |\mu_i^+ - \frac{2m}{n} + 1|.$$

Since average vertex degree of $D^k[G]$ is $k\frac{2m}{n}$, we have

$$
\begin{aligned}
LE(D^k[G]) &= \sum_{i=1}^{n} \left| k\mu_i - k\frac{2m}{n} \right| + (k-1) \sum_{i=1}^{n} \left| kd_i - k\frac{2m}{n} \right| \\
&= k \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| + k(k-1) \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right| \\
&= kLE(G) + k(k-1) \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right|.
\end{aligned}
$$

From this it is clear that $LE(D^k[G]) = kLE(G)$, if G is regular. Also, since the k-fold graph of a regular graph is regular, it follows, if G_1 and G_2 are r-regular L-equienergetic graphs then their k-fold graphs $D^k[G_1]$ and $D^k[G_2]$ are also L-equienergetic. Let $\mathcal{L}(G)$ be the line graph of the graph G. It is shown in [23] that if G_1 and G_2 are r-regular graphs then their k-th ($k \ge 2$) iterated line graphs $\mathcal{L}^k(G_1)$ and $\mathcal{L}^k(G_2)$ are always equienergetic and so L-equienergetic. Therefore it follows that given any two r-regular graphs, we can always construct an infinite family of L-equienergetic graphs.

In case the given r-regular connected graphs are L-equienergetic, the k-fold graph forms a larger family of L-equienergetic graphs than the k-th iterated

line graph. As an example, consider the 4-regular graphs G_1 and G_2 shown in Figure 1 on 9-vertices. It can be seen that the L-spectra of G_1 and G_2 are respectively as $0, 3^4, 6^4$ and $0, 2, 3^2, 5^2, 6^3$ (where a^s means a occurs s times in the spectrum). Therefore $LE(G_1) = 16 = LE(G_2)$. This shows that the graphs G_1 and G_2 are regular L-equienergetic graphs, so their k-fold graphs $D^k[G_1]$ and $D^k[G_2]$ and their k-th ($k \geq 2$) iterated line graphs are also L-equienergetic. In fact the k-fold graph gives an infinite family of L-equienergetic graph pairs of order $n \equiv 0 \pmod 9$, whereas the k-th iterated line graph gives an infinite family of L-equienergetic graph pairs of orders $n = 542\,702\,430$, and so on, from this the assertion follows.
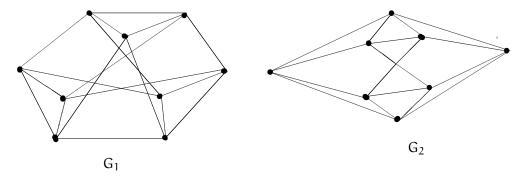

Figure 1

We have seen that the Laplacian energy of the graph $D[G]$ is twice the Laplacian energy of G when G is regular. But this need not be true for the graph G^* as seen from the Laplacian energy of G^* given above. However we have the following observation.

Theorem 19 *Let* G^* *be the extended double cover of the bipartite graph* G. *Then* $LE(G^*) = 2LE(G)$ *if and only if* $\left|\mu_i - \frac{2m}{n}\right| \geq 1$ *for* $1 \leq i \leq n$.

Proof. Let μ_i for $1 \leq i \leq n$ be the L-spectra of the graph G. Then by Corollary 13, the L-spectra of G^* is μ_i, $\mu_i + 2$ for $1 \leq i \leq n$. Assume that $\left|\mu_i - \frac{2m}{n}\right| \geq 1$, for all $i = 1, 2, \ldots, n$. Then since average vertex degree of G^* is $\frac{2m}{n} + 1$, we have

$$\left|\mu_i - \frac{2m}{n} + 1\right| = \begin{cases} \left|\mu_i - \frac{2m}{n}\right| + 1, & \text{if } \mu_i \geq \frac{2m}{n} \\ |\mu_i - \frac{2m}{n}| - 1, & \text{if } \mu_i < \frac{2m}{n}, \end{cases}$$

$$\left|\mu_i - \frac{2m}{n} - 1\right| = \begin{cases} \left|\mu_i - \frac{2m}{n}\right| - 1, & \text{if } \mu_i \geq \frac{2m}{n} \\ \left|\mu_i - \frac{2m}{n}\right| + 1, & \text{if } \mu_i < \frac{2m}{n}. \end{cases}$$

Therefore,

$$LE(G^*)$$

$$= \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} - 1 \right| + \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} + 1 \right|$$

$$= \sum_{i=1}^{n} \left(\left| \mu_i - \frac{2m}{n} - 1 \right| + \left| \mu_i - \frac{2m}{n} + 1 \right| \right)$$

$$= \sum_{\mu_i \geq \frac{2m}{n}} \left(\left| \mu_i - \frac{2m}{n} - 1 \right| + \left| \mu_i - \frac{2m}{n} + 1 \right| \right)$$

$$+ \sum_{\mu_i < \frac{2m}{n}} \left(\left| \mu_i - \frac{2m}{n} - 1 \right| + \left| \mu_i - \frac{2m}{n} + 1 \right| \right)$$

$$= \sum_{\mu_i \geq \frac{2m}{n}} \left(\left| \mu_i - \frac{2m}{n} \right| - 1 + \left| \mu_i - \frac{2m}{n} \right| + 1 \right)$$

$$+ \sum_{\mu_i < \frac{2m}{n}} \left(\left| \mu_i - \frac{2m}{n} \right| + 1 + \left| \mu_i - \frac{2m}{n} \right| - 1 \right)$$

$$= 2 \sum_{\mu_i \geq \frac{2m}{n}} \left| \mu_i - \frac{2m}{n} \right| + 2 \sum_{\mu_i < \frac{2m}{n}} \left| \mu_i - \frac{2m}{n} \right| = 2LE(G).$$

Conversely, suppose that $LE(G^*) = 2LE(G)$. We will show that $\left| \mu_i - \frac{2m}{n} \right| \geq 1$ for all $1 \leq i \leq n$. We prove this by contradiction. Assume that $\left| \mu_i - \frac{2m}{n} \right| < 1$, for some λ_j. Putting $\beta_i = \mu_i - \frac{2m}{n}$, and using the same argument as used in the converse of Theorem 8 in [4] we arrive at a contradiction. $\qquad \square$

If G is a graph satisfying the conditions of Theorem 19, then clearly the graphs G^* and $G \cup G$ are L-equienergetic. We now obtain some new families of L-equienergetic graphs by means of the graphs G^*, G^{k*}, $D[G]$ and $D^k[G]$.

Theorem 20 *Let $G_1(n, m)$ be a graph having L-spectra and Q-spectra respectively as μ_i and μ_i^+ and let $G_2(n, m)$ be another graph having L-spectra and Q-spectra respectively as λ_i and λ_i^+ for $i = 1, 2, \ldots, n$. Then for $p \geq 2n + k$ and $m \leq \frac{(k-1)n}{2} + \frac{k^2}{4}$, $k \geq 3$, we have $LE(G_1^* \vee \bar{K}_p) = LE(G_2^* \vee \bar{K}_p)$.*

Proof. Let G_1^* be the extended double cover of the graph G_1. Then by Theorem 11, the L-spectra of G_1^* is μ_i, $\mu_i^+ + 2$ for $1 \leq i \leq n$ and so by Lemma 3, the L-spectra of $G_1^* \vee \bar{K}_p$ is $p + 2n$, $p + \mu_i(1 \leq i \leq n - 1)$, $p + \mu_i^+ + 2(1 \leq i \leq n)$, $2n$

$((p-1)$ times), 0, with average vertex degree

$$\frac{2m'}{n'} = \frac{4m + 4pn + 2n}{p + 2n}.$$

Therefore,

$$LE(G_1^* \vee \bar{K}_p) = \left|p + 2n - \frac{2m'}{n'}\right| + \sum_{i=1}^{n-1}\left|p + \mu_i - \frac{2m'}{n'}\right| + \left|0 - \frac{2m'}{n'}\right|$$

$$+ \sum_{i=1}^{n}\left|p + \mu_i^+ + 2 - \frac{2m'}{n'}\right| + (p-1)\left|2n - \frac{2m'}{n'}\right|.$$

Now, if $p \geq 2n + k$ and $m \leq \frac{(k-1)n}{2} + \frac{k^2}{4}$, $k \geq 3$, we have for $i = 1, 2, \ldots, n$,

$$p + \mu_i - \frac{2m'}{n'} = p + \mu_i - \frac{4m + 4pn + 2n}{p + 2n} = \frac{p(p - 2n) + (2n + p)\mu_i - 4m - 2n}{p + 2n}$$

$$\geq \frac{k(2n + k) - 2(k-1)n - k^2 - 2n}{p + 2n} = 0,$$

and

$$p + \mu_i^+ + 2 - \frac{2m'}{n'} = p + \mu_i^+ + 2 - \frac{4m + 4pn + 2n}{p + 2n}$$

$$= \frac{p(p - 2n) + (2n + p)\mu_i^+ + 2(p + n) - 4m}{p + 2n}$$

$$\geq \frac{k(2n + k) - 2(k-1)n - k^2 + 2(3n + k)}{p + 2n} = \frac{8n + 2k}{p + 2n} \geq 0.$$

So we have

$$LE(G_1^* \vee \bar{K}_p) = \left(p + 2n - \frac{2m'}{n'}\right) + (n-1)\left(p - \frac{2m'}{n'}\right) + n\left(p + 2 - \frac{2m'}{n'}\right)$$

$$+ (p-1)\left(\frac{2m'}{n'} - 2n\right) + \frac{2m'}{n'} + 4m = 6n + (p - 2n)\frac{2m'}{n'} + 4m.$$

From this it is clear that the Laplacian energy of G_1^* depends only on the parameters p, m and n. Since these parameters are also same for G_2^*, it follows that $LE(G_1^* \vee \bar{K}_p) = LE(G_2^* \vee \bar{K}_p)$. In fact all the graphs of the family $(G_i^* \vee \bar{K}_p)$, $i = 1, 2, \ldots$, having the same parameters n, p and m satisfying the conditions in the hypothesis are mutually L-equienergetic. $\square$

Let G^{t*} be the t-th iterated extended double cover of the graph G. We have the following generalization of Theorem 20.

Theorem 21 *Let* $G(n, m)$ *be a graph having* L-*spectra and* Q-*spectra respectively as* μ_i *and* μ_i^+ *for* $1 \leq i \leq n$. *For* $p \geq 2^t n + k$ *and* $m \leq \frac{(k-t)n}{2} + \frac{k^2}{2^{t+1}}$, $k \geq t + 2, t \geq 1$, *we have* $LE(G^{t*} \vee \bar{K}_p) = 2^t n(t+2) + (p - 2^t n)\frac{2m'}{n'} + 2^t(2m)$.

Proof. Let G^{t*} be the t-th iterated extended double cover of the graph G. Then by Theorem 12, the L-spectra of G^{t*} is μ_i ($\binom{t}{0}$ times), $\mu_i + 2$ $\left(\binom{t-1}{1}\right.$ times$\left.\right)$, $\mu_i^+ + 2$ $\left(\binom{t-1}{0}\right.$ times$\left.\right)$, $\mu_i + 4$ $\left(\binom{t-1}{2}\right.$ times$\left.\right)$, $\mu_i^+ + 4$ $\left(\binom{t-1}{1}\right.$ times$\left.\right)$, ..., $\mu_i + 2(t-2)$ $\left(\binom{t-1}{t-2}\right.$ times$\left.\right)$, $\mu_i^+ + 2(t-2)$ $\left(\binom{t-1}{t-3}\right.$ times$\left.\right)$, $\mu_i + 2(t-1)$ $\left(\binom{t-1}{t-1}\right.$ times$\left.\right)$, $\mu_i^+ + 2(t-1)$ $\left(\binom{t-1}{t-2}\right.$ times$\left.\right)$, $\mu_i^+ + 2t$ ($\binom{t}{t}$ times), where $1 \leq i \leq n$. So by Lemma 2.3, the L-spectra of $G^{t*} \vee \bar{K}_p$ is $0, p+2^t n, 2^t n$ ($p-1$ times), $p+\mu_i$ ($\binom{t}{0}$ times) ($1 \leq i \leq n-1$), $p + \mu_i + 2$ ($\binom{t-1}{1}$ times), $p + \mu_i^+ + 2$ ($\binom{t-1}{0}$ times), $p + \mu_i + 4$ ($\binom{t-1}{2}$ times), $p + \mu_i^+ + 4$ ($\binom{t-1}{1}$ times), ..., $p + \mu_i + 2(t-2)$ ($\binom{t-1}{t-2}$ times), $p + \mu_i^+ + 2(t-2)$ ($\binom{t-1}{t-3}$ times), $p + \mu_i + 2(t-1)$ ($\binom{t-1}{t-1}$ times), $p + \mu_i^+ + 2(t-1)$ ($\binom{t-1}{t-2}$ times), $p + \mu_i^+ + 2t$($\binom{t}{t}$ times), $1 \leq i \leq n$, with average vertex degree

$$\frac{2m'}{n'} = \frac{2^{t+1}m + 2^t tn + 2^{t+1}pn}{p + 2^t n}.$$

Therefore,

$$LE(G^{t*} \vee \bar{K}_p)$$

$$= \sum_{i=1}^{n-1} \left| p + \mu_i - \frac{2m'}{n'} \right| + \sum_{r=1}^{t-1} \sum_{i=1}^{n} \binom{t-1}{r} \left| p + \mu_i + 2r - \frac{2m'}{n'} \right|$$

$$+ \sum_{r=1}^{t-1} \sum_{i=1}^{n} \binom{t-1}{r-1} \left| p + \mu_i^+ + 2r - \frac{2m'}{n'} \right| + \sum_{i=1}^{n} \left| p + \mu_i + 2t - \frac{2m'}{n'} \right|$$

$$+ \left| p + 2^t n - \frac{2m'}{n'} \right| + (p-1)\left| 2^t n - \frac{2m'}{n'} \right| + \left| 0 - \frac{2m'}{n'} \right|.$$

Now, if $p \geq 2^t n + k$ and $m \leq \frac{(k-t)n}{2} + \frac{k^2}{2^{t+1}}$, $k \geq t + 2, t \geq 1$, we have for $i = 1, 2, \ldots, n$ and $r = 0, 1, \ldots, t$

$$p + \mu_i + 2r - \frac{2m'}{n'} = p + \mu_i + 2r - \frac{2^{t+1}m + 2^t tn + 2^{t+1}pn}{p + 2^t n}$$

$$= \frac{p(p - 2^t n) + 2r(p + 2^t n) + (p + 2^t n)\mu_i - 2^{t+1}m - 2^t tn}{p + 2^t n}$$

$$\geq \frac{k(2^t n + k) - k(2^t n + k) + 2^t t n - 2^t t n}{p + 2^t n} = 0.$$

Similarly, it can be seen that $p + \mu_i^+ + 2r - \frac{2m'}{n'} \geq 0$. So we have

$LE(G^{t*} \vee \bar{K}_p)$

$$= (n-1)\left(p - \frac{2m'}{n'}\right) + \sum_{r=1}^{t-1}\left(n\left(p + 2r - \frac{2m'}{n'}\right) + 2m\right)\left[\binom{t-1}{r} + \binom{t-1}{r-1}\right]$$

$$+ \left(p + 2^t n - \frac{2m'}{n'}\right) + (p-1)\left(\frac{2m'}{n'} - 2^t n\right)$$

$$+ \left(n\left(p + 2t - \frac{2m'}{n'}\right) + 2m\right) + \frac{2m'}{n'} + 2m$$

$$= 2^{t+1} n - pn(2^t - 1) + (p-n)\frac{2m'}{n'}$$

$$+ \sum_{r=1}^{t}\binom{t}{r}\left(n\left(p + 2r - \frac{2m'}{n'}\right) + 2m\right) + 2m$$

$$= 2^{t+1} n - pn(2^t - 1) + (p-n)\frac{2m'}{n'} + n(2^t - 1)\left(p - \frac{2m'}{n'}\right)$$

$$+ (2^t - 1)2m + 2^t t n + 2m$$

$$= 2^t n(t+2) + (p - 2^t n)\frac{2m'}{n'} + 2^t(2m),$$

where we have made use of the fact $\left[\binom{t-1}{r} + \binom{t-1}{r-1}\right] = \binom{t}{r}$ and $\sum_{r=1}^{t} r\binom{t}{r} = t 2^{t-1}$.

Clearly the Laplacian energy of the graph $(G^{t*} \vee \bar{K}_p)$ depends only on the parameters p, m, t and n. Therefore all the graphs of the families $(G_i^{t*} \vee \bar{K}_p)$, where $t, i = 1, 2, \ldots$, with the same parameters p, m, t and n satisfying the conditions in the hypothesis are mutually L-equienergetic. □

Theorem 21 gives an infinite family of L-equienergetic graphs in various ways, firstly fix the value of t and allow p to vary we obtain families of L-equienergetic graphs with same t, secondly fix the value of p and allow t to vary we obtain families of L-equienergetic graphs with same p and so on.

Corollary 22 *Let $G(n, m)$ be a bipartite graph having L-spectra μ_i for $1 \leq i \leq n$. For $p \geq 2^t n + k$ and $m \leq \frac{(k-t)n}{2} + \frac{k^2}{2^{t+1}}$, $k \geq t + 2, t \geq 1$, we have $LE(G^{t*} \vee \bar{K}_p) = 2^t n(t+2) + (p - 2^t n)\frac{2m'}{n'} + 2^t(2m)$.*

Proof. The proof follows the proof of Corollary 13 and the same argument as in the proof Theorem 21. □

From Theorem 21, it is clear if G_1 and G_2 are any two graphs with the same parameters, then we can always find tripartite graphs $(G_1^* \vee \bar{K}_p)$ and $(G_2^* \vee \bar{K}_p)$ having the same Laplacian energy. Next we show the construction of L-equienergetic graphs by means of graphs $D[G]$ and $D^k[G]$.

Theorem 23 *Let* $D[G]$ *be the double graph of the graph* G. *Then, for* $p \geq 2n + k$ *and* $m \leq \frac{k(2n+k)}{8}$, $k \geq 4$, *we have*

$$LE(D[G] \vee \bar{K}_p) = 4n + (p - 2n)\frac{2m'}{n'} + 8m.$$

Proof. Let μ_i and d_i for $i = 1, 2, \ldots, n$ be respectively the L-spectra and the degree sequence of the graph G. Then by Theorem 21, the L-spectra of the graph $D^k[G]$ is $k\mu_i$, kd_i $((k-1)n$ times) and so by Lemma 3, the L-spectra of the graph $D^k[G] \vee \bar{K}_p$ is $p + kn$, $p + k\mu_i$ $(1 \leq i \leq n-1)$, $p + kd_i$ $((k-1)n$ times) $(1 \leq i \leq n)$, kn $((p-1)$ times), 0, with average vertex degree

$$\frac{2m'}{n'} = \frac{2k^2m + 2pkn}{p + kn}.$$

So, if $p \geq kn + t$ and $m \leq \frac{t(kn+t)}{2k^2}$, $t \geq 2k, k \geq 2$, we have for $i = 1, 2, \ldots, n$

$$p + k\mu_i - \frac{2m'}{n'} = p + k\mu_i - \frac{2k^2m + 2pkn}{p + kn}$$

$$= \frac{p(p - kn) - 2k^2m + k(p + kn)\mu_i}{p + kn}$$

$$\geq \frac{t(kn + t) - t(kn + t)}{p + kn} = 0.$$

Similarly, we see that

$$p + 2d_i - \frac{2m'}{n'} \geq 0.$$

Therefore,

$$LE(D[G] \vee \bar{K}_p) = \left| p + 2\mu_i - \frac{2m'}{n'} \right| + \sum_{i=1}^{n-1} \left| p + 2\mu_i - \frac{2m'}{n'} \right| + \sum_{i=1}^{n} \left| p + 2d_i - \frac{2m'}{n'} \right|$$

$$+ (p - 1)\left| 2n - \frac{2m'}{n'} \right| + \left| 0 - \frac{2m'}{n'} \right|$$

$$= 4n + (p - 2n)\frac{2m'}{n'} + 8m.$$

Clearly the Laplacian energy of the graph $D[G] \vee \bar{K}_p$ depends only on the parameters p, m and n. Therefore all the graphs of the family $(D[G_i] \vee \bar{K}_p)$, $i = 1, 2, \ldots$ with the same parameters p, m and n satisfying the conditions of the theorem, are mutually L-equienergetic. □

If $D^k[G]$ is the k-fold graph of the graph G, we have the following generalization of Theorem 23.

Theorem 24 *Let $D^k[G]$ be the k-fold graph of the graph G. Then for $p \geq kn + t$ and $m \leq \frac{t(kn+t)}{2k^2}$, $t \geq 2k, k \geq 2$, we have $LE(D^k[G] \vee \bar{K}_p) = 2kn + (p - nk)\frac{2m'}{n'} + 2mk^2$.*

Proof. Let μ_i and d_i for $i = 1, 2, \ldots, n$ be respectively the L-spectra and the degree sequence of the graph G. Then by Theorem 21, the L-spectra of the graph $D^k[G]$ is $k\mu_i$, kd_i $((k-1)n$ times) and so by Lemma 3, the L-spectra of the graph $D^k[G] \vee \bar{K}_p$ is $p + kn$, $p + k\mu_i$ $(1 \leq i \leq n-1)$, $p + kd_i$ $((k-1)n$ times) $(1 \leq i \leq n)$, kn $((p-1)$ times), 0, with average vertex degree

$$\frac{2m'}{n'} = \frac{2k^2m + 2pkn}{p + kn}.$$

So, if $p \geq kn + t$ and $m \leq \frac{t(kn+t)}{2k^2}$, $t \geq 2k, k \geq 2$, we have for $i = 1, 2, \ldots, n$

$$p + k\mu_i - \frac{2m'}{n'} = p + k\mu_i - \frac{2k^2m + 2pkn}{p + kn}$$

$$= \frac{p(p - kn) - 2k^2m + k(p + kn)\mu_i}{p + kn} \geq \frac{t(kn + t) - t(kn + t)}{p + kn} = 0.$$

Similarly, we see that

$$p + kd_i - \frac{2k^2m + 2pkn}{p + kn} \geq 0.$$

Therefore,

$LE(D^k[G] \vee \bar{K}_p)$

$$= \left| p + kn - \frac{2m'}{n'} \right| + \sum_{i=1}^{n-1} \left| p + k\mu_i - \frac{2m'}{n'} \right| + (k-1) \sum_{i=1}^{n} \left| p + 2d_i - \frac{2m'}{n'} \right|$$

$$+ (p - 1) \left| kn - \frac{2m'}{n'} \right| + \left| 0 - \frac{2m'}{n'} \right| = 2kn + 2mk^2 + (p - nk)\frac{2m'}{n'}.$$

From this it is clear the Laplacian energy of the graph $(D^k[G] \vee \bar{K}_p)$ depends on the parameters p, k, m and n. Therefore all the graphs of the families $(D^k[G_i] \vee \bar{K}_p)$ where $i = 1, 2, \ldots$, and $k = 2, 3, \ldots$ having the same parameters p, m, k and n satisfying the conditions of the Theorem, are mutually L-equienergetic. $\square$

Theorem 24 generates families of L-equienergetic graphs in various ways. If we allow p to vary and keep k fixed, we obtain an infinite family of L-equienergetic graphs with same k and if we allow k to vary and keep p fixed, we obtain an infinite family of L-equienergetic graphs with same p and so on.

If $D[G]$ and G^* are respectively the double graph and the extended double cover of the graph G, then the following result gives the construction of L-equienergetic graphs with different number of edges.

Theorem 25 *Let $G_1(n, m_1)$ and $G_2(n, m_2)$ be two graphs of order $n \equiv 0$ (mod 4) with $m_2 = m_1 + \frac{n}{4}$. Then for $p \geq 4n + k$ and $m_2 \leq \frac{n(k-2)}{4} + \frac{k^2}{16}$, $k \geq 4$, we have*

$$LE(D(G_1^*) \vee \bar{K}_p) = LE(D(G_2)^* \vee \bar{K}_p).$$

Proof. Let μ_i, d_i and μ_i^+ for $i = 1, 2, \ldots, n$ be respectively the L-spectra, degree sequence and Q-spectra of G_1 and let λ_i, d_i' and λ_i^+ be the L-spectra, degree sequence and Q-spectra of the graph G_2. Then by Theorems 11 and 18 and Lemma 3, the L-spectra of the graphs $D(G_1^*) \vee \bar{K}_p$ and $D(G_2)^* \vee \bar{K}_p$ are respectively as $p + 4n$, $p + 2\mu_i$ ($1 \leq i \leq n - 1$), $p + 2\mu_i^+ + 4, p + 2d_i + 2$ (2 times) ($1 \leq i \leq n$), $4n$ (($p - 1$) times), 0 and $p + 4n$, $p + 2\lambda_i$ ($1 \leq i \leq n - 1$), $p + 2\lambda_i^+ + 4$, $p + 2d_i' + 2$ (2 times) ($1 \leq i \leq n$), $4n((p - 1)$ times), 0, with average vertex degrees

$$\frac{2m_1'}{n'} = \frac{16m_1 + 8n + 8pn}{p + 4n}, \quad \frac{2m_2'}{n'} = \frac{16m_2 + 8n + 8pn}{p + 4n}.$$

Now, if $p \geq 4n + k$ and $m_2 \leq \frac{n(k-2)}{4} + \frac{k^2}{16}$, $k \geq 4$, we have for $i = 1, 2, \ldots, n$

$$p + 2\mu_i - \frac{2m_1'}{n'} = p + 2\mu_i - \frac{16m_1 + 8n + 8pn}{p + 4n}$$

$$= \frac{p(p - 4n) + 2(p + 4n)\mu_i - 16m_1 - 8n - 8pn}{p + 4n}$$

$$\geq \frac{k(4n + k) - 4n(k - 2) - k^2 - 8n}{p + 4n} = 0.$$

Similarly, we can show that

$$p + 2\mu_i^+ + 4 - \frac{2m_1'}{n'} \geq 0, p + 2d_i + 2 - \frac{2m_1'}{n'} \geq 0.$$

Therefore,

$$LE(D(G_1^*) \vee \bar{K}_p) = \left| p + 4n - \frac{2m_1'}{n'} \right| + \sum_{i=1}^{n-1} \left| p + 2\mu_i - \frac{2m_1'}{n'} \right|$$

$$+ \sum_{i=1}^{n} \left| p + 2\mu_i^+ + 4 - \frac{2m_1'}{n'} \right|$$

$$+ 2 \sum_{i=1}^{n} \left| p + 2d_i + 2 - \frac{2m_1'}{n'} \right|$$

$$+ (p-1) \left| 4n - \frac{2m_1'}{n'} \right| + \left| 0 - \frac{2m_1'}{n'} \right|$$

$$= 16n + 16m_1 + (p - 4n)\frac{2m_1'}{n'}.$$

Proceeding similarly for the graph $D(G_2)^* \vee \bar{K}_p$ it can be seen that

$$LE(D(G_2)^* \vee \bar{K}_p) = 12n + 16m_2 + (p - 4n)\frac{2m_2'}{n'}.$$

Using the fact $m_2 = m_1 + \frac{n}{4}$, the result follows.   □

Let $D[G_1]$ be the double graph of the graph $G_1(n, m_1)$ and let G_2^* be the extended double cover of the graph $G_2(n, m_2)$, then for $p \geq 2n + k$ and $m_1 \leq \frac{k(2n+k)}{8}$, $k \geq 4$, we have from Theorem 23

$$LE(D[G_1] \vee \bar{K}_p) = 4n + 8m_1 + (p - 2n)\frac{2m_1'}{n'}. \tag{3}$$

Also, for $p \geq 2n + k$ and $m_2 \leq \frac{n(k-1)}{2} + \frac{k^2}{4}$, $k \geq 4$, we have by Theorem 20

$$LE(G_2^* \vee \bar{K}_p) = 6n + 4m_2 + (p - 2n)\frac{2m_2'}{n'}. \tag{4}$$

If we suppose that $4m_1 = 2m_2 + n$, then it follows from (3) and 4 that

$$LE(D[G_1] \vee \bar{K}_p) = LE(G_2^* \vee \bar{K}_p).$$

This gives another construction of families of graphs with same Laplacian energy, same number of vertices but different number of edges. Next we give another way of constructing a family of graphs having same number of vertices, same Laplacian energy but different number of edges.

Theorem 26 *Let* $G_1(n, m_1)$ *and* $G_2(n, m_2)$ *be two graphs with* $m_2 = 2m_1$. *Then for* $p \geq 4n + k$ *and* $m_2 \leq \frac{k(4n+k)}{8} - n$, $k \geq 4$, *we have* $LE(D(G_1^*) \vee \bar{K}_p) = LE(G_2^{**} \vee \bar{K}_p)$.

Proof. Let μ_i, μ_i^+ and d_i for $i = 1, 2, \ldots, n$ be respectively the L-spectra, Q-spectra and the degree sequence of the graph G_1 and let λ_i and λ_i^+ be the L-spectra and Q-spectra of the graph G_2. Then by Theorems 11 and 18 and Lemma 3, the L-spectra of $D(G_1^*) \vee \bar{K}_p$ is $p + 4n$, $p + 2\mu_i$ $(1 \leq i \leq n-1)$, $p + 2\mu_i^+ + 4$, $p + 2d_i + 2$ (2 times) $(1 \leq i \leq n)$, $4n$ $((p-1)$ times), 0. Also by Theorem 12 and Lemma 3, the L-spectra of the graph $G_2^{**} \vee \bar{K}_p$ is $p + 4n$, $p + \lambda_i$ $(1 \leq i \leq n-1)$, $p + \lambda_i + 2$, $p + \lambda_i^+ + 2$, $p + \lambda_i^+ + 4$ $(1 \leq i \leq n)$, $4n((p-1)$ times), 0, with average vertex degrees

$$\frac{2m_1'}{n'} = \frac{16m_1 + 8n + 8pn}{p + 4n}, \frac{2m_2'}{n'} = \frac{8m_2 + 8n + 8pn}{p + 4n}.$$

So, if $p \geq 4n + k$ and $m_2 \leq \frac{k(4n+k)}{8} - n$, $k \geq 4$, we have for $i = 1, 2, \ldots, n$

$$p + 2\mu_i - \frac{2m_1'}{n'} = p + 2\mu_i - \frac{16m_1 + 8n + 8pn}{p + 4n}$$

$$= \frac{p(p - 4n) + 2(p + 4n)\mu_i - 16m_1 - 8n - 8pn}{p + 4n}$$

$$= \frac{k(4n + k) - k(4n + k) + 8n - 8n}{p + 4n} = 0.$$

Similarly, we can show

$$p + 2\mu_i^+ + 4 - \frac{2m_1'}{n'} \geq 0, p + 2d_i + 2 - \frac{2m_1'}{n'} \geq 0.$$

Therefore,

$$LE(D(G_1^*) \vee \bar{K}_p)$$

$$= |p + 4n - \frac{2m_1'}{n'}| + \sum_{i=1}^{n-1} |p + 2\mu_i - \frac{2m_1'}{n'}| + \sum_{i=1}^{n} |p + 2\mu_i^+ + 4 - \frac{2m_1'}{n'}|$$

$$+ 2\sum_{i=1}^{n} |p + 2d_i + 2 - \frac{2m_1'}{n'}| + (p-1)|4n - \frac{2m_1'}{n'}| + |0 - \frac{2m_1'}{n'}|$$

$$= 16n + (p - 4n)\frac{2m_1'}{n'} + 16m_1.$$

Proceeding similarly as above for the graph $G_2^{**} \vee \bar{K}_p$, we can see that

$$LE(G_2^{**} \vee \bar{K}_p) = 16n + (p - 4n)\frac{2m_2'}{n'} + 8m_2.$$

Using $m_2 = 2m_1$, the result follows. □

Theorem 26 generates L-equienergetic graphs with same number of vertices but different number of edges, infact when one graph contains twice the number of edges as contained in other. Lastly we give the construction of family of graphs with same number of vertices, edges and Laplacian energy by means of Cartesian product and extended double cover.

Theorem 27 *Let* $G_1(n, m)$ *and* $G_2(n, m)$ *be two connected non-bipartite graphs. Then for* $p \geq n + 2$, *and* $\min(\mu_n^+, \lambda_n^+) \geq \frac{2m}{n} - 2$ *we have* $LE(G_1^* \times K_p) = LE(G_2^* \times K_p)$ *if and only if* $LE(G_1) = LE(G_2)$.

Proof. Let $0 = \mu_n < \mu_{n-1} \leq \cdots \leq \mu_1$ and $0 < \mu_n^+ < \mu_{n-1}^+ \leq \cdots \leq \mu_1^+$ be respectively the L-spectra and the Q-spectra of the graph G_1 and let $0 = \lambda_n < \lambda_{n-1} \leq \cdots \leq \lambda_1$ and $0 < \lambda_n^+ < \lambda_{n-1}^+ \leq \cdots \leq \lambda_1^+$ be respectively the L-spectra and Q-spectra of the graph G_2. Then by Theorem 11 and Lemma 1, the L-spectra of the graphs $G_1^* \times K_p$ and $G_1^* \times K_p$ are respectively as $\gamma_i + q_j$ and $\theta_i + q_j$, $i = 1, 2, \ldots, 2n$, $j = 1, 2, \ldots, n$, where

$$\gamma_i = \begin{cases} \mu_i, & \text{if } i = 1, 2, \ldots, n \\ \mu_i^+ + 2, & \text{if } i = n + 1, n + 2, \ldots, 2n, \end{cases}$$

$$\theta_i = \begin{cases} \lambda_i, & \text{if } i = 1, 2, \ldots, n \\ \lambda_i^+ + 2, & \text{if } i = n + 1, n + 2, \ldots, 2n \end{cases}$$

and $p = q_1 = q_2 = \cdots = q_{p-1}$, $q_p = 0$ with average vertex degree

$$\frac{2m'}{n'} = \frac{2m}{n} + p.$$

Therefore,

$$\begin{aligned} LE(G_1^* \times K_p) &= \sum_{i=1}^{2n} \sum_{j=1}^{n} \left| \gamma_i + q_j - \frac{2m'}{n'} \right| \\ &= (p - 1) \sum_{i=1}^{2n} \left| p + \gamma_i - \frac{2m'}{n'} \right| + \sum_{i=1}^{n} \left| \gamma_i - \frac{2m_1'}{n'} \right| \\ &= (p - 1)LE(G_1) + 4pn - 4n. \end{aligned}$$

Similarly it can be seen that

$$LE(G_2^* \times K_p) = (p - 1)LE(G_2) + 4pn - 4n.$$

It is now clear that $LE(G_1^* \times K_p) = LE(G_2^* \times K_p)$ if and only if $LE(G_1) = LE(G_2)$, therefore the result follows. □

Since G^* is always bipartite, Theorem 27 gives the construction of connected graphs from a given pair of L-equienergetic bipartite graphs having same number of vertices, edges and Laplacian energy. Moreover if t is the first value of p satisfying the conditions in Theorem 27, then every value greater than t also satisfies this condition, therefore we obtain an infinite family of L-equienergetic graph pairs.

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