PARALLEL PROCESSING OF 0–1 SEQUENCES

By

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§. 1. Introduction

This paper is devoted to the investigation of parallel processing of binary sequences.

Processing proceeds in the points of time $1, 2, \ldots$, and it is sequential for every sequence.

The sequences have priorities: processing algorithm considers the elements when choosing them for processing at every point of time in the order first, second. ..., last sequence.

The processing in every point of time is characterized by using the vector of the first non-processed elements of the sequences.

In §. 2. the formal description of the considered problem is given, later (§. 3.) the limit distribution of the vectors of the first non-processed elements is computed.

Finally (§. 4.) the limit distribution is used to get the processing speed.

§. 2. Formulation of the problem

Let

$$F_1 = f_{11}, f_{12}, \ldots$$
$$\vdots \tag{2.1}$$
$$F_r = f_{r1}, f_{r2}, \ldots$$

r $(r \geq 2)$ infinite binary sequences that is $f_{ij} \in \{0, 1\}$ $(i = 1, \ldots, r;\ j = 1, 2, \ldots)$.

We process these sequences using the following algorithm [1].

1. Processing proceeds in the discrete points of time $1, 2, \ldots$. Let $t = 1$.

2. Let $B_t = (f_{11}, f_{21}, \ldots, f_{r1})$.

3. In the moment of time t let us distinguish three cases:

a) $f_{12} \neq f_{11}$;
b) there exists an index k ($1 \leq k < r$) for which $f_{12} = f_{11} = f_{21} = \ldots = f_{k1}$ and $f_{k+1,1} \neq f_{11}$;
c) $f_{12} = f_{11} = f_{21} = \ldots = f_{r1}$.

In the case *a)* the elements f_{11} and f_{12}, in the case *b)* the elements f_{11} and $f_{k+1,1}$, and in the case *c)* only the element f_{11} will be processed.

4. We omit the processed elements, reduce the second index of the remaining elements in the i-th sequence by the number of the omitted ones in the t-th point of time elements of the *i-th* ($i = 1, \ldots, r$) sequence.

5. We add 1 to t and continue the processing from the Step 2.

Of course, the element f_{i1} of B_t ($t = 1, 2, \ldots$) is the starting (first non-processed) element of F_i in the t-th point of time; we call $B_1, B_2, \ldots$ the state sequence of processing.

Let now

$$\begin{aligned} \Theta_1 &= \xi_{11}, \xi_{12}, \ldots \\ &\vdots \\ \Theta_r &= \xi_{r1}, \xi_{r2}, \ldots \end{aligned} \tag{2.2}$$

be r sequences of independent random variables with common distributions

$$P(\xi_{ij} = 0) = \mathrm{P}(\xi_{ij} = 1) = \frac{1}{2} \quad (i = 1, \ldots, r;\ j = 1, 2, \ldots). \tag{2.3}$$

Let us suppose that the realisations of the sequences are processed using the algorithm given above.

If the vectors $\sigma_t (t = 1, 2, \ldots)$ are determined by the distribution of the possible values of B_t's, then σ_t's are random variables.

Our algorithm and scheme is a special case of those studied in [2]; in that work ergodicity was proved for the studied systems.

In [3] the variables ξ_{ij} are independent, uniformly distributed on the set $\{1, \ldots, N\}$, and the question was considered: how many rows are needed to process N (different) elements? Limit distribution was given for this random variable for $t = 1$, $N \to \infty$. In [5] limit distribution is given for it if $t > 1$ fixed and $N \to \infty$.

For any such system the problem of the speed, especially the asymptotic speed (when $t \to \infty$) of the algorithm is of basic importance, as Katai pointed out [4]. The asymptotic speed depends on the ergodic behaviour of σ_t. As we mentioned already, σ_t is an ergodic Markov-chain, and to derive its ergodic distribution is our next aim in the case $N = 2$, $p = \frac{1}{2}$.

First of all let us compute the transition probabilities for this sequence in the case $r = 3$.

Then there are $2^3 = 8$ possible vectors: $(0, 0, 0)$, $(0, 0, 1)$, $\ldots$, $(1, 1, 1)$. For the sake of simplicity we denote the vector (x_1, x_2, x_3) by

$$V = 4x_1 + 2x_2 + x_3, \quad V = 0, 1, \ldots, 7 \quad (x_i = 0, 1). \tag{2.4}$$

The transition probabilities (only the positive ones) are presented in the following table.

Table 1.

Transition probabilities for $r = 3$

	0	1	2	3	4	5	6	7
0	3/4	–	–	–	1/4	–	–	–
1	1/4	2/4	–	–	–	1/4	–	–
2	1/4	–	2/4	–	–	–	1/4	–
3	–	1/4	–	2/4	–	–	–	1/4
4	1/4	–	–	–	2/4	–	1/4	–
5	–	1/4	–	–	2/4	–	1/4	–
6	–	–	1/4	–	–	–	2/4	1/4
7	–	–	1/4	–	–	–	–	3/4

We get these probabilities as follows. For the vectors 0 and 7 we have $f_{11} = f_{21} = \ldots = f_{r1}$, therefore only the cases *a)* and *c)* are possible in Step 3. of the algorithm. With probability $P(f_{11} \neq f_{12} \text{ and } f_{12} = f_{13}) = \frac{1}{4}$ we get the transitions $0 \to 4$ or $7 \to 3$ and with

$$P(f_{11} = f_{12}) + P(f_{11} \neq f_{12} \text{ and } f_{13} = f_{11}) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \tag{2.5}$$

we have the transitions $0 \to 0$ or $7 \to 7$, that is

$$P(0 \to 4) = P(7 \to 3) = \frac{1}{4} \quad \text{and} \quad P(0 \to 0) = P(7 \to 7) = \frac{3}{4}. \tag{2.6}$$

For the remaining vectors only the cases *a)* and *b)* are possible. Let

$$\bar{f}_{ij} = \begin{cases} 1, & \text{if } f_{ij} = 0, \\ 0, & \text{if } f_{ij} = 1, \end{cases} \tag{2.7}$$

and

$$\overline{f_1, f_2, \ldots, f_r} = \bar{f}_1, \bar{f}_2, \ldots, \bar{f}_r. \tag{2.8}$$

Then for the vector s $(1 \leq s \leq 6)$ let $f_{11} = f_{21} = \ldots = f_{k1}$ and $f_{k+1,1} \neq f_{11}$. Now

$$P(s \to s) = P(f_{11} = f_{12} \text{ and } f_{k+1,1} = f_{k+1,2}) + \\ + P(f_{11} \neq f_{12} \text{ and } f_{13} = f_{11}) = \frac{2}{4}, \tag{2.9}$$

(2.10)
$$P((f_{11}, \ldots, f_{r1}) \to (f_{11}, f_{21}, \ldots, f_{r1})) =$$
$$= P(f_{11} \neq f_{12} \text{ and } f_{13} = f_{12}) = \frac{1}{4},$$

and

$$P((f_{11}, \ldots, f_{k,1}, f_{k+1,1}, \ldots, f_{r1}) \to (f_{11}, \ldots, f_{k,1}, f_{k+1,1}, \ldots, f_{r1})) =$$
$$= P(f_{11} = f_{12} \text{ and } f_{k+1,1} \neq f_{k+1,2}) = \frac{1}{4}.$$

It is easy to see that the sequence $\sigma_1, \sigma_2, \ldots$ is a homogeneous ergodic Markov-chain.

If

$$\sigma_t = (f_{11}, \ldots, f_{r1}),$$

and

$$f_{11} = f_{12} = \ldots = f_{1,r+1} = 0,$$

(2.11)
$$f_{22} = f_{32} = \ldots = f_{r2} = 0,$$

then $\sigma_{t+r} = (0, 0, \ldots, 0)$, that is from any vector σ_t we can reach the vector $(0, 0, \ldots, 0)$ in r transitions with a probability which is not less than $\left(\frac{1}{2}\right)^{2r}$, and this fact is enough to guarantee ergodicity due to the Markov-theorem.

In the following paragraph we shall determine the ergodic distribution, that is the limit probabilities

$$\lim_{t \to \infty} P(\sigma_t = j) = p_j \quad (j = 0, 1, \ldots, 2^r - 1).$$

§. 3. The ergodic probabilities

Now we are going to determine the limit distribution of the vectors of the first non-processed elements.

If we process r sequences, then the vectors have 2^r possible values. The ergodic probabilities have to satisfy the following equations:

(3.1)
$$p_j = \sum_{i=1}^{n} p_i p_{ij} \quad (j = 0, 1, \ldots, 2^r - 1)$$

and

(3.2)
$$\sum_{j=0}^{2^r - 1} p_j = 1.$$

Before the general case let us consider the case $r = 3$, whose transition probabilities are given in Table 1.

In this case

$$p_0 = \frac{3}{4}p_0 + \frac{1}{4}p_1 + \frac{1}{4}p_2 + \qquad + \frac{1}{4}p_4 \qquad , \tag{3.3}$$

$$p_1 = \qquad \frac{2}{4}p_1 + \qquad + \frac{1}{4}p_3 + \qquad + \frac{1}{4}p_5 \qquad , \tag{3.4}$$

$$p_2 = \qquad \frac{2}{4}p_2 + \qquad + \frac{1}{4}p_6 \qquad , \tag{3.5}$$

$$p_3 = \qquad + \frac{2}{4}p_3 + \qquad + \frac{1}{4}p_7 , \tag{3.6}$$

$$p_4 = \frac{1}{4}p_0 + \qquad + \frac{2}{4}p_4 \qquad , \tag{3.7}$$

$$p_5 = \qquad \frac{1}{4}p_1 + \qquad + \frac{2}{4}p_5 \qquad , \tag{3.8}$$

$$p_6 = \qquad \frac{1}{4}p_2 + \qquad + \frac{1}{4}p_4 + \qquad + \frac{2}{4}p_6 \qquad , \tag{3.9}$$

$$p_7 = \qquad + \frac{1}{4}p_3 + \qquad + \frac{1}{4}p_5 + \frac{1}{4}p_6 + \frac{3}{4}p_7 , \tag{3.10}$$

and

$$p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 = 1 . \tag{3.11}$$

In consequence of the symmetry we have

$$p_7 = p_0, \quad p_6 = p_1, \quad p_5 = p_2 \quad \text{and} \quad p_4 = p_3 , \tag{3.12}$$

and so instead of the system (3.3)–(3.11) we get

$$4p_1 = 2p_1 + p_2 + p_3 , \tag{3.13}$$

$$4p_2 = p_1 + 2p_2 \qquad , \tag{3.14}$$

$$4p_3 = p_0 \qquad + 2p_3 , \tag{3.15}$$

and

$$2p_0 + 2p_1 + 2p_2 + 2p_3 = 1 . \tag{3.16}$$

(We remark, that (3.3) was omitted because of the redundancy.)

From (3.14) and (3.15) we get

$$2p_2 = p_1, \quad 2p_3 = p_0 . \tag{3.17}$$

Substituting (3.17) into (3.16) we get

$$3p_0+3p_1 = 1\,. \tag{3.18}$$

Substituting (3.17) into (3.13)

$$3p_1 = p_0\,. \tag{3.19}$$

Substituting (3.19) into (3.18)

$$4p_0 = 1\,, \tag{3.20}$$

and then

$$\begin{aligned} p_0 = p_7 = \frac{1}{4}, \quad p_1 = p_6 = \frac{1}{4\cdot 3}, \\ p_2 = p_5 = \frac{1}{4\cdot 3\cdot 2}, \quad p_3 = p_4 = \frac{1}{4\cdot 2}. \end{aligned} \tag{3.21}$$

Generalizing the previous computations we find a connection among the p's.

Let A_j denote any sequence of j $(1\le j\le r)$ binary digits, that is let

$$A_j = i_1, i_2, \ldots, i_j \quad (i_1, \ldots, i_j \in \{0, 1\})\,. \tag{3.22}$$

We shall use also the notation

$$\overline{A_j} = \bar{i}_1, \bar{i}_2, \ldots, \bar{i}_j\,. \tag{3.23}$$

Let $\alpha_r(A_r)$ denote the ergodic probability of the vector A_r in the case of r processable sequences.

At first we prove the following assertion.

Lemma 1. *If* $r\ge 2$, $1\le k\le r$, *then*

$$k\,\alpha_r(0\,0\,\ldots\,0\,0\,1\,A_{r-k}) = \alpha_r(0\,0\,\ldots\,0\,0\,0\,A_{r-k})\,. \quad \square \tag{3.24}$$

According to this assertion the ergodic probability of a binary vector consisting of $(k-1)$ zeros, one 1 and any sequence A_{r-k} of length $(r-k)$ is k times smaller than the ergodic one of the binary vector in which k ones are continued by the sequence A_{r-k}.

Proof of Lemma 1.

For $k = 1$ we assert

$$\alpha_r(1\,A_{r-1}) = \alpha_r(0\,A_{r-1})\,, \tag{3,25}$$

which holds in consequence of the symmetry $\overline{1\,A_{r-1}} = 0\,\overline{A_{r-1}}$.

Let us suppose, (3.24) holds for $k = 1, \ldots, s$, where $s<r$. Then we show that (3.25) holds for $k = s+1$, too.

Let us consider the binary vectors

$$0\,0 \dots 0\,0\,1\,A_{r-s-1}\,. \tag{3.26}$$

We get this vector with a probability $\frac{2}{4}$ from itself, and with a probability $\frac{1}{4}$ from the vectors, whose first s elements contain punctually one 1, and the last $r-s$ elements are $1\,A_{r-s-1}$.

Let B_j $(j=1, \dots, s)$ be the binary vector of length s, in which the j-th element is 1, the remaining elements are 0's.

Then

$$\begin{aligned}&\alpha_r(0\,0 \dots 0\,0\,1\,A_{r-s-1} = \\ &= \frac{2}{4}\alpha_r(0\,0 \dots 0\,0\,1\,A_{r-s-1}) + \sum_{j=1}^{s}\frac{1}{4}\alpha_r(B_j\,1\,A_{r-s-1}),\end{aligned} \tag{3.27}$$

and from here

$$2\alpha_r(0\,0 \dots 0\,0\,1\,A_{r-s-1}) = \sum_{j=1}^{s}\alpha_r(B_j\,1\,A_{r-s-1})\,. \tag{3.28}$$

Using twice the induction hypothesis we get for $j=1, \dots, s-1$

$$\begin{aligned}&\alpha_r(B_j\,1\,A_{r-s-1}) = \alpha_r(0\,0 \dots 0\,1\,0 \dots 0\,1\,A_{r-s-1}) = \\ &= \frac{1}{j}\alpha_r(0\,0 \dots 0\,0\,1 \dots 1\,0\,A_{r-s-1}) = \\ &= \frac{1}{(j+1)\,j}\alpha_r(0\,0 \dots 0\,0\,0 \dots 0\,1\,\overline{A_{r-s-1}})\,.\end{aligned} \tag{3.29}$$

For $j=s$ the induction hypothesis gives

$$\begin{aligned}&\alpha_r(0\,0 \dots 0\,0\,0 \dots 1\,1\,A_{r-s-1}) = \\ &= \frac{1}{s}\alpha_r(0\,0 \dots 0\,0\,0 \dots 0\,0\,\overline{A_{r-s-1}})\,.\end{aligned} \tag{3.30}$$

Let us substitute (3.29) and (3.30) into (3.28):

$$\begin{aligned}&\alpha_r(0\,0 \dots 0\,0\,1\,A_{r-s-1})\left[2-\sum_{j=1}^{s-1}\frac{1}{j\,(j+1)}\right] = \\ &= \frac{1}{s}\alpha_r(0\,0 \dots 0\,0\,\overline{A_{r-s-1}})\,.\end{aligned} \tag{3.31}$$

Taking into account

$$\frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1}, \tag{3.32}$$

we get

$$\left(1+\frac{1}{s}\right)\alpha_r(0\,0\,\ldots\,0\,0\,1\,A_{r-s-1}) = \frac{1}{s}\,\alpha_r(0\,0\,\ldots\,0\,0\,\overline{A_{r-s-1}}), \tag{3.33}$$

that is

$$(s+1)\,\alpha_r(0\,0\,\ldots\,0\,0\,1\,A_{r-s-1}) = \alpha_r(0\,0\,\ldots\,0\,0\,\overline{A_{r-s-1}}). \quad \Box \tag{3.34}$$

Now we compute the ergodic probability of the vectors consisting of identical elements.

Lemma 2. *If $r \geq 2$ then*

$$\alpha_r(0\,0\,\ldots\,0\,0) = \alpha_r(1\,1\,\ldots\,1\,1) = \frac{1}{r+1}. \quad \Box \tag{3.35}$$

Proof. We begin with the equality

$$\sum_{j=0}^{2^r-1} p_j = 1. \tag{3.36}$$

In consequence of the symmetry we have for $j = 2^{r-1}, \ldots, 2^r-1$

$$p_j = p_{2^r-1-j}, \tag{3.37}$$

and therefore

$$\sum_{j=0}^{2^r-1} p_j = \sum_{j=0}^{2^{r-1}-1} p_j + \sum_{j=2^{r-1}}^{2^r-1} p_j = 2\sum_{j=0}^{2^{r-1}} p_j. \tag{3.38}$$

We have got that for $k = 1$ holds

$$(k+1)\sum_{j=0}^{2^{r-k}-1} p_j = 1. \tag{3.39}$$

Let us suppose that (3.39) holds for $k = s$, where $s < r$. We show that than (3.39) holds for $k = s+1$ too.

By the induction hypothesis we have

$$(s+1)\sum_{j=0}^{2^{r-s}-1} p_j = 1. \tag{3.40}$$

We divide the numbers $0, 1, \ldots, 2^{r-s}-1$ into two groups: $0, 1, \ldots, \ldots, 2^{r-s-1}-1$ and $2^{r-s-1}, \ldots, 2^{r-s}-1$. The elements of the groups have the form $0\,0\,\ldots\,0\,0\,A_{r-s-1}$ and $0\,0\,\ldots\,0\,1\,\overline{A_{r-s-1}}$ resp. and we have a one to one correspondence among them (the sum of the corresponding elements equal to $2^{r-s}-1$.) Therefore by Lemma 1 we get

$$(3.41)\qquad (s+1)\sum_{j=0}^{2^{r-s}-1} p_j = (s+1)\sum_{j=0}^{2^{r-s-1}-1} p_j + (s+1)\sum_{j=2^{r-s-1}}^{2^{r-s}-1} p_j =$$

$$(3.42)\qquad = (s+1)\sum_{j=0}^{2^{r-s-1}-1} p_j + (s+1)\sum_{j=0}^{2^{r-s-1}-1} \frac{p_j}{s+1} = (s+2)\sum_{j=0}^{2^{r-s-1}} p_j .$$

Hence (3.42) holds for every $k = 1, \ldots, r$, among them for $k = r$, therefore

$$(3.43)\qquad (r+1)\, p_0 = 1 .$$

From (3.43) and the symmetry we get (3.35). □

Now we can determine the ergodic probabilities.

Theorem 1. *Let us process the binary sequences $\xi_{i1}, \xi_{i2}, \ldots, (i = 1, \ldots, r)$ of independent random variables with common distribution*

$$(3.44)\qquad P(\xi_{ij} = 0) = P(\xi_{ij} = 1) = \frac{1}{2} \quad (i = 1, \ldots, r;\ j = 1, 2, \ldots)$$

using the algorithm defined above.Then the ergodic probability of the vector $(i_0, \ldots, i_{r-1})$ equals to

$$(3.45)\qquad \alpha_r(i_0, \ldots, i_{r-1}) = \frac{1}{(r+1)\prod\limits_{\substack{2\le k\le r\\ i_{k-1}\ne i_k}} k} . \ \square$$

According to this theorem we get $\alpha_r\ (i_0, \ldots, i_{r-1})$ dividing the basic value $\alpha_r(0\,0 \ldots 0\,0) = \frac{1}{r+1}$ by the product of the indeces of i's differing from the previous one.

Proof of theorem 1. Let

$$(3.46)\qquad j = \sum_{k=0}^{r-1} i_k \cdot 2^k$$

be the binary form of j for $j = 0, 1, \ldots, 2^r - 1$. Due to the symmetry we have to deal only with the values $j = 0, 1, \ldots, 2^{r-1} - 1$.

If $r = 1$, then a simple direct calculation gives $\alpha_1(0) = \frac{1}{2}$. Let now $r \ge 2$ be.

As for $j = 0$ the product in (3.45) is empty (i.e. there is no a change in the corresponding binary vector), Lemma 2. gives the same value, as (3.45).

Let now be s $(1 \le s \le r-1)$ changes in j, that is let differ punctually the elements on the h_1-th, $\ldots$, h_s-th places from the elements preceeding them.

Using Lemma 1 for $k = h_1, h_2, \ldots, h_s$, we get

$$\alpha_r(j) \prod_{i=1}^{s} h_i = \alpha_r(0). \tag{3.47}$$

That was to be proved. □

§. 4. On the processing speed

Processing binary sequences as before let the random variable η_{ij} denote the number of the processed elements of the i-th sequence in the j-th point of time.

According to the processing algorithm we have

$$1 \le \eta_{1j} \le 2 \quad (j = 1, 2, \ldots), \tag{4.1}$$

$$0 \le \eta_{ij} \le 1 \quad (i = 2, \ldots, r;\ j = 1, 2, \ldots), \tag{4.2}$$

$$1 \le \sum_{i=1}^{r} \eta_{ij} \le 2 \quad (j = 1, 2, \ldots). \tag{4.3}$$

The processing speed $S^{(r)}$ is defined by

$$S^{(r)} = \lim_{t \to \infty} \frac{\sum_{j=1}^{t} M\left(\sum_{i=1}^{r} \eta_{ij}\right)}{t}. \tag{4.4}$$

There is a close connection between the distributions of σ_j and $\sum_{i=1}^{r} \eta_{ij}$:

$$P\left(\sum_{i=1}^{r} \eta_{ij} = 1\right) = \frac{1}{2} P(\sigma_j = (0, 0, \ldots, 0, 0)) + \frac{1}{2} P(\sigma_j = (1, 1, \ldots, 1, 1)). \tag{4.5}$$

As $\sigma_1, \sigma_2, \ldots$ is ergodic, the right side of (4.5) has a limit, whose value is $\frac{1}{r+1}$ according to Lemma 2. Therefore

$$\lim_{j \to \infty} M\left(\sum_{i=1}^{r} \eta_{ij}\right) = \frac{1}{r+1} \cdot 1 + \frac{r}{r+1} \cdot 2 = 2 - \frac{1}{r+1}, \tag{4.6}$$

and from where

$$S^{(r)} = 2 - \frac{1}{r+1}, \tag{4.7}$$

as the convergence of the sequence (4.6) implies the convergence (to the same limit) of the middle values of its elements in (4.4).

If we define the processing speed S_i for the i-th sequence by

$$(4.8) \qquad S_1 = S^{(1)} \quad \text{and} \quad S_i = S^{(i)} - S^{(i-1)} \quad (i = 2, \ldots, r),$$

then from (4.7) we get

$$(4.9) \qquad S_1 = \frac{3}{2} \quad \text{and} \quad S_i = \frac{1}{i(i+1)} \quad (i = 2, \ldots, r).$$

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