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Recognition of split-graphic sequences

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Abstract. Using different definitions of split graphs we propose quick algorithms for the recognition and extremal reconstruction of split sequences among integer, regular, and graphic sequences.

1 Basic definitions

In this paper a, b, l, m, n, p and q denote nonnegative integers with $b \geq a$ and $l + m \geq 1$. We follow the terminology of *Handbook of Graph Theory* [28] written by Gross, Yellen and Zhang.

An (a, b, n)-***graph*** is a loopless graph in which different vertices are connected at least by a and at most by b edges [43, 44]. A (b, b, l)-***graph*** is denoted by K_l^b and is called a b-***clique*** or b-***complete graph***. Clearly, $K_l^1 = K_l$, where K_l is the ***complete graph*** on l vertices. Its complement, $\overline{K}_l$ is called ***independent graph*** on l vertices.

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The ***join*** [12, 28, 66] of two graphs G and H is denoted by $G + H$. It has the following vertex set and edge set:

$$V(G + H) = V(G) \cup V(H)$$

and

$$E(G + H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G) \text{ and } v \in V(H)\}.$$

A nonincreasing integer sequence $\sigma = (s_1, \dots, s_n)$ with $s_1 \le b(n-1)$ and $s_n \ge a(n-1)$ is said (a, b, n)***-regular*** [43, 44]. A $(0, b, n)$-regular sequence shortly is said b***-regular*** [17]. An integer sequence σ is said (a, b, n)***-graphic***, if it is the degree sequence of an (a, b, n)-graph G [43, 44], and such a graph G is referred to as a ***realization*** of σ. An integer sequence is called ***even,*** if the sum of its elements is even.

In this paper we denote the integer sequences by σ and the degree sequences by δ.

In 1965 Fulkerson, Hoffman and McAndrew [24] proposed the following definition of (γ, δ)***-multigraphs with capacity bounds***. Let $n \ge 1$, $\delta = (d_1, \dots, d_n)$ and $\gamma = (c_{11}, \dots, c_{1n}, c_{21} \dots, c_{2n}, \dots, c_{n-1,n}, c_{n,n})$ sequences of nonnegative integers with $c_{ii} = 0$ and $c_{ij} = c_{ji}$ for $1 \le i < j \le n$. Fulkerson and his coathors call δ ***degree vector***, while γ is the ***capacity vector***. Let G_γ denote the graph in which there is an edge between the vertex with degree d_i and vertex with degree d_j, if $c_{ij} = 1$. The capacity vector γ has the ***odd-cycle condition*** if the graph G_γ has the property that any two of its odd length (simple) cycles either have a common vertices or there exists a pair of vertices, one vertex from each cycle, which are connected with an edge.

With other words, the distance between two odd length cycles is at most 1. In particular, if G_γ is bipartite (has no odd length cycle) or G_γ is complete (all c_{ij} equals to 1) then γ obviously satisfies the odd-cycle condition.

An (a, b, n)-regular sequence is said ***potentially*** K_l^b***-graphic,*** if it has a realization G containing K_l^b as a subgraph. If $b = 1$, then we write simply K_l instead of K_l^1.

An (a, b, n)-regular sequence $\sigma = (s_1, \dots, s_n)$ is said ***potentially*** A_l^b***-graphic,*** if it has a realization G containing J_l^b (definition see later) on vertices having degrees $s_1, \dots,$ s_{l+m}.

An (a, b, n)-regular sequence $\sigma = (s_1, \dots, s_n)$ is said ***potentially*** $A_{l,m}^b$***-graphic,*** if it has a realization G containing $J_{l,m}^b$ (definition see later) on vertices having degrees $s_1, \dots, s_{l+m}$.

A $(0, b, n)$-regular sequence $\sigma = (s_1, \dots, s_n)$ is said ***potentially*** $J_{l,m}^b$***-graphic*** if it has a realization G containing $J_{l,m}^b$ (definition see later) on vertices having degrees $s_1, \ldots, s_{l+m}$. If $b = 1$, then we write simply A_l insted $A^1_{b},$, $A^1_{l,m}$ instead of $A^b_{l,m}$, $J_{l,m}$ instead of $J^1_{l,m}$, and $J_{l,m}$ instead of $J^1_{l,m}$.

Let $n \geq 2$ and $\sigma = (s_1, \ldots, s_n)$ be a nonnegative integer sequence, and k be any integer $1 \leq k \leq n$. Let $\sigma' = (s'_1, \ldots, s'_n)$ be the sequence obtained from s by setting $s_k = 0$ and $s'_i = s_i - 1$ for the s_k largest elements of s other than s_k. Let H_k be the graph obtained on the vertex set $V = \{v_1, \ldots, v_n\}$ by joining v_k to the s_k vertices corresponding to the s_k elements used to obtain s'. This operation of getting s' and H_k is called ***laying off*** s_k, s' is called ***residual sequence***, and H_k is called the ***subgraph obtained by laying off*** s_k [51].

Now we formulate several definitions of split graphs.

The classical and most distributed definition of split graphs was introduced by Földes and Hammer in 1977 [21, 22, 26, 28].

Definition 1 (Földes, Hammer [21, 22]) *An* (l, m)-***partitioned split graph*** (*shortly: psplit graph*) *is one whose vertex set can be partitioned into two disjoint subsets spanning a clique* K_l *and an independent graph* $\overline{K}_m$. *It is denoted by* $S_{l,m}$.

It is worth to mention that one of l and m can be zero, that is if $l \geq 1$, then $S_{l,0}$ is also a psplit graph, and if $m \geq 1$, then $S_{0,m}$ is also a psplit graph, and the independent graph $\overline{K}_{0,m} = S_{0,m}$ are also psplit graphs. For the number of edges $|E(S_{l,m})|$ of an $S_{l,m}$ hold the inequalities $l(l-1)/2 \leq |E(S_{l,m})| \leq l(l-1)/2 + l \cdot m$ and between these bounds every integer value is realizable.

Consider the following example (Figure 1). Let $G = (V, E)$, where $V = \{v_1, \ldots, v_5\}$ and $E = (v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$, that is G contains six edges. Then G is $S_{3,2}$ and also $S_{4,1}$ due to the following two partitions of V: $\{v_1, v_2, v_3, v_4\}$ plus $\{v_4, v_5\}$ (Figure 1a)) containing all six edges and also is $S_{3,2}$ due to the partition $\{v_1, v_2, v_3\}$ plus $\{v_4, v_5\}$ (Figure 1b)) containing only three edges.

In 1996 Brandstädt introduced the following definition of (l, m)-multipartitioned split graphs. Let $G = (V, E)$ with $|V| = n$. $V_1, \ldots, V_k$ is a ***partition*** of V, if and only if for all $u, v \in \{1, \ldots, k\}$ with $i < j$ $V_i \cap V_j = \emptyset$ and $\bigcup_{i=1}^{k} V_i = V$. A partition $C_1, \ldots, C_m, I_1, \ldots, I_m$, with cliques C_i, $i \in \{1, \ldots, l\}$ and independent sets I_j, $j \in \{1, \ldots, m\}$ is an (l, m)-partition of V.

Definition 2 (Brandstädt [7, 8, 10]) *A graph* $G = (V, E)$ *is called an* (l, m)-***split graph***, *if its vertex set has an* (l, m)-*partition.*

In 1998 Gyárfás generalized (l, m)-psplit graphs to (l, m)-bsplit graphs.

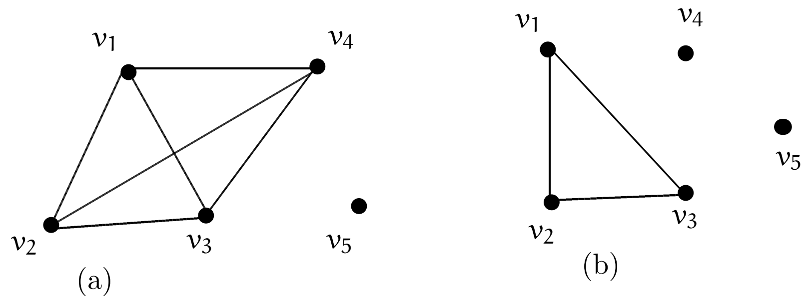


Figure 1: Partition of a psplit graph is not unique.

Definition 3 (Gyárfás [30]) *A graph* G *is called* (l, m)***-bounded split graph*** (*shortly: bsplit graph*) *if its vertex set can be partitioned into* A *and* B *so that the order of the largest clique graph in* A *is* l *and the order of the largest complete subgraph in* B *is* m*.*

In 2001 Hell, Klein, Protti, and Tito [41] defined (k, l)-graphs so, that their vertex set can be partitioned into k cliques and l independent sets.

In 2005 Bradstädt, Hammer, Le and Lozin studied bisplit graphs, defined as follows.

Definition 4 (Brandstädt, Hammer, Le, Lozin [9]) *A graph* G *is called* (l, m)***-bisplit graph*** (*shortly: bisplit graph*) *if its vertex set can be partitioned into a complete bipartite graph and an independent graph. It is denoted by* $B_{l,m}$*.*

For the number $|E(B_{l,m})|$ of edges of a bisplit graph $B_{l,m}$ hold the inequalities $l^2 \leq |E(B_{l,m})| \leq l^2 + 2lm$.

In 2007 Le and Ritter [55] defined probe split graphs (modifying the definition of interval split graphs [61]).

Definition 5 (Le, Ritter [55]) *A* $G(V, E)$ *graph is* ***probe split graphs****, if* V *can be partitoned into two parts* N *(nonprobes) and* P *(probes) where* N *is an independent set and there exists* $E' \subset N \times N$ *such that* $G' = (V, E \cup E')$ *is a psplit graph.*

In 2009 [6] Boros, Gurvich and Zverovich [6] proposed the definition of ***almost CIS-graphs***.

The following simple definition appeared in 2011 and later in the papers of different authors as Chat, Pirzada, and Yin [67, 86, 87].

Definition 6 (Yin [86, 87]) *An* (l, m)*-**join split graph** (shortly: jsplit graph) is the join of* K_l *and* $\overline{K}_m$*. It is denoted by* $J_{l,m}$.

It is worth to remark, that jsplit graphs are special cases of psplit graphs: if G is a jsplit graph, then any vertex of K_l is connected with any vertex of $\overline{K}_m$, while in the corresponding psplit graph even all such edges can be absent.

Consider the following example Let $H = (V, E)$, where $V = \{v_1, \ldots, v_5\}$ and $E = (v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_3, v_2v_4, v_2v_5, v_3v_4, v_3v_5, v_4v_5\}$, that is H is a clique on 5 vertices and so it contains ten edges. Then H is $J_{5,0}$ and also $J_{4,1}$ due to the following partition of V: $\{v_1, v_2, v_3.v_4, v_5\}$ plus $\emptyset$ and is $J_{4,1}$ for example due to the partition $\{v_1, v_2, v_3, v_4\}$ plus $\{v_5\}$. We can remark that these partitions at the same time give psplit graphs with the same size parameters.

Let K_l^b and K_m^b be b-cliques, and let $\overline{K}_m^b$ be the complement of K_m^b, that is an empty graph on m vertices. We propose the following generalization of psplit-graphs.

Definition 7 *A* (b, l, m)*-**partitioned split graph** (shortly:* b*-psplit graph) is one whose vertex set can be partitioned into two disjoint subsets spanning a* b*-clique* K_l^b *and an empty graph* $\overline{K}_m^b$*. It is denoted by* $S_{l,m}^b$.

Clearly, $S_{l,m}^1 = S_{l,m}$.

In 2011 Yin extended the definition of the jsplit-graphs to b-jsplit graphs.

Definition 8 (Yin [87]) *A* (b, l, m)*-**join-split graph** (shortly:* b*-jsplit graph) is the join of* K_l^b *and* $\overline{K}_m^b$*. It is denoted by* $J_{l,m}^b$.

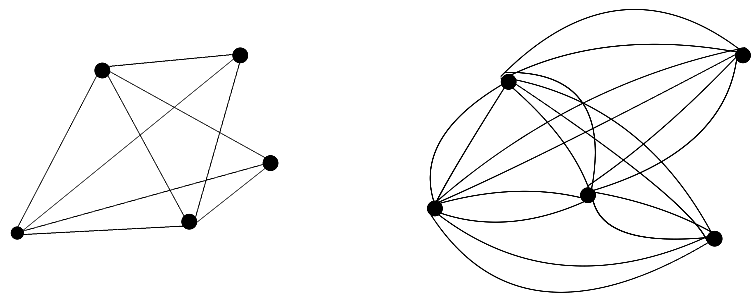
Clearly, $J_{l,m}^1 = J_{l,m}$.

Figure 2 shows $J_{3,2}$ (part a) and $J_{3,2}^2$ (part b).

The structure of the paper is as follows. After the basic definition (Section 1) in Section 2, 3 and 4 the most important mathematical background results connected with graphical, potentially graphical and potentially split graphical sequences are reviewed, then in Sections 5, 6 the new mathematical results are presented.

We review the known algorithms in Sections 7 and 8, while the now proposed algorithms are presented in Section 9.

The main results of the paper are that using different definitions of split graphs [4, 5, 7, 8, 9, 10, 21, 22, 23, 26, 28, 30, 67, 82, 86, 87, 88] we propose quick algorithms for the recognition and extremal reconstruction of split sequences among integer, regular [17, 45] and graphic [43, 45, 48] sequences.



a) b)

Figure 2: Jsplit graphs $J_{3,2}$ (a)) and $J^2_{3,2}$($\mathfrak{b}$)).

2 Known results on graphic sequences

In 1955 Havel, in 1962 Hakimi proposed the following necessary and sufficient condition for n-regular sequences to be graphic.

Theorem 9 (Havel [32], Havel [36]) *Let* $n \geq 2$. *An* n*-regular sequence* $\sigma = (s_1, \ldots, s_n)$ *is graphical if and only if the sequence* $\sigma' = (s_2 - 1, s_3 - 1, \ldots, s_{s_1} - 1, s_{s_1+1} - 1, s_{s_1+2}, \ldots, s_{n-1}, s_n)$ *is graphical.*

Proof. See Hakimi[32], Havel [36]. □

The recursive algorithm implementing this theorem requires in worst case $\Theta(n^2)$ time. It is worth to remark, that this algorithm not only tests the sequences, but if they are graphic, the algorithm constructs a realization of the tested sequence.

In 1973 Kleitman and Wang improved the Havel-Hakimi theorem: according to their following theorem it is sufficient to test *any* nonzero element of the input sequence. The central element of their proof is the laying off the tested sequence.

Theorem 10 (Kleitman, Wang [51]) *Let* $n \geq 2$. *A nonnegative integer sequence* σ *is graphic if and only if the residual sequence obtained by laying off any nonzero element of* s *is graphic.*

Proof. See Kleitman [51]. □

In 1974 Chungphaisan [13] extended the definition of laying off and residual sequence to b-***laying off*** and b-***residual sequence*** as follows. Let $\sigma =$

$(s_1, \ldots, s_n)$ be an n-regular sequence and $1 \le k \le n$. Define $\sigma_k^{'} = (s_1', \ldots, s_{n-1}')$ to be the nonincreasing rearrangement of the sequence obtained from $(s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_n)$ reducing by 1 the remaining largest term that has not already been reduced b times, and repeating the procedure s_k times. $s_k^{'}$ is called the *b-residual sequence* obtained from σ by b-***laying off*** s_k.

Using the b-laying off operation Chungphaisan proved the following generalization of Kleitman-Wang theorem.

Theorem 11 (Chungphaisan [13]) *Let* $n \ge 2$*. A nonnegative integer sequence* σ *is* b*-graphic if and only if the* b*-residual sequence obtained by* b*-laying off any nonzero element of* σ *is graphic.*

Proof. See Chungphaisan [13]. □

In 1960 Erdős and Gallai gave the following necessary and sufficient condition.

Theorem 12 (Erdős, Gallai [17]) *Let* $n \ge 1$*. An* n*-regular even sequence* $\sigma = (s_1, \ldots, s_n)$ *is graphical if and only if*

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$$

is satisfied for each integer k*,* $1 \le k \le n$*.*

Proof. See Erdős, Gallai [17]. □

Later several new proofs of this theorem were published, among others by to Gasharov in 1997, [25], by Tripathi and Tiagy in 2008 [77], by Tripathi, Venugopalan and West in 2010 [78].

In 1974 Chungphaisan extended Erdős-Gallai theorems to $(0, b, n)$-graphs.

Theorem 13 (Chungphaisan [13]). *Let* $\sigma = (s_1, \ldots, s_n)$ *be an* (a, b, n)*-regular even sequence. Then* σ *is* $(0, b, n)$*-graphic if and only if for each positive integer* $t \le n$*,*

$$\sum_{i=1}^{t} s_i \le bt(t-1) + \sum_{i=t+1}^{n} \min(bt, s_i).$$

Proof. See Chungphaisan [13]. □

We remark then if we use the theorems of Erdős-Gallai [17], Havel-Hakimi [32, 36], Kleitman-Wang [51] or Chungphaisan [13] to decide whether an integer sequence is graphic, the decision requires quadratic time. In 2012 Iványi proposed an algorithm for $(0, b, n)$ graphs, then in 2012 [45] for (a, b, n)-graphs allowing the decision in worst case in $O(n)$ time.

In the worst case the algorithm based on Theorem 13 requires quadratic time, but the following assertion allows us to test the sequences in linear time. Since this is an important result, we repeat its proof.

Theorem 14 (Iványi [45]) *If* $n \geq 1$*, then the* $\sigma = (s_1, \dots, s_n)$ $(0, b, n)$*-regular sequence is* $(0, b, n)$*-graphic if and only if*

$$\sum_{i=1}^{n} s_i \quad \textit{is even}$$

and

$$H_i > bi(y_i - 1) + H_n - H_y \quad (i = 1, \dots, n-1), \tag{1}$$

where

$$y_i = \max(i, w_i) \quad (i = 1, \dots, n-1). \tag{2}$$

Proof. This proof is an improved version of the proof of linearity of EGL in [48] and was published in 2012 [45]

We exploit that s is monotone and determine the capacity estimations $c_k = \min(jb, s_k)$ appearing in (1) in constant time. The base of the quick computation is again the sequence of the weight points $w(\sigma) = (w_1, \dots, w_{n-1})$ containing the *weight points* belonging to the elements of σ, and the sequence $y(\sigma) = (y_1, \dots, y_n)$ containing the cutting points of the elements of s. For given s_i the ***weight point*** w_i is the largest k $(1 \leq k \leq n)$ having the property $s_k \geq i$. . The ***cutting point*** y_i belonging to s_i is the maximum of i and w_i, see (2).

During the testing of the elements of s there are two cases:

a) if $i > w_i$, then the maximal contribution $C_i = \sum_{k=i+1}^{n} \min(i, s_k)$ of the actual tail of s is at most $H_n - H_i$, since the maximal contribution $c_k = \min(i, s_k)$ of the element s_k is only s_k, and so

$$C_i = \sum_{k=i+1}^{n} c_k = H_n - H_i,$$

implying the requirement

$$H_i \leq bi(i-1) + H_n - H_i; \tag{3}$$

b) if $i \le w_i$, then the maximal contribution C_i of the actual tail of s consists of contributions of two types: $c_{i+1}, \ldots, c_{w_i}$ are equal to bi, while $c_j = s_j$ for $j = w_i + 1, \ldots, n$, therefore we have

$$C_i = bi(w_i - i) + H_n - H_{w_i}, \tag{4}$$

implying the requirement

$$H_i = bi(i-1) + bi(w_i - i) + H_n - H_{w_i}. \tag{5}$$

Transforming (5) we get

$$H_i = bi(w_i - 1) + H_n - H_{w_i}. \tag{6}$$

Considering the definition of y_i given in (2), further (4) and (5) we get the required (1). □

In 1981 Rao [69] analyzed the conditions for graphic sequences to be P-graphic.

In 2009 Hell Hell and Kirkpatrick [40] extended the concept of graphic sequences defining quasi-graphic sequences and proposing a linear time algorithm for their certifying. A state of art of certifying algorithms was published in 2011 [59] by McConell, Mehlhorn, Näher and Schweitzer.

The following assertion is the base of the quick testing of integer sequences whether they are (a, b, n)-graphic or not. In 2012 Iványi proved that theorem of Chungphaisan has the following consequence allowing the quick test of potential (a, b, n)-graphic sequences..

Corollary 15 (Iványi [45]) *If* $n \ge 2$, *then an* (a, b, n)*-regular sequence* $\sigma = (s_1, \ldots, s_n)$ *is* (a, b, n)*-graphic if and only if the sequence* $s' = (s_1 - a, s_2 - a, \ldots, s_n - a)$ *is* $(0, b - a, n)$*-regular.*

Proof. See [45] □

This corollary allows the testing of (a, b, n)-regular sequences in worst case in $O(n)$ time using algorithm Erdős-Gallai-Linear [48] or algorithm Havel-Hakimi-Chungphaisan [45].

The following sources contain results on the enumeration of graphic and b-graphic sequences [35, 45, 46, 47, 48, 49, 74].

3 Known results on A-graphic sequences

The following two results due to J. H. Yin are generalizations from 1-graphs to b-graphs of two well-known results given by A. R. Rao [50, 68, 70]

Theorem 16 (Yin [84]). *Let $n \geq l+1$ and $\sigma = (s_1, \ldots, s_n)$ be a b-graphic sequence with $s_{l+1} \geq bl$. Then σ is potentially A_{l+1}^b-graphic if and only if s'_{l+1} is b-graphic.*

Proof. See [84]. □

Theorem 17 (Yin [84]) *Let $n \geq l+1$ and $\sigma = (s_1, \ldots, s_n)$ be a b-graphic sequence with $s_{l+1} \geq 2bl - 1$, then s is potentially A_{l+1}^b-graphic.*

Proof. See [84]. □

In 1978 Hakimi and Schmeichel [33] studied potentially and forcibly P-graphic sequences.

In 2009 Yin generalized a result Gould, Jacobson and Lehel [27].

Theorem 18 (Yin [85]) *If $\delta = (d_1, \ldots, d_n)$ is is a b-graphic sequence with a realization G containing a b-graph H as a subgraph, then there exists a realization G' of δ so that the vertices of H have the largest degrees of δ.*

Proof. See [85]. □

In 2009 Yin wrote [84] that the following assertion is a special case of Theorem 18.

Corollary 19 *A b-graphic sequence is potentially K_l^b-graphic, if and only if it is potentially A_l^b-graphic.*

Proof. We prove a bit stronger assertion.

It is trivial, that if an integer sequence is A_l^b-graphic, then it is b-graphic.

The sufficiency can be proved following the proof of Lemma 2.1. in [85]. □

We remark, that Theorem 18 contains only a sufficient condition.

Let $l, m,\ r$ and n be positive integers, $n \geq l+m$, and let $\sigma = (s_1,\ \ldots,\ s_n)$ be an n-regular sequence with $s_l \geq l+m-1$ and $s_{l+m} \geq l$. We construct the sequences $\sigma_1,\ \ldots,\ \sigma_l$ as follows. At first we construct the sequence

$$\sigma_1 = (s_1 - 1, \ldots, s_l - 1, s_{m+1}, s_{l+1}, \ldots, s_{l+m+1}^1, \ldots, s_n^1)$$

from σ by deleting s_1, reducing the first s_1 remaining elements of σ by one, and then reordering the last $n-l-m$ elements to be nonincreasing. For $2 \leq i \leq l$, we recursive construct

$$\sigma_i = (s_{i+1} - i, \ldots, s_l - i, s_{l+1} - i, \ldots, s_{l+1} - i, s_{l+m+1}^{i-1}, \ldots, s_n^i)$$

from σ_{i-1} by deleting $s_i - i + 1$, reducing the first $s_i - i + 1$ remaining elements of σ_{i-1} by one, and then reordering the last $n - l - m$ elements to be nonincreasingIn 2012 Yin proved the two following theorems.

In 2012 Yin proved the following two theorems.

Theorem 20 (Yin [87]) *σ is potentially A_b-graphic if and only if σ_b is graphic.*

Proof. See [87]. □

Theorem 21 (Yin [87]) *Let $n \geq l + m$ and let $\sigma = (s_1, \ldots, s_n)$ be a nonincreasing graphic sequence. If $s_{l+m} \geq l + m - 2$, then σ is potentially $A_{l,m}$-graphic.*

Proof. See [87]. □

Using the algorithm Erdős-Gallai-Linear [48] or algorithm Havel-Hakimi-Linear [45] we can decide in worst case in $O(n)$ time whether π is graphic.

The following theorem allows to decrease the expected running time of Havel-Hakimi-Split.

Theorem 22 (Yin [87]). *Let $n \geq l + m$ and let $\sigma = (s_1, \ldots, s_n)$ be an n-regular sequence. If $s_{r+s} \geq 2l + m - 2$, then σ is A_{l+m}-graphic.*

Proof. See [87]. □

In the same paper Yin [87] published a Havel-Hakimi type algorithm which constructs the corresponding $J_{l,m}$-graph.

Let $A_n = (a_1, \ldots, a_n)$ be an n-regular sequence, and $B_n = (b_1, \ldots, b_n)$ a sequence of nonnegative integers with $a_i \leq b_i$ and $a_i + b_i \geq a_{i+1} + b_{i+1}$ for $i = 1, \ldots, n - 1$. $(A_n; B_n)$ is said to be *potentially K_{m+1}-graphic* (resp. A_{m+1}-graphic) if there exists a graph G with vertices $v_1, \ldots, v_n$ such that $a_i \leq v_i(G) \leq b_i$ for $i = 1, \ldots, n$ and G contains K_{m+1} as a subgraph. In 2013 Yin [88] characterized $(A_n; B_n)$ so, that it is potentially A_{m+1}-graphic and potentially K_{m+1}-graphic.

In 2014 Yin [89] characterized the sequences having a realization containing an arbitrary subgraph.

In 2014 Pirzada and Chat proved the following assertion.

Theorem 23 (Pirzada, Chat [67]) *If G_1 is a realization of $\sigma_1 = (s_1^1, \ldots, s_l^1)$, containing K_l as a subgraph and G_2 is a realization of $\sigma_2 = (s_1^2, \ldots, s_m^2)$ containing K_m as a subgraph, then the degree sequence $\sigma = (s_1, \ldots, s_{l+m})$ of the join of G_1 and G_2 is K_{l+m}-graphic.*

Proof. See [67] □

4 Known results on split sequences

The ***girth*** $g(G)$ of a graph G containing at last one cycle is the length of its shortest cycle. The girth of an acyclic graph is infinite. A graph G is called ***chordal***, if it does not contain an induced subgraph with finite girth greater then 3.

In 1977 Földes and Hammer gave the following characterization of psplit graphs.

Theorem 24 (Földes, Hammer [22]) *For any graph G the following three conditions are equivalent:*
(i) G and $\overline{G}$ both are chordal;
(ii) $V(G)$ can be partitioned into a complete and an empty set;
(iii) G *does not contain an induced subgraph isomorphic to* $2K_2$*,* C_4 *or* C_5*.*

In 1993 Blázsik, Hujter, Pluhár and Tuza [5] characterized the *pseudo split graphs* defined as graphs with no induced C_4 and $2K_2$ (see also [2]). In 1998 Maffray and Preissmann [58] proved the following assertion.

Theorem 25 (Maffray, Preissmann [58]) G *is a pseudo split graph with a nonincreasing degree sequence* $\delta = (d_1, \ldots, d_n)$*, then* G *is a pseudo split graph, if* G *is a split graph or*

$$\sum_{i=1}^{q} d_i = q(q+4) + \sum_{i=m+1}^{n} d_i \tag{7}$$

and

$$d_{q+1} = d_{q+2} = d_{q+3} = d_{q+4} = d_{q+5} = q+2, \tag{8}$$

where $q = \max(i \mid d_i \geq q+4)$.

Proof. See [58]. □

The following theorem allows to design a linear time algorithm recognising the psplit graphs in linear time.

Theorem 26 (Golumbic [26]; Hammer, Simeone [34]; Tyshkevich [79]; Tyshkevich, Melnikow, Kotov [81], Wikipedia [82]) *Let the nonincreasing degree sequence of a graph* G *be* $\delta = (d_1, \ldots, d_n)$*, and let* m *be the largest value of* i

such that $d_i \geq i-1$. *Then* G *is a psplit graph if and only if*

$$\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i. \tag{9}$$

If this is the case, then the m *vertices with the largest degrees form a maximum clique in* G*, and the remaining vertices constitute and independent set.*

Proof. See [34]. □

An extremal problem for 1-graphic sequences to be potentially K_l^1-graphic was considered by Erdős, Jacobson and Lehel [19], and solved by Gould et al. [27] and Li et al. [56, 57]. Recently, Yin [85] generalized this extremal problem and the Erdős-Jacobson-Lehel conjecture from 1-graphs to b-graphs.

Different split graphs are closely connected with the problems of colorings of graphs, since the clique number gives a lower bound of coloring number. E.g. Erdős and Gyárfás [18], Gyárfás and Lehel [29] deal with coloring of psplit graphs. Yin and Li [90] gave sufficient conditions for graphic sequences to have a realization with prescribed clique size.

There are many publications on the maximal clique algorithms. Recently Zavalnij [91] analysed parallel algorithms for the the solution of the maximal clique problem. This problem was earlier studied e.g. in [60, 63, 64, 75, 72, 73, 76].

In 2000 observed a bijection between nonisomorphic psplit graphs and minimal covers of a set by its subsets. Using the formula proved by Clarke [15] for the number of minimal covers, Royle [71] proved the following assertion, giving the number $p(n)$ of the nonisomorphic psplit graphs on n vertices. This result was published also by Tyshkevich and Chernak in 1990.

Theorem 27 (Royle [71], Tyshkevich, Chernak [80]) *If* $n \geq 1$*, then*

$$p(n) = \sum_{k=1}^{n} t(n-k,k), \tag{10}$$

where

$$t(n,k) = \frac{1}{n!k!} \sum_{\alpha\in P_n,\beta\in P_k} \binom{n}{\alpha}\binom{\beta}{k} \prod_i \left(\left(\prod_j 2^{(\alpha_i,\beta_j)}\right)\right), \tag{11}$$

$$\binom{n}{\alpha} = \frac{n!}{\prod_i \mu_i! i^{\mu_i}}, \tag{12}$$

where μ_i *is the number of* i*'s in the partition* α, (u, v) *denotes the greatest common divisor of* u *and* v, P_n *is the set of all partitions of* n.

Figure 3 contains the values of $p(n)$ for $n = 1, \ldots, 20$. This data are taken from *Encyclopedia of Interger Sequences* [39] containing the values of $p(n)$ for $n = 1, \ldots, 40$ vertices.

n	$p(n)$	n	$p(n)$
1	1	11	64 956
2	2	12	501 696
3	4	13	5 067 146
4	9	14	67 997 750
5	21	15	1 224 275 498
6	56	16	29 733 449 510
7	164	17	976 520 265 678
8	557	18	43 425 320 764 422
9	2 223	19	2 616 632 636 247 976
10	10 766	20	213 796 933 371 366 930

Figure 3: The number $p(n)$ of nonisomorphic psplit graphs for $n = 1, \ldots, 20$ vertices.

In 1995 Nikolopoulos proposed a constant-time parallel algorithm for the recognition of psplit graphs.

Theorem 28 (Nikolopoulos [62]) *Let* $G(V, E)$ *be a graph with* $|V| = n$ *and* $|E| = m$*. Then algorithm* SPLIT-RECOGNITION *decides—using* $O(nm)$ *processors— in* $O(1)$ *time whether* G *is a psplit graph.*

Proof. See [62]. □

Several papers deals with the hamiltonicity of split graphs. E.g. in 1980 Burkard and Hammer [11] gave a necessary condition of the hamiltonicity of psplit graphs.

In 1988 Peemüller analyzed the condition of Burkard and Hammer and proved new necessary conditions for hamiltonian psplit graphs.

Theorem 29 (Peemüller [65]) *Let* $G = (C, I, E_1, E_2)$ *be a psplit graph with* $|C| < |I|$*. If* G *is hamiltonian, then*

$$2|X'| - m(X', Y') + f(X', Y') \leq m(Y', \overline{Y}) - fY', \overline{Y}), \tag{13}$$

for all $X' \subset X,\ X' \neq \emptyset, ,$ *while* m, f *and* N *is defined in* [65].

Proof. See [65]. □

In 1998 Woeginger proved the following property of the taughness [14] of psplit graphs solving a problem posed by Kratsch, Lehel and Müller in 1996 [53].

Theorem 30 (Woginger [83]) *The toughness of psplit graphs can be computed in polynomial time.*

Proof. See [83]. □

It is worth to remark that in 1990 Burkard, Hakimi and Schmeichel [3] that recognising of the toughness of a graph is NP-hard.

In 1999 Brandstädt, Le and Spinrad [10], in 2012 Almeida, Mello and Morgana [1] studied the classification problem of split graphs.

In 2006 Kratsch, McConnell, Mehlhorn, and Spinrad [52] reviewed certifying algorithms for recognizing interval graphs and permutation graphs

In 2008 Ibarra [42] studied fully dynamical algorithms of maintenance of psplit graphs.

In 2009 Heggernes and Mancini [38] analysed the minimal completion of psplit graphs.

In 2012 LaMar [54] defined directed psplit graphs and derived conditions for integer sequences to be degree sequences of directed psplit graphs.

In 2014 Habib and Mamcarz [31] investigated split decompositions of graphs.

5 New results for A-graphic sequences

In the next result, we use the Havel-Hakimi procedure to test whether a b-graphic sequence δ is potentially $A^b_{l,m}$-graphic.

Theorem 31 *Let $b \geq 1$ and $n \geq 1$. A b-graphic sequence $\sigma = (s_1, \ldots, s_n)$ is potentially $A^b_{l,m}$-graphic if and only if σ_l is b-graphic.*

Proof. Assume that σ is potentially $A^b_{l,m}$-graphic. Then σ has a realization G with the vertex set $V(G) = \{v_1, \ \mathtt{ldots}, v_n\}$ such that $d_G(v_i) = s_i$ for $(1 \leq i \leq n)$ and G contains $J^b_{l,m}$ on the vertices $v_1, \ldots, v_{l+m}$, where $l + m \leq n$, so that $V^b(K_l) = \{v_1, \ldots, v_l\}$ and $V(\overline{K}^b_m) = \{v_{l+1}, \ldots, v_{l+m}\}$. We will show that by applying a sequence of b-exchanges to G in order that there is one such realization G' such that $G' \setminus v_1$ has degree sequence σ_1. If not, we may choose such a realization H of b-graphic sequence σ such that the number of vertices adjacent to v_1 in $\{v_{l+m+1}, \ldots, v_{s_1+1}\}$ is maximum. Let $v_i \in \{v_{l+m+1}, \ldots, v_{s_1+1}\}$

and assume that there is no edge between ν_1 and ν_i and let $\nu_j \in \{\nu_{s_1+2}, \dots, \nu_n\}$ and there are b edges between $\nu_1\nu_j$. We may assume that $s_i > s_j$, since the order of i and j can be interchanged if $s_i < s_j$. Hence there is a vertex $\nu_t, t \neq i, j$ such that there are b edges between ν_i and ν_t and no edge between ν_j and ν_t. Clearly $G = \big(H \setminus \{\nu_1^b\nu_j, \nu_i^b\nu_t\}\big) \bigcup \{\nu_1^b\nu_i, \nu_j^b\nu_t\}$—where $\nu_i^b\nu_j$ means that there are b edges between ν_i and ν_j—is a realization of σ such that $d_G(\nu_i) = s_i$ for $1 \leq i \leq n$, G contains $S_{l,m}^b$ on $\nu_1, \dots, \nu_{l+m}$ with $V^b(K_l) = \{\nu_1, \dots, \nu_l\}$ and $V(\overline{K}_m^b) = \{\nu_{l+1}, \dots, \nu_{l+m}\}$ and G has the number of vertices adjacent to ν_1 in $\{\nu_{l+m+1}, \dots, \nu_{s_1+1}\}$ larger than that of H. This contradicts the choice of H. Repeating this procedure, we can see that σ_i is potentially A_{l-i}^b-graphic successively for $i = 2, \dots, l$. In particular, σ_l is b-graphic.

Conversely suppose that σ_l is b-graphic and is realized by a graph G_l with a vertex set $V(G_l) = \{\nu_{l+1}, \dots, \nu_n\}$ such that $d_{G_l}(\nu_i) = s_i$ for $l + 1 \leq i \leq n$. For $i = l, \dots, 1$ form G_{i-1} from G_i by adding a new vertex ν_i that is adjacent to each of $\nu_{i+1}, \dots, \nu_{l+m}$ with b-edges and also to the vertices of G_i with degrees $s_{l+m+1}^{i-1} - b, \dots, s_{d_i+1}^{i-1} - b$. Then for each i, G_i has degrees given by π_i and G_i contains $J_{l-i,m}^b$ on $l + m - i$ vertices $\nu_{i+1}, \dots, \nu_{l+m}$ whose degrees are $s_{i+1} - ib, \dots, s_{l+m} - ib$ so that $V(K_{l-i}^b) = \{\nu_{i+1}, \dots, \nu_l\}$ and $V(\overline{K}_m^b) = \{\nu_{l+1}, \dots, \nu_{l+m}\}$. In particular, G_0 has degrees given by σ and contains $S_{l,m}^b$ on $l + m$ vertices $\nu_1, \dots, \nu_{l+m}$ whose degrees are $s_1, \dots, s_{l+m}$ so that $V(K_l^b) = \{\nu_1, \dots, \nu_l\}$ and $V(\overline{K}_m^b) = \{\nu_{l+1}, \dots, \nu_{l+m}\}$. Hence the result follows. □

Now we prove a sufficient condition for a b-graphic sequence to be potentially A_l^b-graphic.

Theorem 32 *Let $n \geq l + m$ and let $\sigma = (s_1, \dots, s_n)$ be a b-graphic sequence. If $s_{l+m} \geq 2bl + bm - 2$, then σ_i is potentially $\overline{A}_{l,m}^b$-graphic.*

Proof. Let $n \geq l + m$ and let $\sigma = (s_1, \dots, s_n)$ be a nonincreasing b-graphic sequence with $s_{l+m} \geq 2bl + m - 2$. By Theorem 17, σ is potentially K_l^b-graphic and hence by Lemma 33, A_l^b-graphic. Therefore, we may assume that G is a realization of σ with a vertex set $V(G) = \nu_1, \dots, \nu_n$ such that $d_G(\nu_i) = s_i$, $(1 \leq i \leq n)$ and G contains K_l^b on $\nu_1, \dots, \nu_l$, that is, $V(K_l^b) = \{\nu_1, \dots, \nu_l\}$ and $M = e_G(\{\nu_1, \dots, \nu_l\}, \{\nu_{l+1}, \dots, \nu_{l+m}\})$ (that is, the number of edges between $\{\nu_1, \dots, \nu_l\}$ and $\{\nu_{l+1}, \dots, \nu_{l+m}\}$) is maximum. If $M = blm$, then G contains $\overline{S}_{l,m}^b$ on $\nu_1, \dots, \nu_{l+m}$ with $V(\overline{K}_m^r) = \{\nu_{l+1}, \dots, \nu_{l+m}\}$. In other-words, $sigma$ is potentially $\overline{A}_{l,m}^b$-graphic. Assume that $M < blm$. Then there exists a $\nu_k \in \{\nu_1, \dots, \nu_l\}$ and $\nu_m \in \{\nu_{l+1}, \dots, \nu_{l+m}\}, (i \neq j)$ such that $e_G(\nu_k, \nu_m) < b$. Let

$$A = N_{G \setminus \{\nu_l, \dots, \nu_{l+m}\}}(\nu_k) \setminus N_{G \setminus \{\nu_1, \dots, \nu_l\}}(\nu_m),$$

$$B = N_{G\setminus\{v_1,\dots,v_{l+m}\}}(v_k) \cap N_{G\setminus\{v_1,\dots,v_l\}}(v_m).$$

Then $e_G(x,y) = b$ for $x \in N_{G\setminus\{v_1,\dots,v_l\}}(v_m)$ and $y \in N_{G\setminus\{v_1,\dots,v_{l+m}\}}(v_k)$. Otherwise, if $e_G(x,y) < b$, then $G' = (G \setminus \{vy, v_m x\}) \cup \{v_k v_m, xy\}$ is a realization of π and contains $\overline{J}^b_{l,m}$ on $v_1,\dots,v_{l+m}$ with $V(K^b_l) = \{v_1,\dots,v_l\}$ and $(\overline{K}^b_m) = \{v_{l+1},\dots,v_{l+m}\}$ such that

$$e_{G'}(\{v_1,\dots,v_l\},\{v_{l+1},\dots,v_{l+m}\}) > M,$$

which contradicts the choice of G. Thus B is b-complete. We consider the following two cases.

Case 1. Let $A = \emptyset$. Then $2bl + bm - 2 \le d_k = d_G(v_k) < bl + bm - 1 + b|B|$, and so $|B| \ge bl$. Since each vertex in $N_{G\setminus v_1,\dots,v_l}(v_m)$ is adjacent to each vertex in B by b edges and $|N_{G\setminus\{v_1,\dots,v_l\}}(v_m)| \ge 2bl + bm - 2 = bl + bm - 1$. It can be easily seen that the b induced subgraph of $N_{G\setminus\{v_1,\dots,v_l\}}(v_m) \cup \{v_m\}$ in G contains $\overline{J}^b_{l,m}$ as a subgraph. Thus π is potentially $\overline{A}^b_{l,m}$- graphic.

Case 2. Let $A \ne \emptyset$. Let $a \in A$. If there are $x, y \in N_{G\setminus\{v_1,\dots,v_l\}}(v_m)$ such that $e_G(x,y) < b$ then $G' = (G_{\setminus\{v_m x, v_m y, v_k a\}}) \cup \{v_k v_m, a v_m, xy\}$ is a realization of σ and contains $\overline{J}^b_{l,m}$ on $v_1,\dots,v_{l+m}$ with $V(K^b_l) = \{v_1,\dots,v_l\}$ and $(\overline{K}^r_m) = \{v_{l+1},\dots,v_{l+m}\}$ such that $e_{G'}(\{v_1,\dots,v_l\},\{v_{l+1},\dots,v_{l+m}\}) > M$ which contradicts the choice of G. Thus $N_{G\setminus\{v_1,\dots,v_l\}}(v_m)$ is b-complete. Since

$$|N_{G\setminus\{v_1,\dots,v_l\}}(v_m)| \ge bl + bm - 1 \text{ and } e_G(v_m, z) = b$$

for any $z \in N_{G\setminus\{v_1,\cdots,v_l\}}(v_m)$, it is easy to see that the induced b-subgraph of $N_{G\setminus\{v_1,\dots,v_l\}}(v_m) \cup \{v_m\}$ in G is b-complete, and so contains $\overline{J}^b_{l,m}$ as a b-subgraph. Thus σ is potentially $\overline{A}^b_{l,m}$-graphic. □

6 New results for split sequences

Let $n \ge l + m$ and let $\sigma = (s_1,\dots,s_n)$ be a nonincreasing sequence of nonnegative integers with $s_l \ge b(l+m) - 1$ and $s_{l+m} \ge bl$. We define sequences $\sigma_1,\dots,\sigma_l$ as follows. We first construct the sequence

$$\sigma_1 = (s_2 - b, \dots, s_l - b, s_{l+1} - b, \dots, s_{l+m} - b, s^1_{l+m+1}, \dots, s^1_n)$$

from σ by reducing 1 the largest term that has not already been reduced b times, and then reordering the last $n - l - m$ terms to be nonincreasing. For $2 \le i \le b$, we construct

$$\sigma_i = (s_{i+1} - ib, \dots, s_l - ib, s_{l+1} - br, \dots, s_{l+m} - ib, s^i_{l+m+1}, \dots, s^i_n)$$

from

$$\sigma_{i-1} = (s_i - (i-1)b, \ldots, s_l - (i-1)b, s_{l+1} - (i-1)b, \ldots, \\ s_{l+m} - (i-1)b, s^{i-1}_{l+m+1}, \ldots, s^{i-1}_n)$$

by deleting $s_i - (i-1)b$, reducing the first $s_i - (i-1)b$ remaining terms of s_{i-1} by one that has not already been reduced b times, and then reordering the last $n - l - m$ terms to be nonincreasing.

We start with the following lemma.

Lemma 33 *A nonincreasing integer sequence* $\sigma = (s_1, \ldots, s_n)$ *is potentially* $A^b_{l,m}$*-graphic if and only if it is potentially* $J^b_{l,m}$*-graphic.*

Proof. We only need to prove that if $\sigma = (s_1, \ldots, s_n)$ is potentially $J^b_{l,m}$-graphic, then it is potentially $A^b_{l,m}$-graphic. We may choose a realization G of σ with vertex set $V(G) = \{v_1, \ldots, v_n\}$ such that $d_G(v_i) = s_i$ for $1 \leq i \leq n$, the induced b-subgraph $G[\{v_1, \ldots, v_{l+m}\}]$ of $\{v_1, \ldots, v_{l+m}\}$ in G contains $J^b_{l,m}$ as its b-subgraph and $|V(K^b_l) \cap \{v_1, \ldots, v_l\}|$ is maximum. Denote $H = G[\{v_1, \ldots, v_{l+m}\}]$. If $|V(K^b_l) \cap \{v_1, \ldots, d_l\}| = l$, that is, $V(K^b_l) = \{v_1, \ldots, v_l\}$, then σ is potentially $A^b_{l,m}$-graphic. Assume that $|V(K^b_l) \cap \{v_1, \ldots, v_l\}| < l$. Then there exists $v_i \in \{v_1, \ldots, v_l\} \setminus V(K^b_l)$ and a $v_j \in V(K^b_l) \setminus \{v_1, \ldots, v_l\}$. Let $A = N_H(v_j) \setminus (\{v_i\} \cup N_H(v_i))$ and $B = N_G(v_i) \setminus (\{v_j\} \cup N_G(v_j))$. Since $d_G(v_i) \geq d_G(v_j)$. We have $|B| \geq |A|$. Let us choose any subset $C \subseteq B$ such that $|C| = |A|$. Now form a new realization G' of s by a sequence of b-exchanges the b-edges of the star centralized at v_j with end vertices in A with the non b-edges of the star centralized at v_j with end vertices in C, and by a sequence of b-exchange the b-edges of the star centralized at v_i with end vertices in C with the non b-edges of the star centralized at v_i with end vertices in A. It is easy to see that G' contains $J^b_{l,m}$ on $\{v_1, \ldots v_{l+m}\}$ so that $|V(K^b_l) \cap \{v_1, \ldots, v_l\}|$ is larger than that of G, which contradicts to the choice of G. □

In the next result, we use the result of Fulkerson et al. [24] and prove a necessary and sufficient condition for a b-graphic sequence s to be potentially $J^b_{l,m}$-graphic.

Theorem 34 *Let* $n \geq l + m$ *and* $\sigma = (s_1, \ldots, s_n)$ *be a nonincreasing even sequence of nonnegative integers, where* $s_l \geq b(l + m - 1)$ *and* $s_{l+m} \geq lb$.

Then σ *is potentially* $J^b_{l,m}$*-graphic if and only if*

$$\sum_{i=1}^{p}(s_i - b(l+m-1)) + \sum_{i=b+1}^{l+p'}(s_i - bl) + \sum_{i=l+m+1}^{l+m+q} s_i \leq$$
$$r(p+p'+q)(p+p'+q-1) - rp(p-1) - 2bpp'$$
$$+ \sum_{i=p+1}^{r} \min\{bq, s_i - b(l+m-1)\}$$
$$+ \sum_{i=l+p'+1}^{l+m} \min\{b(p'+q), s_i - bl\} + \sum_{i=l+m+q+1}^{n} \min\{b(p+p'+q), s_i\}$$

for any $1 \leq l \leq n$, $1 \leq m \leq n$, *for any* p, p' *with* $0 \leq p \leq l$, $0 \leq p' \leq m$ *and* $0 \leq q \leq n-l-m$.

Proof. To prove the necessity, by Lemma 33, let G be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ such that $d_G(v_i) = s_i$ for $1 \leq i \leq n$ and G contains $J^b_{l,m}$ on $v_1, \ldots, v_{l+m}$ with $V(K^b_l) = \{v_1, \ldots, v_l\}$ and $V(\overline{K}^b_m) = \{v_{l+1}, \ldots, v_{l+m}\}$. The removal of the b edges induced by $\{v_1, \ldots, v_l\}$ and the b-edges between $\{v_1, \ldots, v_l\}$ and $\{v_{l+1}, \ldots, v_{l+m}\}$ results in a graph G' in which all degrees in $\{v_1, \ldots, v_l\}$ are reduced by $b(l+m-1)$ and all degrees in $\{v_{l+1}, \ldots, v_{l+m}\}$ are reduced by lb. For $0 \leq p \leq l, 0 \leq p' \leq m$ and $0 \leq q \leq n-l-m$, denote $P = \{v_i | 1 \leq i \leq p\}, P' = \{v_i | l+1 \ leqi \leq l+p'\}, R = \{v_i | p+1 \leq i \leq l\}, R' = \{v_i | l+p'+1 \leq i \leq l+m\}, Q = \{v_i | l+m+1 \leq i \leq q+l+m\}$ and $S = \{v_i | q+l+m+1 \leq i \leq n\}$. The degree sum in the b-subgraph induced by $P \cup P' \cup Q$ is at most $b(p+p'+q)(p+p'+q-1) - bp(p-1) - 2bpp'$. Therefore,

$$m = \sum_{i=1}^{p}(s_i - b(l+m-1)) + \sum_{i=r+1}^{b+p'}(s_i - bl) + \sum_{i=l+m+1}^{l+m+q} s_i$$
$$- b(p+p'+q)(p+p'+q-1) - bp(p-1) - 2bpp'$$

is the minimum number of edges of G' with exactly one end vertex in $P \cup P' \cup Q$. On the other hand, the maximum number of edges of G' with exactly one end vertex in $R \cup R' \cup S$ is

$$M = \sum_{i=p+1}^{b} \min\{bq, s_i - b(l+m-1)\} + \sum_{i=b+p'+1}^{l+m} \min\{b(p'+q), s_i - bl\}$$

$$+ \sum_{i=l+m+q+1}^{n} \min\{b(p+p'+q), s_i\}$$

We observe that in the graph G', $m \leq M$ is true. This proves the necessity.

To prove the sufficiency, we shall use the following well-known result of Fulkerson et al. [24]. Let H be a b-graph on the vertex set $V(H) = \{v_1, \ldots, v_n\}$. There exists a b-subgraph $G \subseteq H$ such that every vertex v_i has degree s_i, if and only if

$$\sum_{i=1}^{n} s_i \text{ is even,} \tag{14}$$

and for every $A, B \subseteq V(H)$ such that $A \cap B = s$, we have

$$\sum_{v_i \in A} s_i \leq \sum_{v_i \in A, v_j \in V(H) \setminus B} e_H(v_i, v_j) + \sum_{v_i \in B} s_i. \tag{15}$$

We now continue to proceed with the proof of sufficiency. Let $n \geq l+m$ and $\sigma = (s_1, \ldots, s_n)$ be a nonicreasing sequence of nonnegative integers, where $s_l \geq l+m-1, s_{l+m} \geq l$ and $\sum\limits_{i=1}^{n} s_i$ is even. Let $s' = (s'_1, \ldots, s'_n)$, where $s'_i = s_i - l - m + 1$ for $1 \leq i \leq l, s'_i = s_i - b$ for $l+1 \leq i \leq l+m$ and $s'_i = s_i$ for $l+m+1 \leq i \leq n$. Let H be the graph obtained from K_n^b with vertex set $V(K_n^b) = \{v_1, \ldots, v_n\}$ by deleting all edges of the complete b-subgraph induced by $\{v_1, \ldots, v_l\}$ and all edges between $\{v_1, \ldots, v_l\}$ and $\{v_{1+1}, \ldots, v_{l+m}\}$. It is easy to see that s is potentially $A_{l,m}^b$-graphic if and only if H has a subgraph G with the degree sequence s' such that every vertex v_i has degree s'_i. Observe that between two disjoint odd cycles of H there is an edge. Therefore, H satisfies the odd-cycle condition and we apply (14) and (15).

Let $K = \{v_1, \ldots, v_l\}, K' = \{v_{l+1}, \ldots, v_{l+m}\}$ and $A, B \subseteq V(H)$ such that $A \cap B = s$. Let $A_1 = A \cap K, A'_1 = A \cap K', A_2 = A \setminus (K \cup K'), B_1 = B \cap K, B'_1 = B \cap K,\ B_2 = B \setminus (K \cup K')$ and set $p = |A_1|,\ p' = |A'_1|,\ q = |A_2|, b_1 = |B_1|, b'_1 = |B'_1|, b_2 = |B_2|$. For convenience, we denote

$$L(p, p', q) = \sum_{i=1}^{p} (s_i - b(l+m-1)) + \sum_{i=r+1}^{r+p'} (s_i - bl) + \sum_{i=r+s+1}^{r+s+q} s_i, \tag{16}$$

$$
\begin{aligned}
R(p,p',q) &= b(p+p'+q)(p+p'+q-1) - bp(p-1) - 2bpp' \\
&\quad + \sum_{i=p+1}^{b} \min\{bq, s_i - b(l+m-1)\} + \sum_{i=r+p'+1}^{l+m} \min\{b(p'+q), s_i - bl\} \\
&\quad + \sum_{i=l+m+q+1}^{n} \min\{b(p+p'+q), s_i\},
\end{aligned}
$$

$$
L'(A,B) = \sum_{v_i \in A} s'_i = \sum_{v_i \in A_1} \{s_i - b(l+m-1)\} + \sum_{v_i \in A'_1} \{s_i - bl\} + \sum_{v_i \in A_2} s_i,
$$

$$
\begin{aligned}
R'(A,B) &= \sum_{v_i \in A, v_j \in V(H)\setminus B} e_H(v_i,v_j) + \sum_{v_i \in B} s'_i \\
&= \sum_{v_i \in A, v_j \in V(H)\setminus B} e_H(v_i,v_j) + \sum_{v_i \in B_1} (s_i - b(l+m-1)) + \sum_{v_i \in B'_1} (s_i - lb) + \sum_{v_i \in B_2} s_i.
\end{aligned}
$$

Clearly, $L'(A,B) \le L(p,p',q)$. Further $\sum_{v_i \in A, v_j \in V(H)\setminus B} e_H(v_i,v_j)$ is the number of counting the edges of H between A and $V(H) \setminus (A \cup B)$ and double counting the edges induced by A. Thus we get

$$
\begin{aligned}
&\sum_{v_i \in A, v_j \in V(H)\setminus B} e_H(v_i,v_j) \\
&= r(p+p'+q)(p+p'+q-1) - bp(p-1) - 2bpp' + qb(l-p-b_1) \\
&+ b(p'+q)(m-p'-b'_1) + b(p+p'+q)(n-l-m-q-b_2) \\
&= b(p+p'+q)(p+p'+q-1) - bp(p-1) - 2bpp' + \sum_{i=p+1}^{l-b_1} q \\
&+ \sum_{i=l+p'+1}^{l+m-b'_1} (p'+q) + \sum_{i=l+m+q+1}^{n-b_2} (p+p'+q).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
R'(A,B) = & \sum_{v_i \in A, v_j \in V(H)\setminus B} e_H(v_i, v_j) + \sum_{v_i \in B_1} (s_i - b(l+m-1)) \\
& + \sum_{v_i \in B_1'} (s_i - lb) + \sum_{v_i \in B_2} s_i \\
& \geq b(p+p'+q)(p+p'+q-1) - bp(p-1) - 2bpp' + \sum_{i=p+1}^{l-b_1} q \\
& + \sum_{i=l+p'+1}^{l+m-b_1'} (p'+q) + \sum_{i=l+m+q+1+1}^{n-b_2} (p+p'+q) \\
& + \sum_{i=l-b_1+1}^{l} (s_i - b(l+m-1)) + \sum_{i=l+m-b_1'+1}^{l+m} (s_i - br) + \sum_{i=n-b_2+1}^{n} s_i \\
& \geq r(p+p'+q)(p+p'+q-1) - bp(p-1) - 2bpp' \\
& + \sum_{i=p+1}^{l} \min\{bq, s_i - b(l+m-1)\} + \sum_{i=l+p'+1}^{l+m} \min\{b(p'+q), s_i - bl\} \\
& + \sum_{i=l+m+q+1}^{n} \min\{r(p+p'+q), s_i\} \\
& = R(p,p',q).
\end{aligned}
$$

It follows from $L(p,p',q) \leq R(p,p',q)$ that $L'(A,B) \leq R'(A,B)$. By (14) and (15) H is a b-subgraph G with the degree sequence s' such that every vertex v_i has degree s_i'. Hence s is potentially $A_{l,m}^b$-graphic. Thus the sufficiency is proved. □

It is easy to enumerate the $J_{l,m}$-split graphs on n vertices.

Theorem 35 *If* $l \geq 0$, $m \geq 1$, $l+m \geq 1$, *and* $b \geq 1$, *then*

1. *there are* $\lambda(b,l,m) = \binom{l+m}{l}$ *labeled* $J_{l,m}^b$ *and they are isomorphic;*
2. *there are* $\beta(b,l,m) = l+m$ *nonisomorphic* $J_{l,m}^b$.

Proof.

1. Since $J_{l,m}$ has $l+m$ vertices, therefore there are $\binom{l+m}{l}$ ways to choose the vertices of K_l. If we consider two different labeled $J_{l,m}$ jsplit graphs, then

the vertices of the clique parts correspond to each other, and the independent vertices of these graphs also correspond to each other, therefore these graphs are isomorphic.

2. Formally $J^b_{0,l+m}$, $J^b_{1,l+m-1}, \ldots,$ $J^b_{0,l+m}$ are $l+m+1$ different possibilities, but the last two split graphs are isomorphic, therefore $\beta(l,m) = l+m$.

$\square$

We can remark, that if $m \geq 1$, then $J_{l,m}$ is also a $J_{l+1,m-1}$ split graph.

7 Known algorithms for graphic sequences

In this section at first we present the classical Havel-Hakimi (HH) algorithm, then its testing version (HHL), which even in the worts case in $O(n)$ time decides whether an integer sequence is $(0,1,n)$-graphic. Then we describe algorithm Havel-Hakimi-pqlm-Split which in $O(n)$ time decides the similar problem for potentially $J^{(p,q)}_{l,m}$-graphic sequences, further a Havel-Hakimi type algorithm for recognition of (a,b,n).

7.1 Havel-Hakimi algorithm (HH)

If $n = 1$, then there exists one $(0,1,n)$-graphic sequence: (0). If $n \geq 2$, then Havel-Hakimi theorem (Theorem 9) gives a necessary and sufficient condition.

Input. n: the length of the sequence s ($n \geq 2$);
$\sigma = (s_1, \ldots, s_n)$: the investigated n-regular sequence.

Output. L: logical variable ($L = 0$ signalizes, that σ is not graphic, while $L = 1$ means, that σ is $(0,1,n)$-graphic).

Working variable. i: cycle variables.

```
Havel-Hakimi(n, σ)
01 L = 0                                  // line 01–07: test of the elements of s
02 for i = 1 to n − 1
03     if s_{s_i+i} == 0                  // lines 03–04: s is not graphic
04         return L
05     for j = i + 1 to s_i + i
06         s_j = s_j − 1
07     sort (s_{i+1}, ..., s_n) in decreasing order
08 L = 1                                  // lines 08–09: s is graphic
09 return L
```

7.2 Havel-Hakimi-Testing-Linear **algorithm (HHTL)**

The original Havel-Hakimi algorithm in worst case requires quadratic time to test the $(0, 1, n)$-regular sequences. Using the concepts weight point, reserve and cutting point we reduced the worst running time to $O(n)$.

The definition of the *weight point* w_i belonging to s_i was introduced in [48] in connection with Erdős-Gallai-Linear and it is as follows. w_i is the largest k $(1 \leq k \leq n)$ having the property $s_k \geq i$. But if $s_1 < i$, then $w_i = 0$. EGL exploits the property w_i ensuring that if $i \leq w_i$, then the key expression $\min j, s_k$ in the Erdős-Gallai theorem equals to i, otherwise equals to s_k.

Here we extend the definition to be applicable also in the proof of the linearity of Chungphaisan-Erdős-Gallai. Now let w_i the largest k $(1 \leq k \leq n)$ having the property $s_k \geq bi$. But if $s_1 < bi$, then let $w_i = 0$. In the case $b = 1$ the new definition results the old one.

In HHL the weight point w_i determines the increment of the tail capacity when we switch to the investigation of the next element of σ.

The *remainder* r_i belonging to s_i is defined as the unused part of the actual tail capacity and can be computed by the formulas

$$r_i = w_1 - 1 - s_1$$

and

$$r_i = w_i - r_{i-1} - s_i \quad \text{for } 1 \leq i \leq n - 1.$$

The *cutting point* y_i belonging to s_i is $\max(i, w_i)$.

The programs of this paper are written using the pseudocode conventions descibed in [16].

Input. n: number of vertices $(n \geq 1)$;
$\sigma = (s_1, \ldots, s_n)$: the investigated n-graphic sequence.

Output. L: logical variable.

Work variables. i: cycle variable;
$r = (r_1, \ldots, r_n)$: r_i the reserve belonging to s_i;
$w = (w_1, \ldots, w_n)$: w_i the weight point belonging to s_i;
$H = (H_1, \ldots, H_n)$: H_i is the sum of the first i elements of s.

```
Havel-Hakimi-Testing-Linear(n, s)
01 L = 0                                  // lines 01: set the probable value
02 if s1 == 0      // lines 02–04: test of the sequence consisting of only zeros
03    L = 1
04    return L
```

```
05 if s_{s_1+1} == 0                     // lines 05–06: test of s_{s_1} in constant time
06    return L
07 H_1 = s_1                             // line 07: initialization of H
08 for i = 2 to n                        // lines 08–09: further H_i's
09      H_i = H_{i-1} + s_i
10 if H_n is odd                         // lines 10–11: test of the parity
11    return L
12 w_1 = n  // lines 12–15: computation of the first weight point and reserve
13 while s_{w_1} < 1
14         w_1 = w_1 - 1
15 r_1 = w_1 - 1 - s_1                   // lines 15–24: testing of σ
16 s_{n+1} = 0
17 for i = 2 to n - 1
18      if s_i ≤ i or s_{i+1} = 0
19         L = 1
20         return L
21      w_i = w_{i-1}
22      while s_{w_i} < i and w_i > 0
23            w_i = w_i - 1
24      if s_i > w_i - 1 + r_{i-1}       // line 24: Is σ graphic?
25         return L                      // line 25: σ is not graphic
26      r_i = w_i + r_{i-1} - s_i
27 L = 1                                 // lines 27–28: σ is graphic
28 return L
```

Theorem 36 *The running time of* Havel-Hakimi-testing-Linear *is in best case* $\Theta(1)$, *and in worst case is* $\Theta(n)$.

Proof. If the condition in line 2 holds, then the running time is $\Theta(1)$. If not, then we reduce the actual w at most n times and the remaining operations require $O(1)$ operations for all reductions. $\square$

7.3 Erdős-Gallai-Chungphaisan-Linear algorithm (EGChL)

The following algorithm tests the potential degree sequences of $(0, b, n)$-graphs. It is based on Theorem 13.

Input. n: number of vertices ($n \geq 1$);
$\sigma = (s_1, \ldots, s_n)$: a $(0, b, n)$-regular sequence;
b: the maximal permitted number of arcs between two vertices.

Output. 1 or 0: 1, if s is $(0, b, n)$-graphic and 0 otherwise.
Work variable. i: cycle variable;
$r = (r_1, \ldots, r_n)$: r_i is the reserve belonging to s_i;
$w = (w_1, \ldots, w_n)$: w_i is the weightpoint belonging to s_i.

Erdős-Gallai-Chungphaisan-Linear(n, σ, b)

01 $H_1 = s_1$ // line 01: initialization of H_1
02 **for** $i = 2$ **to** $n - 1$ // line 02–03: computation of the elements of H
03 $\quad H_i = H_{i-1} + s_i$
04 **if** H_n is odd // line 04–05: test of the parity
05 $\quad$ **return** 0
06 $w = n$ // lines 06: initialization of the first weight point
07 **for** $i = 1$ **to** $n - 1$ // lines 07–12: test of σ
08 $\quad$ **while** $s_w < ib$ and $w > 0$
09 $\quad\quad w = w - 1$
10 $\quad y = \max(i, w)$
11 $\quad$ **if** $H_i > bi(y - 1) + H_n - H_y$
12 $\quad\quad$ **return** 0
13 **return** 1 // line 13: acceptance of σ

Theorem 37 *The running time of* Erdős-Gallai-Chungphaisan-Linear *is* $\Theta(n)$ *in all cases.*

Proof. Lines 01–05 require $\Theta(n)$ time. Since the value of w is strictly decreasing, lines 06–13 require $O(n)$ time, therefore the running time is $\Theta(n)$ in all cases. $\square$

Let us consider two examples. Let $b = 3$ and $\sigma' = (13, 10, 5, 5, 4, 1)$. $H_6 = 38$ is even. If $i = 1$, then $w_i = y = 5$ and the condition in line 11 is not satisfied $(13 \leq 3 \cdot 1 \cdot (5 - 1))$. If $i = 2$, then $w_i = y = 2$ and the condition in line 11 holds $(23 > 3 \cdot 2 \cdot (2 - 1)) + 5 + 5 + 4 + 1$, therefore σ is *not* $(0, 3, 6)$-*graphic.*

Let b remain 3, but change σ to $\sigma' = (13, 10, 5, 5, 4, 3)$. The first difference comparing with the previous example comes when $i = 2$. Now $23 \leq 3 \cdot 2 \cdot (2 - 1)) + 5 + 5 + 4 + 3$, and the condition in line 11 holds for $i = 3$, 4 and 5 too, therefore σ' is $(0, 3, 6)$-graphic.

Using Corollary 15 it is easy to test an (a, b, n)-regular sequence σ whether it is (a, b, n)-graphic. We use EGChL with input sequence $\sigma' = (s_1 - a(n - 1), \ldots, s_n - a(n - 1))$.

8 Known algorithms for split sequences

In this section we describe the linear time algorithm proposed for the recognition and reconstruction of potentially psplit and jsplit sequences.

8.1 Hammer-Simeone-PSplit algorithm (HSPS)

The following algorithms was proposed in 1981 by Hammer and Simeone [34]. Ist base is Theorem 26.

Let G be a graph with degree sequence $d = (d_1, \ldots, d_n)$.

Input. n: number of elements of δ;
$\delta = (d_1, \ldots, d_n)$: a graphic sequence.

Output. 1 or 0: 1, if d is potentially psplit sequence.

Work variable. i, k: cycle variables;
Σ_1, Σ_2: actual sums of the degrees.

```
Hammer-Simeone-Linear(n, δ)
01 k = 0                                      // line 01–02: initialization of k and S
02 Σ1 = 0
03 while d_{k+1} ≥ k − 1 and k < n             // line 03–07: computation of m
04         m = k + 1
05         Σ1 = Σ1 + d_k
06         k = k + 1
07 Σ2 = m(m − 1)
08 for i = m + 1 to n                          // lines 08–09: computation of Σ2
09      Σ2 = Σ2 + d_i
10 if Σ1 ≠ Σ2                                  // lines 10–11: G is not psplit graph
11    return 0
12 return G i'is psplit, maximal clique size is m // line 12: G is psplit graph
```

Theorem 38 *Let* G *a graph with degree sequence* δ. *Algorithm* Hammer-Simeone-Linear *decides, if* G *is a psplit graph and computes the maximal clique size in* $\Theta(n)$ *time.*

Proof. Lines 01–02 require $O(1)$ time, lines 03–09 $\Theta(n)$ time and lines 10–12 $O(1)$ time. □

8.2 Further linear algorithms for psplit sequences

In 1980 Golumbic [26], in 2003 Feder et al. [20], in 2007 Heggernes and Kratsch [37] proposed linear time algorithm for the recognition of psplit graphs.

8.3 Havel-Hakimi-Testing-JSplit algorithm (HHJST)

In 2012 Yin [87] described HHJST, a Havel-Hakimi type linear algorithm for the recognition of potentially jsplit sequences.

9 New algorithms

In this section we present two simple algorithms, which decide whether a sequence of nonnegative integers is A_l^b-graphic or $J_{l,m}^b$-graphic, and if the answer is yes, then they compute the maximal suitable l too.

These algorithms require in worst case only $O(n)$ time even for (a, b, n)-regular input, and are quicker for (a, b, n)-graphic input., since then the sorting can be omitted.

We remark, that earlier only for pseudo-split graphs was published a linear time testing algoritm [58].

9.1 Algorithm Ab-l-Max

For given sequence $\sigma = (s_1, \ldots, s_n)$ of nonnegative integers and given nonnegative integer b algorithm A-b-l-Max computes the maximal l for which the sequence s is A_l^b-graphic.

Input. $n \geq 1$: the length of the sequence s;
$\sigma = (s_1, \ldots, s_n)$: a sequence of nonnegative integers;
b: the maximal permitted number of arcs between two different vertices.

Output. l: the maximal value for which d is A_l^b-graphic.

Work variable. i: cycle variable.

```
A-b-l-Max(n, σ, b)
01 Counting-Sort(n, σ)                   // line 01: sorting of σ
02 l = 1                                 // line 02: initialization of l
03 while s_{l+1} ≤ bl and l < n          // line 03–04: computation of l
04       l = l + 1
05 return l 'is the maximal value'       // line 05: return of the maximal l
```

Theorem 39 *Let* b, l *and* n *be positive integers. Algorithm* A-b-l-Max *computes the maximal* l *for which* $\sigma = (s_1, \ldots, s_n)$ *is* A_l^b*-graphic in* $\Theta(n)$ *time.*

Proof. Let G be a b-graph and $s' = (s'_1, \ldots, s'_n)$ be the nonincreasingly sorted sequence consisting from the elements of s. K_l^b contains l vertices whose degrees are equal to $b(l-1)$. Therefore to find the maximal size K_l^b which is a subgraph of G it is sufficient to find the maximal j satisfying $s'_j \geq b(j-1)$. In lines 01–04 A-b-l-Max computes this maximal l.

Lines 01 of A-b-l-Max requires $\Theta(n)$ time, lines 02 and 05 $O(1)$ time, and lines 03–04 require $O(n)$ time, so the best and worst running times of this algorithm are both $\Theta(n)$. □

As an example consider the sequence $\sigma = (6\ 6\ 1\ 6)$ and $b = 2$. Then $\sigma' = (6\ 6\ 6\ 1)$ and Ab-l-Max returns with $l = 3$. Indeed G contains K_3 as a subgraph but it does not contain K_4 as a subgraph. Since the sum of the elements of s is odd, according to theorem of Erdős and Gallai [17] s is not graphic, that is an A_l^b-graphic sequences are not always graphic.

9.2 Algorithm J-b-l-Max

For given sequence $\sigma = (s_1, \ldots, s_n)$ of nonnegative integers and given nonnegative integer b algorithm S-b-l-Max computes the maximal l for which the sequence σ is $S_{l,n-l}^b$-graphic.

If K_l^b and K_m^b are vertex disjoint and G is the join of K_l^b and $\overline{K}_m$, then in G the degrees of the vertices of K_l are equal to $b(l-1+m)$, while the degrees of the vertices of K_m are equal to bm. This observation is the base of the following algorithm S-b-l-Max.

Input. n: the length of the degree sequence s;
$\sigma = (s_1, \ldots, s_n)$: a sequence of nonnegative integers;
b: the maximal permitted number of edges between two different vertices.

Output. l: the maximal value for which σ is $S_{l,n-l}^b$-graphic or the message 'σ is not $(b, l, n-l)$-graphic'.

Work variable. i: cycle variable.

J-b-l-Max(n, σ, b)

01 **if** s_1/b is not integer // line 01–02: constant time test
02 **return** 'σ is not $(b, l, n-l)$-graphic' // line 02: σ is not $(b, l, n-l)$-graphic
03 Counting-Sort(n, σ) // line 03: sorting of σ
04 $l = 1$ // line 04: initialization of l
05 **while** $s_{l+1} == s_1$ and $l < n$ // line 05–06: computation of l
06 $l = l + 1$
07 **if** $s_{l+1} \neq bl$ // line 07–08: σ is not $(b, l, n-l)$-graphic
08 's is not $(b, l, n-l)$-graphic'
09 **return** l 'is the maximal value'

Theorem 40 *Algorithm* J-b-l-Max *computes the maximal* l *for which* $\sigma = (s_1, \ldots, s_n)$ *is* $S^b_(l, n-l)$*-graphic in* $\Theta(n)$ *time.*

Proof. Let b, l, m and n be positive integers. Let G be a b-graph and $\sigma = (s'_1, \ldots, s'_n)$ be the nonincreasingly sorted sequence consisting from the elements of s'.

The next part of the proof is similar to the corresponding part of Theorem 39.

Line 01 of J-b-l-Max requires $\Theta(n)$ time and lines 02-05 require $O(n)$ time, so the best running time is $\Theta(1)$ and the worst running time is $\Theta(n)$. □

We remark that if the input of J-b-l-Max is sorted, then we can omit lines 03 and 04, and using logarithmic search we can reduce the worst case running time to $\Theta(\log n)$.

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