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# Minimal digraphs with given imbalance sequence

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**Abstract.** Let $a$ and $b$ be integers with $0 \leq a \leq b$. An $(a,b)$-*graph* is such digraph $D$ in which any two vertices are connected at least $a$ and at most $b$ arcs. The *imbalance* $a(v)$ of a vertex $v$ in an $(a,b)$-graph $D$ is defined as $a(v) = d^+(v) - d^-(v)$, where $d^+(v)$ is the outdegree and $d^-(v)$ is the indegree of $v$. The *imbalance sequence* $A$ of $D$ is formed by listing the imbalances in nondecreasing order. A sequence of integers is $(a,b)$-*realizable*, if there exists an $(a,b)$-graph $D$ whose imbalance sequence is $A$. In this case $D$ is called a *realization* of $A$. An $(a,b)$-realization $D$ of $A$ is *connection minimal* if does not exist $(a,b')$-realization of $D$ with $b' < b$. A digraph $D$ is *cycle minimal* if it is a connected digraph which is either acyclic or has exactly one oriented cycle whose removal disconnects $D$. In this paper we present algorithms which construct connection minimal and cycle minimal realizations having a given imbalance sequence $A$.

## 1 Introduction

Let $a$, $b$ and $n$ be nonnegative integers with $0 \leq a \leq b$ and $n \geq 1$. An $(a,b)$-*graph* is a digraph $D$ in which any two vertices are connected at least $a$ and at most $b$ arcs. If $d^-(v)$ denotes the outdegree and $d^+(v)$ denotes of vertex $v$ in an $(a,b)$-graph $D$ then the *imbalance* [14] of $v$ is defined as

$$a(v) = d^+(v) - d^-(v).$$

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Since loops have no influence on the imbalances therefore for the simplicity we suppose everywhere in this paper that the investigated graphs are loopless.

The *imbalance sequence* of $D$ is formed by listing its imbalances in nondecreasing order (although imbalances can be listed in nonincreasing order as well). The set of distinct imbalances of a digraph is called its *imbalance set*. Mostly the literature on imbalance sequences is concerned with obtaining necessary and sufficient conditions for a sequence of integers to be an imbalance sequence of different digraphs [9, 10, 11, 14, 18, 19, 20, 21], although there are papers on the imbalance sets too [15, 16, 17, 19].

If $D$ is an $(a, b)$-digraph and $A$ is its imbalance sequence then $D$ is a *realization* of $A$. If we wish to find a realization of $A$ in any set of directed graphs then

$$\sum_{i=1}^{n} a_i = 0 \tag{1}$$

is a natural necessary condition. If we allow parallel arcs then this simple condition is sufficient to find a realization. If parallel arcs are not allowed then the simple examle $A = [-3, 3]$ shows that (1) is *not sufficient* to find a realization.

Mubayi et al. [14] characterized imbalance sequences of simple digraphs (digraphs without loops and parallel arcs [2, 9, 25]) proving the following necessary and sufficient condition. We remark that simple digraphs are such $(0, 2)$-graphs which do not contain loops and parallel arcs.

**Theorem 1** (Mubayi, Will, West, 2001 [14]) *A nondecreasing sequence* $A = [a_1, \ldots, a_n]$ *of integers is the imbalance sequence of a simple digraph iff*

$$\sum_{i=1}^{k} a_i \leq k(n-k) \tag{2}$$

*for* $1 \leq k \leq n$ *with equality when* $k = n$.

**Proof.** See [14]. □

Mubayi et al. [14] provided a Havel-Hakimi type [3, 4, 7, 8] greedy algorithm Greedy for constructing a simple realization.

The pseudocode of Greedy follows the conventions used in [1].

The *input data* of Greedy are $n$: the number of elements $A$ ($n \geq 2$); $A = (a_1, \ldots, a_n)$: a nondecreasing sequence of integers satisfying (2). Its *output* is $M$: the $n \times n$ sized incidence matrix of a simple directed graph $D$

whose imbalance sequence is $A$. The *working variables* are the cycle variables $i$, $j$, $k$, $l$, $x$ and $y$.

```
GREEDY(n, A)
01 for i ← 1 to n                                  // line 01–03: initialization of M
02     for j ← 1 to n
03         M_ij ← 0
04 i = 1                                           // line 04–10: computation of M
05 while a_i > 0
06         Let k = a_i, j_1 < ··· < j_k, further let a_{j_1}, ..., a_{j_k} be
           the k smallest elements among a_{i+1}, ..., a_n, where
           a_x smaller a_y means that a_x < a_y or a_x = a_y and x < y
07         for l ← 1 to k
08             a_l ← a_l + 1
09             M_{i,a_l} ← 1
10         i ← i + 1
11 return M                                        // line 11: return of the result
```

The running time of Greedy is $\Theta(n^2)$ since the lines 1–3 require $\Theta(n^2)$ time, the **while** cycle executes $O(n)$ times and in the cycle line 06 and line 07 require $O(n)$ time.

Kleitman and Wang in 1973 [12] proposed a new version of Havel-Hakimi algorithm, where instead of the recursive choosing the largest remaining element of the investigated degree sequence it is permitted to choose *arbitrary element*. Mubayi et al. [14] point out an interesting difference between the directed and undirected graphs. Let us consider the imbalance sequence $A = [-3, 1, 1, 3]$ of a transitive tournament. Deleting the element $1$ and adding $1$ to the smallest imbalance leaves us trying to realize $[-2, -1, 3]$, which has no realization among the simple digraphs although it has among the $(0, 2)$-graphs.

Let $\alpha(D)$ denote the number of edges of $D$. It is easy to see the following assertion.

**Lemma 1** *If a directed graph* $D$ *is a realization of a sequence* $A = [a_1, \ldots, a_n]$ *then*

$$\alpha(D) \geq \frac{1}{2}\sum_{i=1}^{n} |a_i|. \tag{3}$$

**Proof.** Any realization has to contain at least so many outgoing arcs as the sum of the positive elements of $A$. Since $S$ is realizable for the corresponding set, according to (1) the sum of the absolute values of the negative elements of $A$ equals to the sum of the positive elements, therefore we have to divide the sum in (3) by 2. □

A realization $D$ of $A$ is called *arc minimal* (for a given set of digraphs) if $A$ has no realization (in the given set) containing less arcs than $D$. Mubayi et al. [14] proved the following characterization of GREEDY.

**Lemma 2** (Mubayi et al., 2001 [14]) *If $A$ is realizable for the simple graphs then the realization generated by* GREEDY *contains the minimal number of arcs.*

**Proof.** See in [14]. □

Lemma 1 and Lemma 2 imply the following assertion.

**Theorem 2** *If $A$ is realizable for simple digraphs then the realization generated by* GREEDY *is arc minimal and the number of arcs contained by the realization is given by the lower bound* (3).

Wang [22] gave an asymptotic formula for the number of labeled simple realizations of an imbalance sequence.

In this paper we deal with the more general problem of $(a, b)$-graphs. The following recent paper characterizes the imbalance sequences of $(0, b)$-graphs.

**Theorem 3** (Pirzada, Naikoo, Samee, Iványi, 2010 [19]) *A nondecreasing sequence $A = [a_1, \dots, a_n]$ of integers is the imbalance sequence of a $(0, b)$-graph iff*

$$\sum_{i=1}^{k} a_i \le bk(n-k) \tag{4}$$

*for $1 \le k \le n$ with equality when $k = n$.*

**Proof.** See [19]. □

We say that a realization $D$ is *cycle minimal* if $D$ is connected and does not contain a nonempty set of arcs $S$ such that deleting $S$ keeps the digraph connected but preserves imbalances of all vertices. Obviously such a set $S$, if it exists, must add 0 to the imbalances of vertices incident to it and hence must be a union of oriented cycles. Thus a cycle minimal digraph is either acyclic or has exactly one oriented cycle whose removal disconnects the digraph. For the sake of brevity we shall use the phrase *minimal realization* to refer a cycle minimal realization of $A$. We denote the set of all minimal realizations of $A$ by $\mathcal{M}(\mathcal{A})$.

A realization $D$ of $A$ is called *connection minimal* (for a given set of directed graphs) if the maximal number of arcs $\gamma(D)$ connecting two different vertices of $D$ is minimal.

The aim of this paper is to construct a connection and a cycle minimal digraph $D$ having a prescribed imbalance sequence $A$. At first we determine the minimal $b$ which allows to reconstruct the given $A$. Then we apply a series of arithmetic operations called *contractions* to the imbalance sequence $A$. This gives us a *chain* $C(A)$ of imbalance sequences. Then by the recursive transformations of $C(A)$ we get a required $D$.

The structure of the paper is as follows. After the introductory Section 1 in Section 2 we present an algorithm which determines the minimal number of arcs which are necessary between the neighboring vertices to realize a given imbalance sequence then in Section 3 we define a contraction operation and show that the contraction of an imbalance sequence produces another imbalance sequence. Finally in Section 4 we present an algorithm which constructs a connection minimal realization of an imbalance sequence.

## 2 Computation of the minimal $r$

According to (1) the sum of elements of any imbalance sequence equals to zero. Let us suppose that according to (1) the sum of the elements of a potential imbalance sequence $P = [p_1, \ldots, p_n]$ is zero and $b = \max(-a_1, a_n)$. Then it is easy to construct such $(0, b)$-digraph $D$ whose imbalance sequence is $A$ connecting the vertices having positive imbalance with the vertices having negative imbalance using the prescribed number of arcs. It is a natural question the value $b_{\mathrm{min}}(P)$ defined as the minimal value of $b$ sufficient for a potential imbalance sequence $P$ to be the imbalance sequence of some $(0, b)$-graph.

$b_{\mathrm{min}}(P)$ has the following natural bounds.

**Lemma 3** *If* $A = [a_1, \ldots, a_n]$ *is an imbalance sequence, then*

$$\left\lceil \frac{a_n - a_1}{n} \right\rceil \le b_{\min} \le \min(-a_1, a_n). \tag{5}$$

The following algorithm BMIN computes $b_{\min}(A)$ for a sequence $A = [a_1, \ldots, a_n]$ satisfying (4). BMIN is based on Theorem 3, on the bounds given by Lemma 3 and on the logarithmic search algorithm described by D. E. Knuth [13, page 410] and is similar to algorithm MINF-MAXG [6, Section 4.2].

*Input.* $n$: the number of elements $A$ ($n \ge 2$);
$A = [a_1, \ldots, a_n]$: a nondecreasing sequence of integers satisfying (4).

*Output.* $b_{min}(A)$: the smallest sufficient value of $b$.

*Working variables.* $k$: cycle variable;
$l$: current value of the lower bound of $b_{min}(A)$;
$u$: current value of the upper bound of $b_{min}(A)$;
$S$: the current sum of the first $k$ elements of $A$.

```
Bmin(n, A)
01 l ← ⌈a_n − a_1⌉                  // line 01–02: initialization of l and u
02 u ← min(a_n, −a_i)
03 while l < u        // line 03–14: computation of the minimal necessary b
04        b ← ⌊(l+u)/2⌋
05        S = S ← 0
06        for k ← 1 to n − 1
07            S ← S + a_i
08            if S < bk(n − k)
09               l ← r
10            if l == r + 1
11               b_min ← l + 1
12               return b
13        go to 03
14        u ← b
15 b_min ← l                  // line 15–16: return of the computed minimal b
16 return b_min
```

The next assertion characterizes Bmin.

**Lemma 4** *Algorithm* Bmin *computes* $b_{min}$ *for a sequence* $A = [a_1, \ldots, a_n]$ *satisfying* (4) *in* $\Theta(n \log n)$ *time.*

**Proof.** Bmin computes $b_{min}$ on the base of Theorem 3 therefore it is correct. Running time of Bmin is $\Theta(n \log n)$ since the **while** cycle executes $\Theta(\log n)$ times and the **for** cycle in it requires $\Theta(n)$ time. □

## 3 Contraction of an imbalance sequence

Let $D$ be a digraph having $n$ vertices and $m$ arcs. Throughout we assume that the vertices of $D$ are labeled $v_1, \ldots, v_n$ according to their imbalances in nondecreasing order while the arcs of $D$ are labeled $e_1, \ldots, e_m$ arbitrarily. Let $A = [a_1, \ldots, a_n] = [a_{n1}, \ldots, a_{nn}]$ be the imbalance sequence of $D$, where

$a_i = a_{ni}$ is the imbalance of vertex $v_i = v_{ni}$. We define an arithmetic operation, called *contraction* on $A$ as follows.

Let $n \geq 2$, an imbalance sequence $A = [a_1, \ldots, a_n]$ and an ordered pair $(a_i, a_j)$ with $1 \leq i,\ j \leq n$ and $i \neq j$ be given. Then the *contraction* of $(a_i, a_j)$ means that we delete $a_i$ and $a_j$ from $A$, add a new element $a'_i = a_i + a_j$ and sort nonincreasingly the received sequence using COUNTING-SORT [1] so that the indices of the elements are updated and the updated index of $a_i + a_j$ is denoted by $k_i$. The new sequence is denoted by $A/(i, j)$.

Note that $j < i$ is permitted. We refer to $A/(i, j)$ as a *minor* of $A$. Our terminology is inspired by the concept of minor and edge contraction from graph theory [23, 24]. The proof of Theorem 5 explains our choice of terminology.

An imbalance sequence $A$ is a $(0, b)$-*imbalance sequence* if at least one of its realizations is a $(0, b)$-graph. We also observe that if $b' > b$ then a $(0, b)$-imbalance sequence is also a $(0, b')$-imbalance sequence.

The next assertion allows us to construct imbalance sequences and their realizations recursively. It also establishes a relation between the arithmetic operation of contraction discussed above and the edge contraction operation of graphs.

**Theorem 4** *If* $A$ *is a* $(0, b)$*-imbalance sequence, then all its minors are* $(0, 2b)$*-imbalance sequences.*

**Proof.** Let $A$ be the imbalance sequence of a $(0, b)$-graph. Suppose that $B = A/(p, q)$ and let $a_p$ and $a_q$ be both negative with $a_p \leq a_q$. Then $b_l = a_p + a_q$ so that $b_l < a_p$. Thus, for all $k \leq q$, we have

$$\begin{aligned}
\sum_{i=1}^{k} b_i &\geq \sum_{i=1}^{k} b_i + (q-k)a_q, && \text{(since all these elements are negative)} \\
&\geq \sum_{i=1}^{q} a_i, && \text{(since } A \text{ is a nondecreasing sequence).}
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{i=1}^{k} b_i &\geq \sum_{i=1}^{q} a_i - (q-k)a_q \\
&\geq \sum_{i=1}^{k} a_i \geq bk(k-n), && \text{(since } A \text{ is an imbalance sequence)} \\
&\geq (2b)k(k-n+1) && \text{(since } n \geq k+2\text{),}
\end{aligned}$$

For $q < k \le n-1$, we have

$$\begin{aligned}\sum_{i=1}^{k} b_i &= \sum_{i=1}^{k} a_i \\ &\geq bk(k-n), \qquad \text{(since A is an imbalance sequence)} \\ &\geq (2b)k(k-n+1)\end{aligned}$$

for $n \geq k = 2$ and equality holds when $k = n-1$. Thus in either case B is an imbalance sequence of a $(0, 2b)$-graph by Theorem 3.

By symmetry, we have that Theorem 4 holds if $a_p$ and $a_q$ are both positive.

Now suppose that $a_p \le 0$ and $a_q \ge 0$ with $|a_p| \ge |a_q|$. If $b_l = a_p + a_q$, then $b_l \le 0$. For all $k \le p$, we have

$$\begin{aligned}\sum_{i=1}^{k} b_i &\geq \sum_{i=1}^{k} a_i \geq bk(k-n), \qquad \text{(since A is an imbalance sequence)} \\ &\geq (2b)k(k-n+1), \qquad \text{(since } n \geq k+2).\end{aligned}$$

For all $p < k \le l$, we have

$$\begin{aligned}\sum_{i=1}^{k} b_i &\geq \sum_{i=1}^{k} a_i - a_p \geq \sum_{i=1}^{k} a_i \\ &\geq bk(k-n), \qquad \text{(since A is an imbalance sequence)} \\ &\geq (2b)k(k-n+1), \qquad \text{(since } n \geq k+2).\end{aligned}$$

For all $k > l$, we have

$$\begin{aligned}\sum_{i=1}^{k} b_i &\geq \sum_{i=1}^{k} a_i + a_q \geq \sum_{i=1}^{k} a_i \\ &\geq bk(k-n), \qquad \text{(since A is an imbalance sequence)} \\ &\geq (2b)k(k-n+1).\end{aligned}$$

The last inequality holds if $n \geq k = 2$, with equality when $k = n-1$. Thus once again B is an imbalance sequence of a $(0, b)$-graph by Theorem 3. By symmetry, Theorem 4 holds if $|a_p| \le |a_q|$. □

Suppose that $D'$ is a cycle minimal realization of $A/(1, n)$. Then the following algorithm algorithm VERTEX constructs $D''$, a cycle minimal realization of A.

*Input parameters* of VERTEX are $n \geq 2$: the number of elements of $A$; $b \geq 1$: the connection parameter of $D'$; $A = [a_1, \ldots, a_n]$: the imbalance sequence; $A' = A/(1, n) = [a'_1, \ldots, a'_{n-1}]$; $k$: index of the element $a'_1 = a_1 + a_n$ in the minor $A'$; $D'$: a $(0, b)$-graph, which is a cycle minimal realization of $A/(1, n)$ ($D'$ is given by an $(n-1) \times (n-1)$ sized incidence matrix $\mathcal{X} = [x_{i,j}]$).

The output of VERTEX is $D''$: a $(0, q)$-graph, which is a cycle minimal realization of $A$, where $q = \max(1, b, a_n)$.

```
VERTEX(n, A, k, X)
01 read n                     // line 01–04: read of the input data
02 for i ← 1 to n − 1
03     for j ← 1 to n − 1
04         read x_ij
05 for i ← 1 to n − 1         // line 05–08: add an isolated vertex to D'
06     x_in ← 0
07 for i ← 1 to n
08     x_ni ← 0
09 if a_1 = 0 and a_n = 0     // line 09–12: if all a's are equal to zero
10    for i ← 1 to n
11        x_i,i+1 ← 1         // line 12: i + 1 is taken mod n
12        return X
13 x_nk ← a_n                 // line 13: if a_1 < 0
14 return X                   // line 14: return the incidence matrix of D''
```

We now show that VERTEX gives a cycle minimal realization of $A$.

**Theorem 5** *The realization* $D''$ *obtained by* VERTEX *is a cycle minimal* $(0, \max(b, 1, a_n))$*-graph. The running time of* VERTEX *is* $\Theta(n^2)$.

**Proof.** If $a_1 = a_n = 0$, then $D''$ is constructed in lines 09–12 and contains exactly one cycle and is a 1-digraph. If we remove this cycle then remain isolated vertices that is a not connected graph.

If $a_1 < 0$, then due to Theorem 3 $a_n > 0$. In this case $D''$ is constructed in line 14 connecting the isolated vertex $v'_n$ with the contracted vertex $v'_k$. So $D''$ contains a cycle only if the cycle minimal $D'$ contained a cycle, and removing this cycle changes $D''$ to a not connected graph. In this case $D''$ is a $\max r, a_n$-graph.

So $D''$ is a $(0, q)$-graph, where $q = \max(b, 1, a_n)$.

The double cycle in lines 02–04 requires $\Theta(n^2)$ time, and the remaining part of the program requires only $O(n)$ time, so the running time of VERTEX is $\Theta(n^2)$. □

# 4 Construction of a cycle minimal realization

A *chain* of an imbalance sequence $A = A_n = [a_{n1}, \dots, a_{nn}]$ is a sequence of imbalance sequences $\mathcal{C}(A_n) = [A_n,\ A_{n-1},\ \dots,\ A_1]$ with $A_n = A$ and $A_{i-1}(A)$ being a minor of $A_i(A)$ for every $1 \le i \le n-1$. *The simple chain* $\mathcal{S}(\mathcal{A})$ of an imbalance sequence $A$ is the sequence of imbalance sequences $[A_n,\ A_{n-1},\ \dots,\ A_1]$ with $A_n = A$ and $A_{i-1} = [a_{i-1,1}, \dots, a_{i-1,i-1}]$ being the minor of $A_i = [a_{i,1}, \dots, a_{i,i}]$ received by the contraction of the first and last element of $A_i$. It is worth to remark that the simple chain of an imbalance sequence is unique.

Chain is an algorithm for constructing the simple recursion chain $\mathcal{C}(A)$ of $A$.

The *input data* of Chain are $n \ge 2$: the length of an imbalance sequence $A = [a_{n1}, \dots, a_{nn}]$; an imbalance sequence $A$. The *output* of Chain is $\mathcal{C}$: the simple chain of $A$. *Working variable* is the cycle variable $i$.

Chain$(n, A_n)$
01 **read** $n$ // line 01–03: read of the input data
02 **for** $i \leftarrow 1$ **to** $n$
03 **read** $a_{ni}$
04 **for** $i \leftarrow n$ **downto** 2 // line 04–05: construction of $\mathcal{C}$
05 delete the first and last elements of $A_i$, add a new element $a_{i1} + a_{ii}$, sort nondecreasingly the received sequence and denote by $k_i$ the index of the new element
06 **return** $\mathcal{C}$ and $k$ // line 06: return of the results

Now, since each contraction in Step 05 of Chain reduces the number of elements of the corresponding imbalance sequence by 1, the last element $A_1(A)$ of the chain contains exactly one element and so the length of the chain is equal to the number of elements $n$ of the imbalance sequence $A$. Thus for all $1 \le i \le n$ the sequence $A_i(A)$ contains $i$ elements. To every chain of an imbalance sequence $A$ of length $n$ we can associate bijectively a chain of $n-1$ ordered pairs with $i$ element equal to $(j,k)$, where $A_{n-i} = A_{n-i+1}/(j,k)$. That is $(v_j, v_k)$ is contracted to obtain $A_{n-i}$ from $A_{n-i+1}$. This bijection allows us to represent every chain of imbalance sequences by the sequence of pairs $(j,k)$.

We present a simple algorithm Realization for associating a small cycle minimal realization $D''$ to any imbalance sequence $A$.

*Input values* are $n \ge 2$: the number of elements of $A$; $A$: an imbalance sequence; $D'$: a cycle minimal $(0,b)$-graph which is a realization of $A/(1,n)$

and is given by its incidence matrix $X$.

The *output* of Realization is $D''$, a $(0, q)$-graph, which is a cycle minimal realization of $A$; $k = [k_1, \ldots, k_{n-1}]$: the sequence of the updated indices of the elements received by contraction. $D''$ is represented by its incidence matrix $X$, and $q = \max(b, 1, a_n)$. *Working variable* is $i$: cyclic variable.

```
Realization(n, A)
01 read n                              // line 01–03: read of the input data
02 for i ← 1 to n
03      read a_ij
04 Chain(n, A)                         // line 04: construction the simple chain C(A)
05 x_11 ← 0                            // line 05: construction of D''_1
06 for i ← 2 to n                      // line 06–07: recursive construction of D''_n
07      Vertex(i, A_i, k, X_{i−1})
08 return D''_n and k   // line 08: return of the constructed minimal digraph
```

The next assertion shows that Realization is correct and constructs a cycle minimal realization of an imbalance sequence in polynomial time.

**Theorem 6** *Let* $A$ *be a* $(0, b)$*-imbalance sequence having* $n$ *entries and let* $C(A) = [(a_1, b_1) \ldots (a_n, b_n)]$ *be a chain of* $A$*. Then there exists a cycle minimal digraph* $D$ *having* $n$ *vertices such that* $D$ *is reconstructible from* $C$ *and* $D$ *is a* $q = \max(r, 1, a_n)$*-realization of* $A$*. Moreover, this reconstruction requires* $O(n^2)n$ *time.*

**Proof.** By Vertex, the digraph $D_n$ which is the output of Realization, is assured to be a cycle minimal realization of $A$. Now, Vertex constructs $D_i$ from $D_{i-1}$ in $O(n)$ time and there are $n - 1$ such constructions. Thus Realization runs in $O(n^2)$ time. □

The following example illustrate the work of algorithms Vertex, Chain and Realization.

**Example 1** *Let* $A = [-2, -2, -2, -1, 3, 4]$*. Figure 1 shows a realization of* $A$*, therefore* $A$ *is an imbalance sequence.*

Figure 1 also shows that this realization is a 1-digraph. Since there are nonzero imbalances therefore all realizations have to contain arcs so this realization is connection minimal. Since all realization of $A$ has to contain at least

$$m_{\min} = \frac{\sum_{i=1}^{n} a_i}{2}$$

arcs, and now $m_{\min} = 7$, so D is also an arc minimal realization.

| Vertex/Vertex | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|---|---|---|---|---|---|---|
| $v_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_3$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_4$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_5$ | 1 | 1 | 1 | 0 | 0 | 0 |
| $v_6$ | 1 | 1 | 1 | 1 | 0 | 0 |

Figure 1: Incidence matrix of a realization of $A = [-2, -2, -2, -1, 3, 4]$

Now we construct a cycle minimal realization of $A$ using Realization. After the reading of the input data in lines 01–03 Chain constructs the simple chain $\mathcal{S} = [A_1, \ldots, A_6]$ and $k = [k_1, k_2, k_3, k_4, k_5] = [1, 1, 2, 3, 4]$, where $A_6 = A = [-2, -2, -2, -1, 3, 4]$, $A5 = [-2, -2, -1, 2, 3]$, $A_4 = [-2, -1, 1, 2]$, $A_3 = [-1, 0, 1]$, $A_2 = [0, 0]$ and $A_1 = [0]$.

After the construction of C Realization sets $x_{11} = 0$ in Step 5 and so it defines $\mathcal{X}_1$, the incidence matrix of $D_1$ consisting of an isolated vertex $v_1$. Then it constructs $D_2$, …, $D_6$ in lines 06–07 calling Vertex recursively: at first $k_1 = 1$ helps to construct $D_2$ having the incidence matrix $X_2$ which is shown in Figure 2.

| Vertex/Vertex | $v_1$ | $v_2$ |
|---|---|---|
| $v_1$ | 0 | 1 |
| $v_2$ | 1 | 0 |

Figure 2: Incidence matrix of $D_2$ ($X_2$)

Now using $k_2 = 1$ $D_3$ is constructed. The result is the incidence matrix $X_3$ shown in Figure 3.

The next step is the construction of $D_4$ using $k_3 = 2$. Figure 4 shows $X_4$.

The next step is the construction of $D_5$ using $k_4 = 3$. Figure 5 shows $X_5$.

The final step is the construction of $D_6$ using $k_5 = 4$. Figure 6 shows $X_6$.

It is worth to remark that $D_6$ contains 9 arcs while the realization of $A$ whose incidence matrix is shown in Figure 6 contains only the necessary 7 arcs and is also a cycle minimal realization of A.

The graph $D'_6$ whose incidence matrix $X'_6$ shown in Figure 7 contains only

| Vertex/Vertex | $v_1$ | $v_2$ | $v_3$ |
|---|---|---|---|
| $v_1$ | 0 | 1 | 0 |
| $v_2$ | 1 | 0 | 0 |
| $v_3$ | 1 | 0 | 0 |

Figure 3: Incidence matrix of $D_3$ ($X_3$)

| Vertex/Vertex | $v_1$ | $v_2$ | $v_3$ | $v_4$ |
|---|---|---|---|---|
| $v_1$ | 0 | 1 | 0 | 0 |
| $v_2$ | 1 | 0 | 0 | 0 |
| $v_3$ | 1 | 0 | 0 | 0 |
| $v_4$ | 2 | 0 | 0 | 0 |

Figure 4: Incidence matrix of $D_4$ ($X_4$)

| Vertex/Vertex | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ |
|---|---|---|---|---|---|
| $v_1$ | 0 | 1 | 0 | 0 | 0 |
| $v_2$ | 1 | 0 | 0 | 0 | 0 |
| $v_3$ | 1 | 0 | 0 | 0 | 0 |
| $v_4$ | 2 | 0 | 0 | 0 | 0 |
| $v_5$ | 3 | 0 | 0 | 0 | 0 |

Figure 5: Incidence matrix of $D_5$ ($X_5$)

| Vertex/Vertex | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|---|---|---|---|---|---|---|
| $v_1$ | 0 | 1 | 0 | 0 | 0 | 0 |
| $v_2$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $v_3$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $v_4$ | 2 | 0 | 0 | 0 | 0 | 0 |
| $v_5$ | 3 | 0 | 0 | 0 | 0 | 0 |
| $v_6$ | 0 | 0 | 0 | 4 | 0 | 0 |

Figure 6: Incidence matrix of $D_6$ ($X_6$)

7 arcs and is also a cycle minimal realization of $A$.

| Vertex/Vertex | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|---|---|---|---|---|---|---|
| $v_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_3$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_4$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_5$ | 1 | 1 | 1 | 0 | 0 | 0 |
| $v_6$ | 1 | 1 | 1 | 1 | 0 | 0 |

Figure 7: Incidence matrix of $D'_6$ $(X'_6)$

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