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The exact annihilating-ideal graph of a commutative ring

Research Article

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Abstract: The rings considered in this article are commutative with identity. For an ideal I of a ring R, we denote the annihilator of I in R by $Ann(I)$. An ideal I of a ring R is said to be an exact annihilating ideal if there exists a non-zero ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$. For a ring R, we denote the set of all exact annihilating ideals of R by $\mathbb{EA}(R)$ and $\mathbb{EA}(R)\backslash\{(0)\}$ by $\mathbb{EA}(R)^*$. Let R be a ring such that $\mathbb{EA}(R)^* \neq \emptyset$. With R, in [Exact Annihilating-ideal graph of commutative rings, *J. Algebra and Related Topics* **5**(1) (2017) 27-33] P.T. Lalchandani introduced and investigated an undirected graph called the exact annihilating-ideal graph of R, denoted by $\mathbb{EAG}(R)$ whose vertex set is $\mathbb{EA}(R)^*$ and distinct vertices I and J are adjacent if and only if $Ann(I) = J$ and $Ann(J) = I$. In this article, we continue the study of the exact annihilating-ideal graph of a ring. In Section 2 , we prove some basic properties of exact annihilating ideals of a commutative ring and we provide several examples. In Section 3, we determine the structure of $\mathbb{EAG}(R)$, where either R is a special principal ideal ring or R is a reduced ring which admits only a finite number of minimal prime ideals.

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1. Introduction

The rings considered in this article are commutative with identity which are not integral domains. Let R be a ring. For an element $a \in R$, *the annihilator of* a *in* R, denoted by $Ann_R(a)$ or simply by $Ann(a)$ is defined as $Ann(a) = \{r \in R \mid ra = 0\}$. Recall from [12] that an element $x \in R$ is said to be an *exact zero-divisor* if there exists $y \in R\backslash\{0\}$ such that $Ann(x) = Ry$ and $Ann(y) = Rx$. It is clear that any exact zero-divisor of R is a zero-divisor of R. We denote the set of all zero-divisors of a ring R by

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$Z(R)$ and $Z(R)\backslash\{0\}$ by $Z(R)^*$. As in [15], we denote the set of all exact zero-divisors of R by $EZ(R)$ and $EZ(R)\backslash\{0\}$ by $EZ(R)^*$. Let R be a ring such that $EZ(R)^* \neq \emptyset$. Recall from [15] that the *exact zero-divisor graph of* R, denoted by $E\Gamma(R)$ is an undirected graph whose vertex set is $EZ(R)^*$ and distinct vertices x and y are adjacent in $E\Gamma(R)$ if and only if $Ann(x) = Ry$ and $Ann(y) = Rx$. Several properties of the exact zero-divisor graph of a commutative ring were investigated in [15, 16]. Let R be a ring. Recall from [7] that an ideal I of R is said to be an *annihilating ideal* if there exists $r \in R\backslash\{0\}$ such that $Ir = (0)$. As in [7], we denote the set of all annihilating ideals of R by $\mathbb{A}(R)$ and $\mathbb{A}(R)\backslash\{(0)\}$ by $\mathbb{A}(R)^*$. The concept of annihilating-ideal graph of a commutative ring was introduced and investigated by M. Behboodi and Z. Rakeei in [7]. Let R be a ring. Recall from [7] that the *annihilating-ideal graph of* R, denoted by $\mathbb{AG}(R)$ is an undirected graph whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I and J are adjacent in this graph if and only if $IJ = (0)$. Motivated by the interesting results proved on the annihilating-ideal graph of a ring in [7, 8], several researchers contributed to the study of annihilating-ideal graphs of commutative rings (for example, refer [1], [2], [11]). Inspired by the above mentioned work on annihilating-ideal graphs of rings and by the work on exact zero-divisor graphs of rings in [15, 16], in [17], P.T. Lalchandani introduced and studied the concept of the exact annihilating-ideal graph of a commutative ring. Let R be a ring. Recall from [17] that an ideal I of R is said to be an *exact annihilating ideal* if there exists a non-zero ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$, where for an ideal A of R, the *annihilator of* A *in* R, denoted by $Ann_R(A)$ or simply by $Ann(A)$ is defined as $Ann(A) = \{r \in R \mid rA = (0)\}$ [4, page 19]. As in [17], we denote the set of all exact annihilating ideals of a ring R by $\mathbb{EA}(R)$ and we denote $\mathbb{EA}(R)\backslash\{(0)\}$ by $\mathbb{EA}(R)^*$. It is clear that for any ring R, $\mathbb{EA}(R)^* \subseteq \mathbb{A}(R)^*$. Let R be a ring such that $\mathbb{EA}(R)^* \neq \emptyset$. Recall from [17] that the *exact annihilating-ideal graph of* R, denoted by $\mathbb{EAG}(R)$ is an undirected graph whose vertex set is $\mathbb{EA}(R)^*$ and distinct vertices I and J are adjacent in $\mathbb{EAG}(R)$ if and only if $Ann(I) = J$ and $Ann(J) = I$. The graphs considered in this article are undirected and simple. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. For a ring R with $\mathbb{EA}(R)^* \neq \emptyset$, it is clear that $V(\mathbb{EAG}(R)) = \mathbb{EA}(R)^* \subseteq \mathbb{A}(R)^* = V(\mathbb{AG}(R))$. Observe that if $I, J \in \mathbb{EA}(R)^*$ are such that I and J are adjacent in $\mathbb{EAG}(R)$, then $Ann(I) = J$ and $Ann(J) = I$. Hence, $IJ = (0)$ and so, I and J are adjacent in $\mathbb{AG}(R)$. Therefore, $\mathbb{EAG}(R)$ is a subgraph of $\mathbb{AG}(R)$. The aim of this article is to continue the study of the exact annihilating-ideal graph of a commutative ring which was carried out in [17].

Throughout this article, we consider rings R such that $\mathbb{EA}(R)^* \neq \emptyset$ (it is noted in a remark which appears just preceding the statement of Corollary 2.2 that for a ring R, $\mathbb{EA}(R)^* \neq \emptyset$ if and only if R is not an integral domain) and study the interplay between the graph-theoretic properties of $\mathbb{EAG}(R)$ and the ring-theoretic properties of R. This article consists of three sections including the introduction. In Section 2 of this article, we discuss some results on the exact annihilating ideals of R, where R is a commutative ring which is not an integral domain. Let $I \in \mathbb{A}(R)^*$. It is proved in Lemma 2.1 that the statements (1) $I \in \mathbb{EA}(R)^*$; (2)$I = Ann(J)$ for some non-zero ideal J of R; and (3) $Ann(Ann(I)) = I$ are equivalent. For a ring R, we denote the set of all proper ideals of R by $\mathbb{I}(R)$ and $\mathbb{I}(R)\backslash\{(0)\}$ by $\mathbb{I}(R)^*$. Many examples of rings R are provided in Section 2 such that $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$ (see Examples 2.3, 2.8, Lemmas 2.4 and 2.6). We denote the cardinality of a set A by $|A|$. Whenever a set A is a subset of a set B and $A \neq B$, then we denote it by $A \subset B$. It is well-known that for a ring T, $|\mathbb{A}(T)^*| = 1$ if and only if $(T, Z(T)))$ is a special principal ideal ring (SPIR) with $(Z(T))^2 = (0)$[7, Corollary 2.9(a)]. For a ring R, we denote the set of all prime ideals of R by $Spec(R)$ and the set of all maximal ideals of R by $Max(R)$. Let I be a non-zero proper ideal of a ring R. Motivated by [7, Corollary 2.9(a)], it is shown in Theorem 2.9 that the statements (1) $\mathbb{EA}(R)^* = \{I\}$ and (2) $I \in Spec(R), I^2 = (0)$, and $Z(R) = I$ are equivalent. In Example 2.14, a ring R is provided to illustrate that (2) $\Rightarrow$ (1) of Theorem 2.9 can fail to hold if the assumption that $I = Z(R)$ is omitted. It is verified that the ring R given in Example 2.14 is such that $|\mathbb{EA}(R)^*| = 2$. Inspired by this example, it is natural to try to determine necessary and sufficient conditions on the ideals I, J of a ring R such that $\mathbb{EA}(R)^* = \{I, J\}$. It is well-known that the set of all nilpotent elements of a ring R is an ideal of R [4, Proposition 1.7] and is called the nilradical of R. We denote the nilradical of a ring R by $nil(R)$. A ring R is said to be *reduced* if $nil(R) = (0)$. We denote the set of all units of R by $U(R)$. We denote the set of all minimal primes ideals of a ring R by $Min(R)$. For non-zero proper ideals I, J of a reduced ring R which is not an integral domain, it is proved in Theorem 2.16 that the statements (1) $\mathbb{EA}(R)^* = \{I, J\}$; (2) $J = Ann(I)$, $I, J \in Spec(R)$; and (3) $Min(R) = \{I, J\}$ are equivalent. Let R be a ring. Let $\mathfrak{p} \in Spec(R)$ be such that $\mathfrak{p} = Rp$ is principal,

$n \geq 2$ is least with the property that $\mathfrak{p}^n = (0)$, and $\mathfrak{p} = Z(R)$. Then it is shown in Proposition 2.10 that $\mathbb{EA}(R)^* = \{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\}$ and moreover, it is verified in Proposition 2.10 that $\mathbb{A}(R)^* = \mathbb{EA}(R)^*$ if and only if $\mathfrak{p} \in Max(R)$. Let R be a reduced ring. Let $n \geq 2$ and let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, \ldots, n\}\}$. Let $\mathcal{C}$ denote the collection of all non-empty proper subsets of $\{1, 2, \ldots, n\}$. It is proved in Proposition 2.15 that $\mathbb{EA}(R)^* = \{\prod_{i \in A} \mathfrak{p}_i \mid A \in \mathcal{C}\}$. Moreover, it is verified in Proposition 2.15 that $\mathbb{A}(R)^* = \mathbb{EA}(R)^*$ if and only if $\mathfrak{p}_i \in Max(R)$ for each $i \in \{1, 2, \ldots, n\}$. Let T be a unique factorization domain (UFD). It is shown in Theorem 2.17 that the statements (1) For each prime element p of T, $\mathbb{A}(\frac{T}{Tp^2})^* = \mathbb{EA}(\frac{T}{Tp^2})^*$; (2) T is a principal ideal domain (PID); and (3) For each $I \in \mathbb{I}(T)^*$ with $I \notin Max(T)$, $\mathbb{A}(\frac{T}{I})^* = \mathbb{EA}(\frac{T}{I})^*$ are equivalent. Let T be a UFD with at least two non-associate prime elements. It is proved in Theorem 2.18 that the statements (1) For all non-associate prime elements p_1, p_2 of T, $\mathbb{A}(\frac{T}{Tp_1p_2})^* = \mathbb{EA}(\frac{T}{Tp_1p_2})^*$; (2) T is a PID; and (3) For any $I \in \mathbb{I}(T)^*$ with $I \notin Max(T)$, $\mathbb{A}(\frac{T}{I})^* = \mathbb{EA}(\frac{T}{I})^*$ are equivalent. Let R be a von Neumann regular ring which is not a field. It is shown in Corollary 2.19 that $|\mathbb{EA}(R)^*| < \infty$ if and only if there exist $n \geq 2$ and fields $F_1, F_2, \ldots, F_n$ such that $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings.

Let R be a ring such that $\mathbb{EA}(R)^* \neq \emptyset$. The aim of Section 3 of this article is to discuss some results regarding the properties of $\mathbb{EAG}(R)$. Let $I, J \in \mathbb{EA}(R)^*$ be such that $I \neq J$. It is proved in Proposition 3.1 that there is a path in $\mathbb{EAG}(R)$ between I and J if and only if I and J are adjacent in $\mathbb{EAG}(R)$. If $I - J$ is an edge of $\mathbb{EAG}(R)$, then for any $A \in \mathbb{EA}(R)^* \backslash \{I, J\}$, it is shown in Lemma 3.2 that I and A are not adjacent in $\mathbb{EAG}(R)$ and J and A are not adjacent in $\mathbb{EAG}(R)$. As a consequence of Lemma 3.2, it is deduced in Corollary 3.3 that if g is any component of $\mathbb{EAG}(R)$, then g is a complete graph with at most two vertices. Let R be a ring. Let $\mathfrak{p} \in Spec(R) \backslash \{(0)\}$ be such that $\mathfrak{p}^2 = (0)$, and $Z(R) = \mathfrak{p}$. It is noted in Proposition 3.4 that $\mathbb{EA}(R)^* = \{\mathfrak{p}\}$ and moreover, it is verified in Proposition 3.4 that its conclusion holds for a SPIR $(R, \mathfrak{m})$ with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$. For a real number x, we denote the integer part of x by $[x]$. Let R be a ring. Let $\mathfrak{p} \in Spec(R)$ be such that $\mathfrak{p} = Rp$ is principal. Let $n \geq 3$ be least with the property that $\mathfrak{p}^n = (0)$ and $Z(R) = \mathfrak{p}$. Then it is proved in Proposition 3.5 that the following statements hold: (1) If n is odd, then $\mathbb{EAG}(R)$ has exactly $[\frac{n}{2}]$ components and each component is a complete graph with two vertices. (2) If $n \geq 4$ is even, then $\mathbb{EAG}(R)$ has exactly $\frac{n}{2}$ components $g_1, \ldots, g_{\frac{n}{2}-1}, g_{\frac{n}{2}}$ such that g_j is a complete graph with two vertices for each $j \in \{1, \ldots, \frac{n}{2} - 1\}$ and $g_{\frac{n}{2}}$ is a complete graph on a single vertex. Moreover, it is noted in Proposition 3.5 that the statements (1) and (2) hold for a SPIR $(R, \mathfrak{m})$ with the property that $\mathfrak{m}^n = (0)$ but $\mathfrak{m}^{n-1} \neq (0)$. Let $R, \mathfrak{p} = Rp = Z(R)$ be as in the statement of Proposition 3.5. Let $n \geq 2$ be least with the property that $\mathfrak{p}^n = (0)$. Then it is shown in Theorem 3.7 that the statements (1) $\mathbb{EAG}(R) = \mathbb{AG}(R)$ and (2) $(R, \mathfrak{p})$ is a SPIR and $n \in \{2, 3\}$ are equivalent. It is verified in Example 3.8 that the ring R provided by D.D. Anderson and M. Naseer in [3, page 501] is such that $\mathbb{EAG}(R) \neq \mathbb{AG}(R)$ which illustrates that (2) $\Rightarrow$ (1) of Theorem 3.7 can fail to hold if the hypothesis that $\mathfrak{p}$ is principal is omitted. Let R be a reduced ring which is not an integral domain. It is shown in Lemma 3.9 that each component of $\mathbb{EAG}(R)$ is a complete graph with two vertices. It is proved in Corollary 3.10 that $\mathbb{EAG}(R)$ is connected if and only if $|Min(R)| = 2$ and it is shown in Corollary 3.11 that $\mathbb{EAG}(R) = \mathbb{AG}(R)$ if and only if $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. If $|Min(R)| = n \geq 2$, then it is proved in Corollary 3.12 that $\mathbb{EAG}(R)$ has exactly $2^{n-1} - 1$ components. Let R be a ring such that $\mathbb{EA}(R)^* \neq \emptyset$. It is shown in Theorem 3.14 that the statements (1) $\mathbb{EAG}(R) = \mathbb{AG}(R)$ and (2) Either $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$ or $(R, \mathfrak{m})$ is a SPIR and if $n \geq 2$ is least with the property that $\mathfrak{m}^n = (0)$, then $n \in \{2, 3\}$ are equivalent. Let R be a ring. The Krull dimension of R is simply referred to as the dimension of R and is denoted by $dim R$. Let R be a ring such that $dim R = 0$. If $\mathbb{EAG}(R)$ is connected, then it is proved in Proposition 3.16 that $|Max(R)| \leq 2$ and if $|Max(R)| = 2$ then it is shown in Corollary 3.17 that $\mathbb{EAG}(R)$ is connected if and only if $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. Let R be a ring such that $\mathbb{EA}(R)^* \neq \emptyset$. It is noted in Corollary 3.20 that $girth(\mathbb{EAG}(R)) = \infty$ and $\mathbb{EAG}(R)$ is perfect.

2. Some basic properties of $\mathbb{EA}(R)^*$

As mentioned in the introduction, the rings considered in this article are commutative with identity. Let R be a ring such that $\mathbb{EA}(R)^* \neq \emptyset$. The aim of this section is to discuss some basic properties of the exact annihilating ideals of R.

Let R be a ring which is not an integral domain. Let $I \in \mathbb{A}(R)^*$. In Lemma 2.1, we provide a necessary and sufficient condition for I to be in $\mathbb{EA}(R)^*$.

Lemma 2.1. *Let R be a ring and let $I \in \mathbb{A}(R)^*$. The following statements are equivalent:*

(1) $I \in \mathbb{EA}(R)^*$.

(2) $I = Ann(J)$ *for some non-zero ideal* J *of* R.

(3) $Ann(Ann(I)) = I$.

Proof. (1) $\Rightarrow$ (2) We are assuming that $I \in \mathbb{EA}(R)^*$. Hence, by definition, there exists a non-zero ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$.

(2) $\Rightarrow$ (3) We are assuming that $I = Ann(J)$ for some non-zero ideal J of R. Note that $Ann(Ann(I)) = Ann(Ann(Ann(J)) = Ann(J) = I$.

(3) $\Rightarrow$ (1) We are assuming that $Ann(Ann(I)) = I$. Let us denote $Ann(I)$ by J. Since $I \in \mathbb{A}(R)^*$ by hypothesis, $Ann(I) \neq (0)$. Thus $J \neq (0)$ and is such that $Ann(I) = J$ and $Ann(J) = I$. This proves that $I \in \mathbb{EA}(R)^*$. □

Let R be a ring which is not an integral domain. Then $\mathbb{A}(R)^* \neq \emptyset$. Let $A \in \mathbb{A}(R)^*$. Then $Ann(A) \neq (0)$ and as $A(Ann(A)) = (0)$, it follows that $Ann(A) \in \mathbb{A}(R)^*$. It follows from (2) $\Rightarrow$ (1) of Lemma 2.1 that $Ann(A) \in \mathbb{EA}(R)^*$. The above arguments imply that for a ring R, $\mathbb{EA}(R)^* \neq \emptyset$ if and only if R is not an integral domain.

Corollary 2.2. *Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. The following statements are equivalent:*

(1) $\mathbb{A}(R)^* = \mathbb{EA}(R)^*$.

(2) *If* $I \in \mathbb{A}(R)^*$, *then* $I = Ann(J)$ *for some non-zero ideal* J *of* R.

(3) *For any* $I \in \mathbb{A}(R)^*$, $Ann(Ann(I)) = I$.

Proof. The statements (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) follow respectively from (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) of Lemma 2.1. For any ring T, as $\mathbb{EA}(T)^* \subseteq \mathbb{A}(T)^*$, the proof of (3) $\Rightarrow$ (1) follows immediately from (3) $\Rightarrow$ (1) of Lemma 2.1. □

We illustrate Corollary 2.2 with the help of the example provided by D.D. Anderson and M. Naseer in [3, page 501]. We verify that $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$ for the ring R provided in [3, page 501] in Example 2.3. For any $n \geq 2$, we denote the ring of integers modulo n by $\mathbb{Z}_n$.

Example 2.3. *Let $T = \mathbb{Z}_4[X, Y, Z]$ be the polynomial ring in three variables X, Y, Z over $\mathbb{Z}_4$. Let I be the ideal of T generated by $\{X^2 - 2, Y^2 - 2, Z^2, XY, YZ - 2, XZ, 2X, 2Y, 2Z\}$. Let $R = \frac{T}{I}$. Then $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$.*

Proof. It is convenient to denote $X + I, Y + I, Z + I$ by x, y, z, respectively. It was already noted in [3, page 501] that R is local with $\mathfrak{m} = Rx + Ry + Rz$ as its unique maximal ideal, $\mathfrak{m}^2 = \{0 + I, 2 + I\}$, $\mathfrak{m}^3 = (0 + I)$, and $|R| = 32$. Observe that $Z(R) = \mathfrak{m}$ and from $\mathfrak{m}^3 = (0)$, we get that each proper ideal of R is an annihilating ideal of R. Therefore, $\mathbb{I}(R)^* = \mathbb{A}(R)^*$. The ring R was also considered in [8, Proposition 2.1] and it was noted there that $\mathbb{I}(R)^* = \{(2+I), (x), (y), (z), (x+y), (y+z), (z+x), (x+y+z), (x, y), (y, z), (z, x), (x, y+z), (y, z+x), (z, x+y), (x+y, y+z), (x, y, z)\}$. From the multiplication table provided in [3, page 503], it follows that $Ann(2+I) = \mathfrak{m}$, $Ann(x) = \{0+I, 2+I, y, y+2, z, z+2, y+z, y+z+2\}$, $Ann(y) = \{0+1, 2+I, x, x+2, y+z, y+z+2, x+y+z, x+y+z+2\}$, $Ann(z) = \{0+I, 2+I, x, x+2, z, z+2, x+z, x+z+2\}$, $Ann(x+y) = \{0+I, 2+I, y+z, y+z+2, z+x, z+x+2, x+y, x+y+2\}$, $Ann(y+z) = \{0+I, 2+I, x, x+2, y, y+2, x+y, x+y+2\}$, $Ann(z+x) = \{0+I, 2+I, z, z+2, x+y, x+y+2, x+y+z, x+y+z+2\}$, and $Ann(x + y + z) = \{0 + I, 2 + I, x + z, x + z + 2, y, y + 2, x + y + z, x + y + z + 2\}$. Note that

$x(x+z) = y(x+z) = z(y+z) = x(x+y) = x(x+y+z) = 2+I$. From the above given arguments, it is clear that $Ann(\mathfrak{m}^2) = \mathfrak{m}, Ann(\mathfrak{m}) = \mathfrak{m}^2, Ann(Rx) = Ry+Rz, Ann(Ry+Rz) = Rx, Ann(Ry) = Rx+R(y+z), Ann(Rx+R(y+z)) = Ry, Ann(Rz) = Rx+Rz, Ann(Rx+Rz) = Rz, Ann(R(x+y)) = R(y+z)+R(z+x), Ann(R(y+z)+R(z+x)) = R(x+y), Ann(R(y+z)) = Rx+Ry, Ann(Rx+Ry) = R(y+z), Ann(R(z+x)) = Rz+R(x+y), Ann(Rz+R(x+y)) = R(z+x), Ann(R(x+y+z)) = Ry+R(x+z)$, and $Ann(Ry+R(x+z)) = R(x+y+z)$.

From the above discussion, we obtain that $\mathbb{I}(R)^* = \mathbb{A}(R)^*$ and each proper A of R is such that $Ann(Ann(A)) = A$. Hence, we obtain from (3) $\Rightarrow$ (1) of Corollary 2.2 that $\mathbb{A}(R)^* = \mathbb{EA}(R)^*$. Therefore, $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$. □

Recall that a principal ideal ring R is said to be a *special principal ideal ring* (SPIR) if R has a unique prime ideal. If $\mathfrak{m}$ is the unique prime ideal of R, then it follows from [4, Proposition 1.8] that $nil(R) = \mathfrak{m}$. Since $\mathfrak{m}$ is principal, we get that $\mathfrak{m}$ is nilpotent. Suppose that R is not a field. Then $\mathfrak{m} \neq (0)$. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Then it follows from the proof of $(iii) \Rightarrow (i)$ of [4, Proposition 8.8] that $\{\mathfrak{m}^i \mid i \in \{1, \ldots, n-1\}\}$ is the set of all non-zero proper ideals of R. If R is a SPIR with $\mathfrak{m}$ as its only prime ideal, then we denote it by the notation $(R, \mathfrak{m})$ is a SPIR. Let $(R, \mathfrak{m})$ be a SPIR which is not a field. We verify in Lemma 2.4 that $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$.

Lemma 2.4. *Let $(R, \mathfrak{m})$ be a SPIR which is not a field. Then $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$.*

Proof. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Note that $\mathbb{I}(R)^* = \{\mathfrak{m}^i \mid i \in \{1, \ldots, n-1\}\}$. From $\mathfrak{m}^n = (0)$, it follows that $\mathbb{I}(R)^* = \mathbb{A}(R)^*$. Let $i \in \{1, \ldots, n-1\}$. Observe that $Ann(\mathfrak{m}^i) = \mathfrak{m}^{n-i}$ and so, $Ann(Ann(\mathfrak{m}^i)) = Ann(\mathfrak{m}^{n-i}) = \mathfrak{m}^i$. Therefore, we obtain from (3) $\Rightarrow$ (1) of Corollary 2.2 that $\mathbb{A}(R)^* = \mathbb{EA}(R)^*$. This proves that $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$. □

Corollary 2.5. *Let T be a PID which is not a field. Let $\mathfrak{m} \in Max(T)$. Let $n \geq 2$ and let $R = \frac{T}{\mathfrak{m}^n}$. Then $\mathbb{I}(R)^* == \mathbb{A}(R)^* = \mathbb{EA}(R)^*$.*

Proof. Let $m \in \mathfrak{m}$ be such that $\mathfrak{m} = Tm$. Observe that R is a principal ideal ring. Note that $\frac{\mathfrak{m}}{\mathfrak{m}^n} \in Spec(R)$. Let $\mathfrak{P} \in Spec(R)$. Then $\mathfrak{P} = \frac{\mathfrak{p}}{\mathfrak{m}^n}$ for some $\mathfrak{p} \in Spec(T)$ with $\mathfrak{p} \supseteq \mathfrak{m}^n$. This implies that $\mathfrak{p} \supseteq \mathfrak{m}$ and so, $\mathfrak{p} = \mathfrak{m}$. Therefore, $\mathfrak{P} = \frac{\mathfrak{m}}{\mathfrak{m}^n}$. Thus R is a principal ideal ring with $Spec(R) = \{\frac{\mathfrak{m}}{\mathfrak{m}^n}\}$. Hence, $(R, \frac{\mathfrak{m}}{\mathfrak{m}^n})$ is a SPIR. Therefore, we obtain from Lemma 2.4 that $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$. □

We provide some more examples in Example 2.8 to illustrate Corollary 2.2. We use Lemmas 2.6 and 2.7 in the verification of Example 2.8.

Lemma 2.6. *Let $n \geq 2$ and let R_i be a ring for each $i \in \{1, 2, \ldots, n\}$. Let $R = R_1 \times R_2 \times \cdots \times R_n$. Suppose that for any $i \in \{1, 2, \ldots, n\}$ and any ideal I_i of R_i, $Ann_{R_i}(Ann_{R_i}(I_i)) = I_i$. Then $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$.*

Proof. Let $I \in \mathbb{I}(R)^*$. Then for each $i \in \{1, 2, \ldots, n\}$, there exists an ideal I_i of R_i such that $I = I_1 \times I_2 \times \cdots \times I_n$. Since $I \neq R$, it follows that $I_i \neq R_i$ for at least one $i \in \{1, 2, \ldots, n\}$. If $I_i = (0)$, then $I(Re_i) = (0) \times (0) \times \cdots \times (0)$, where e_i is the element of R whose i-th coordinate equals 1 and whose j-th coordinate equals 0 for all $j \in \{1, 2, \ldots, n\} \backslash \{i\}$. As Re_i is a non-zero ideal of R and $I(Re_i)$ equals the zero ideal of R, it follows that $I \in \mathbb{A}(R)^*$. Suppose that $I_i \neq (0)$. Then from the hypothesis, $Ann_{R_i}(Ann_{R_i}(I_i)) = I_i$, we get that $I_i \in \mathbb{A}(R_i)^*$ and so, $I \in \mathbb{A}(R)^*$. This shows that $\mathbb{I}(R)^* \subseteq \mathbb{A}(R)^*$ and so, $\mathbb{I}(R)^* = \mathbb{A}(R)^*$. Let $I = I_1 \times I_2 \times \cdots \times I_n \in \mathbb{I}(R)^* = \mathbb{A}(R)^*$. Observe that $Ann(Ann(I)) = Ann_{R_1}(Ann_{R_1}(I_1)) \times Ann_{R_2}(Ann_{R_2}(I_2)) \times \cdots \times Ann_{R_n}(Ann_{R_n}(I_n)) = I_1 \times I_2 \times \cdots \times I_n = I$. Thus for each $I \in \mathbb{A}(R)^*$, $Ann(Ann(I)) = I$. Hence, we obtain from (3) $\Rightarrow$ (1) of Corollary 2.2 that $\mathbb{A}(R)^* = \mathbb{EA}(R)^*$. Therefore, $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$. □

Lemma 2.7. *Let R be a ring and let $\mathfrak{m} \in Max(R)$. Let $\mathfrak{q}$ be a $\mathfrak{m}$-primary ideal of R. Then $\frac{R}{\mathfrak{q}} \cong \frac{R_\mathfrak{m}}{\mathfrak{q}_\mathfrak{m}}$ as rings.*

Proof. This is well-known. We provide a proof of this lemma for the sake of completeness. Let $f : R \to R_{\mathfrak{m}}$ be the usual homomorphism of rings defined by $f(r) = \frac{r}{1}$. Using the hypothesis that $\mathfrak{q}$ is $\mathfrak{m}$-primary, it can be shown that $f^{-1}(\mathfrak{q}_{\mathfrak{m}}) = \mathfrak{q}$. Hence, f induces an injective ring homomorphism $\overline{f} : \frac{R}{\mathfrak{q}} \to \frac{R_{\mathfrak{m}}}{\mathfrak{q}_{\mathfrak{m}}}$ defined by $\overline{f}(r + \mathfrak{q}) = f(r) + \mathfrak{q}_{\mathfrak{m}}$. We verify that $\overline{f}$ is onto. Let Y be any element of $\frac{R_{\mathfrak{m}}}{\mathfrak{q}_{\mathfrak{m}}}$. Then there exist $r \in R, s \in R\backslash\mathfrak{m}$ such that $Y = \frac{r}{s} + \mathfrak{q}_{\mathfrak{m}}$. Since $s \in R\backslash\mathfrak{m}$ and $\mathfrak{m} \in Max(R)$, we get that $\mathfrak{m} + Rs = R$. Hence, $\sqrt{\mathfrak{q}} + \sqrt{Rs} = R$ and so, we obtain from [4, Proposition 1.16] that $\mathfrak{q} + Rs = R$. Therefore, there exist $x \in R$ and $q \in \mathfrak{q}$ such that $q + xs = 1$. Hence, $r = rq + rxs$ and so, $Y = \frac{r}{s} + \mathfrak{q}_{\mathfrak{m}} = \frac{rsx+rq}{s} + \mathfrak{q}_{\mathfrak{m}} = \frac{rx}{1} + \mathfrak{q}_{\mathfrak{m}}$, since $\frac{rq}{s} \in \mathfrak{q}_{\mathfrak{m}}$. Thus $Y = \overline{f}(rx + \mathfrak{q})$. This shows that $\overline{f}$ is onto. Hence, $\overline{f} : \frac{R}{\mathfrak{q}} \to \frac{R_{\mathfrak{m}}}{\mathfrak{q}_{\mathfrak{m}}}$ is an isomorphism of rings. Therefore, $\frac{R}{\mathfrak{q}} \cong \frac{R_{\mathfrak{m}}}{\mathfrak{q}_{\mathfrak{m}}}$ as rings. □

Example 2.8. (1) *Let $n \geq 2$. Let $R = F_1 \times F_2 \times \cdots \times F_n$, where F_i is a field for each $i \in \{1, 2, \ldots, n\}$. Then $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$.*

(2) *Let T be a Dedekind domain and let I be a non-zero proper ideal of T such that $I \notin Max(T)$. Let $R = \frac{T}{I}$. Then $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$.*

(3) *Let T be a principal ideal domain. Let I be a non-zero proper ideal of T such that $I \notin Max(T)$. Let $R = \frac{T}{I}$. Then $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$.*

Proof. (1) Let $i \in \{1, 2, \ldots, n\}$. Observe that F_i and (0) are the only ideals of F_i and for each ideal I_i of F_i, $Ann_{F_i}(Ann_{F_i}(I_i)) = I_i$. Now, it follows from Lemma 2.6 that $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$.

(2) Since T is a Dedekind domain, T is Noetherian, $dimT = 1$, and T is integrally closed. Thus any non-zero prime ideal of T is maximal. It follows from [4, Corollary 9.4] that there exist distinct maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of T and positive integers $k_1, \ldots, k_n$ such that $I = \prod_{i=1}^{n} \mathfrak{m}_i^{k_i}$. Observe that for each $i \in \{1, \ldots, n\}$, $\sqrt{\mathfrak{m}_i^{k_i}} = \mathfrak{m}_i \in Max(T)$ and so, we obtain from [4, Proposition 4.2] that $\mathfrak{m}_i^{k_i}$ is a $\mathfrak{m}_i$-primary ideal of T. We know from Lemma 2.7 that $\frac{T}{\mathfrak{m}_i^{k_i}} \cong \frac{T_{\mathfrak{m}_i}}{(\mathfrak{m}_i^{k_i})_{\mathfrak{m}_i}}$ as rings. We know from $(i) \Rightarrow (iii)$ of [4, Theorem 9.3] that $T_{\mathfrak{m}_i}$ is a discrete valuation ring and so, it is a PID. Now, for all distinct $i, j \in \{1, \ldots, n\}$, $\sqrt{\mathfrak{m}_i^{k_i}} + \sqrt{\mathfrak{m}_j^{k_j}} = T$ and so by [4, Proposition 1.16] that $\mathfrak{m}_i^{k_i} + \mathfrak{m}_j^{k_j} = T$. Hence, we obtain from [4, Proposition 1.10] that $\frac{T}{I} \cong \frac{T}{\mathfrak{m}_1^{k_1}} \times \cdots \times \frac{T}{\mathfrak{m}_n^{k_n}}$. Note that for each $i \in \{1, \ldots, n\}$, $(\mathfrak{m}_i)_{\mathfrak{m}_i}$ is the unique maximal ideal of $T_{\mathfrak{m}_i}$ and $(\mathfrak{m}_i^{k_i})_{\mathfrak{m}_i} = ((\mathfrak{m}_i)_{\mathfrak{m}_i})^{k_i}$. Therefore, we obtain that $\frac{T}{I} \cong \frac{T_{\mathfrak{m}_1}}{((\mathfrak{m}_1)_{\mathfrak{m}_1})^{k_1}} \times \cdots \times \frac{T_{\mathfrak{m}_n}}{((\mathfrak{m}_n)_{\mathfrak{m}_n})^{k_n}}$ as rings. Let $i \in \{1, \ldots, n\}$ and let us denote the ring $\frac{T_{\mathfrak{m}_i}}{((\mathfrak{m}_i)_{\mathfrak{m}_i})^{k_i}}$ by R_i and the unique maximal ideal $(\mathfrak{m}_i)_{\mathfrak{m}_i}$ of $T_{\mathfrak{m}_i}$ by $\mathfrak{n}_i$. If $k_i = 1$, then R_i is a field. If $k_i \geq 2$, then as $T_{\mathfrak{m}_i}$ is a PID, we obtain from the proof of Corollary 2.5 that $(R_i, \frac{\mathfrak{n}_i}{\mathfrak{n}_i^{k_i}})$ is a SPIR. From the proof of Lemma 2.4, we know that $Ann_{R_i}(Ann_{R_i}(I_i)) = I_i$ for each ideal I_i of R_i. Now, $R \cong R_1 \times \cdots \times R_n$ as rings. Suppose that $n = 1$. Since $I \notin Max(T)$ by hypothesis, $k_1 \geq 2$ and so, it follows from Lemma 2.4 that $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$. Suppose that $n \geq 2$. As $Ann_{R_i}(Ann_{R_i}(I_i)) = I_i$ for each ideal I_i of R_i and for each $i \in \{1, 2, \ldots, n\}$, we obtain from Lemma 2.6 that $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$.

(3) If T is a PID which is not a field, then we know from [4, Example (1), page 96] that T is a Dedekind domain. Therefore, the conclusion of (3) follows immediately from (2). □

Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. It was shown in [7, Corollary 2.9(a)] that $\mathbb{A}(R)^* = \{I\}$ if and only if (R, I) is a SPIR with $I^2 = (0)$ (see also, [19, Lemma 2.6]). Let R be a ring and let $I \in \mathbb{A}(R)^*$. In Theorem 2.9, we determine necessary and suffcient conditions on I such that $\mathbb{EA}(R)^* = \{I\}$.

Theorem 2.9. *Let R be a ring and let I be a non-zero ideal of R. The following statements are equivalent:*

(1) *$\mathbb{EA}(R)^* = \{I\}$.*

(2) *$I \in Spec(R), I^2 = (0)$, and $Z(R) = I$.*

Proof. (1) $\Rightarrow$ (2) As $I \in \mathbb{EA}(R)^*$, $I \in \mathbb{A}(R)^*$ and $Ann(I) \in \mathbb{A}(R)^*$. Hence, we obtain from (2) $\Rightarrow$ (1) of Lemma 2.1 that $Ann(I) \in \mathbb{EA}(R)^* = \{I\}$ and so, $Ann(I) = I$. This proves that $I^2 = (0)$. We next verify that $I \in Spec(R)$. It is clear that $I \neq R$. If B is any non-zero ideal of R with $Ann(B) \neq (0)$, then $Ann(B) \in \mathbb{A}(R)^*$ and it follows from (2) $\Rightarrow$ (1) of Lemma 2.1 that $Ann(B) \in \mathbb{EA}(R)^* = \{I\}$. Hence, $Ann(B) = I$. Let $a, b \in R$ be such that $ab \in I = Ann(I)$. This implies that $Iab = (0)$. Suppose that $a \notin I = Ann(I)$. Then $Ia \neq (0)$ and from $I(Ia) = (0)$, it follows that $Ann(Ia) \neq (0)$. Hence, $Ann(Ia) = I$. From $b \in Ann(Ia)$, we get that $b \in I$. This proves that $I \in Spec(R)$. As any member of $\mathbb{A}(R)$ is a subset of $Z(R)$, it follows that $I \subseteq Z(R)$. Let $r \in Z(R)^*$. Then there exists $s \in R\backslash\{0\}$ such that $rs = 0$. As $Rs \neq (0)$ and $Ann(Rs) \neq (0)$, it follows that $Ann(Rs) = I$. From $rs = 0$, we obtain that $r \in I$. This shows that $Z(R) \subseteq I$ and so, $Z(R) = I$.

(2) $\Rightarrow$ (1) We claim that for any non-zero ideal J of R with $J \subseteq I$, $Ann(J) = I$. Let $r \in Ann(J)$. Then $rJ = (0)$. From $J \neq (0)$, it follows that $r \in Z(R) = I$. This shows that $Ann(J) \subseteq I$. From $I^2 = (0)$ and $J \subseteq I$, we get that $JI \subseteq I^2 = (0)$. This shows that $I \subseteq Ann(J)$. Therefore, $Ann(J) = I$ and so, in particular, $Ann(I) = I$. Hence, $I \in \mathbb{EA}(R)^*$. Let $A \in \mathbb{EA}(R)^*$. Then there exists a non-zero ideal B of R such that $Ann(A) = B$ and $Ann(B) = A$. This implies that $AB = (0)$ and so, $A \cup B \subseteq Z(R) = I$. Hence, $Ann(A) = Ann(B) = I$. From $Ann(B) = A$, we obtain that $A = I$. This proves that $\mathbb{EA}(R)^* = \{I\}$. $\square$

Let R be a ring such that it admits $\mathfrak{p} \in Spec(R)$ with $\mathfrak{p} = Rp$ is principal, $\mathfrak{p} \neq (0)$ but $\mathfrak{p}$ is nilpotent. Let $n \geq 2$ be least with the property that $\mathfrak{p}^n = (0)$. If $Z(R) = \mathfrak{p}$, then we prove in Proposition 2.10 that $\mathbb{EA}(R)^* = \{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\}$.

Proposition 2.10. *Let R be a ring. Let $\mathfrak{p} \in Spec(R)$ be such that $\mathfrak{p} = Rp$ is principal, $\mathfrak{p} \neq (0)$ but $\mathfrak{p}$ is nilpotent, and $Z(R) = \mathfrak{p}$. Let $n \geq 2$ be least with the property that $\mathfrak{p}^n = (0)$. Then $\mathbb{EA}(R)^* = \{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\}$. Moreover, $\mathbb{A}(R)^* = \mathbb{EA}(R)^*$ if and only if $\mathfrak{p} \in Max(R)$.*

Proof. Let $i \in \{1, \ldots, n-1\}$. As $\mathfrak{p} = Rp$, we get that $\mathfrak{p}^i = Rp^i$. From $p^n = 0$, it follows that $p^{n-i} \in Ann(Rp^i)$. Hence, $Rp^{n-i} \subseteq Ann(Rp^i)$. Let $r \in Ann(Rp^i)$. Then $rp^i = 0$. As $p^i \neq 0$, we obtain that $r \in Z(R) = Rp$. We claim that $r \in Rp^{n-i}$. This is clear if $r = 0$. Suppose that $r \neq 0$. It is possible to find $j \in \{1, \ldots, n-1\}$ such that $r \in Rp^j \backslash Rp^{j+1}$. Hence, there exists $s \in R\backslash Z(R)$ such that $r = p^j s$. From $rp^i = 0$, we get that $sp^{i+j} = 0$. As $s \in R\backslash Z(R)$, it follows that $p^{i+j} = 0$. Since n is least with the property that $p^n = 0$, we obtain that $i + j \geq n$ and so, $j \geq n - i$. Therefore, $r \in Rp^j \subseteq Rp^{n-i}$. This proves that $Ann(Rp^i) \subseteq Rp^{n-i}$ and so, we obtain that $Ann(Rp^i) = Rp^{n-i}$. As $n - i \in \{1, \ldots, n-1\}$, it follows that $Ann(Rp^{n-i}) = Rp^i$. Thus for any $i \in \{1, \ldots, n-1\}$, $Ann(\mathfrak{p}^i) = \mathfrak{p}^{n-i}$ and $Ann(\mathfrak{p}^{n-i}) = \mathfrak{p}^i$. This proves that $\{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\} \subseteq \mathbb{EA}(R)^*$. Let $A \in \mathbb{EA}(R)^*$. Then there exists a non-zero ideal B of R such that $Ann(A) = B$ and $Ann(B) = A$. From $AB = (0)$, we get that $A \cup B \subseteq Z(R) = Rp$. It is possible to find $j \in \{1, \ldots, n-1\}$ such that $B \subseteq Rp^j$ but $B \not\subseteq Rp^{j+1}$. Note that $Rp^{n-j} \subseteq Ann(B) = A$. Let $b \in B\backslash Rp^{j+1}$. As $B \subseteq Rp^j$, it follows that $b = sp^j$ for some $s \in R\backslash Z(R)$. From $AB = (0)$, we obtain that for any $a \in A$, $a(sp^j) = 0$ and so, $ap^j = 0$. This implies that $a \in Ann(Rp^j) = Rp^{n-j}$. This shows that $A \subseteq Rp^{n-j}$ and so, $A = Rp^{n-j}$. Hence, $\mathbb{EA}(R)^* \subseteq \{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\}$. Therefore, we obtain that $\mathbb{EA}(R)^* = \{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\}$.

We next verify the moreover part of this proposition. Assume that $\mathbb{A}(R)^* = \mathbb{EA}(R)^*$. As $\mathbb{EA}(R)^* = \{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\}$, we obtain that $\mathbb{A}(R)^*$ is finite. Hence, R satisfies descending chain condition (d.c.c.) on $\mathbb{A}(R)^*$. Therefore, it follows from [7, Theorem 1.1] that R is Artinian and so, we obtain from [4, Proposition 8.1] that $\mathfrak{p} \in Max(R)$. We also include a direct argument to show that $\mathfrak{p} \in Max(R)$. Let $\mathfrak{m} \in Max(R)$ be such that $\mathfrak{p} \subseteq \mathfrak{m}$. Let $m \in \mathfrak{m}$. If $pm = 0$, then $m \in Z(R) = \mathfrak{p}$. Suppose that $pm \neq 0$. Note that $Rpm \in \mathbb{A}(R)^*$. Therefore, $Rpm = \mathfrak{p}^i = Rp^i$ for some $i \in \{1, \ldots, n-1\}$. If $i = 1$, then $p = rpm$ for some $r \in R$. Hence, $p(1 - rm) = 0$. This implies that $1 - rm \in Z(R) = \mathfrak{p} \subseteq \mathfrak{m}$ and so, $1 \in \mathfrak{m}$. This is impossible and therefore, $i \geq 2$. From $p^n = 0$ and $Rpm = Rp^i$, it follows that $p^{n-i+1}m = 0$. As $i \geq 2$, it follows that $p^{n-i+1} \neq 0$. Hence, $m \in Z(R) = \mathfrak{p}$. This proves that $\mathfrak{m} \subseteq \mathfrak{p}$ and so, $\mathfrak{p} = \mathfrak{m} \in Max(R)$.

Conversely, assume that $\mathfrak{p} \in Max(R)$. Let $\mathfrak{P} \in Spec(R)$. Now, $\mathfrak{P} \supseteq (0) = \mathfrak{p}^n$. This implies that $\mathfrak{P} \supseteq \mathfrak{p}$. Since $\mathfrak{p} \in Max(R)$, it follows that $\mathfrak{P} = \mathfrak{p}$. Therefore, $Spec(R) = Max(R) = \{\mathfrak{p}\}$. Now, $\mathfrak{p} = Rp$ is principal and $n \geq 2$ is least with the property that $\mathfrak{p}^n = (0)$. Hence, we obtain from the proof of $(iii) \Rightarrow (i)$

of [4, Proposition 8.8] that $\{\mathfrak{p}^i = Rp^i \mid i \in \{1, \ldots, n-1\}\}$ is the set of all non-zero proper ideals of R. Therefore, it follows that $(R, \mathfrak{p})$ is a SPIR and so, $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^* = \{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\}$. □

We provide Example 2.12 to illustrate Theorem 2.9 and Proposition 2.10. We use Lemma 2.11 in the verification of Example 2.12.

Lemma 2.11. *Let p be a prime element of an integral domain T. Let $n \geq 2$. Let $R = \frac{T}{Tp^n}$. Let $\mathfrak{p} = \frac{Tp}{Tp^n}$. Then $\mathbb{EA}(R)^* = \{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\}$.*

Proof. By hypothesis, p is a prime element of T. Hence, $Tp \in Spec(T)$ and so, $\mathfrak{p} = \frac{Tp}{Tp^n} \in Spec(R = \frac{T}{Tp^n})$. It is clear that $\mathfrak{p} = R(p + Tp^n)$ is principal. Observe that $n \geq 2$ is least with the property that $\mathfrak{p}^n = (0 + Tp^n)$. Note that Tp^n is a Tp-primary ideal of T. Hence, the zero ideal $(0 + Tp^n)$ of R is a $\mathfrak{p}$-primary ideal of R. Therefore, we obtain from [4, Proposition 4.7] that $Z(R) = \mathfrak{p}$. Now, it follows from Proposition 2.10 that $\mathbb{EA}(R)^* = \{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\}$. □

Example 2.12. *Let $T = \mathbb{Z}[X]$ be the polynomial ring in one variable X over $\mathbb{Z}$. Let $n \geq 2$ and let $R = \frac{T}{TX^n}$. Let $\mathfrak{p} = \frac{TX}{TX^n}$. Then $\mathbb{EA}(R)^* = \{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\}$, $\mathbb{A}(R)^* = \{I \in \mathbb{I}(R)^* \mid I \subseteq \mathfrak{p}\}$, and $\mathbb{A}((R)^* \neq \mathbb{EA}(R)^*$.*

Proof. Note that T is an integral domain. Indeed, T is a unique factorization domain and X is a prime element of T. Therefore, we obtain from Lemma 2.11 that $\mathbb{EA}(R)^* = \{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\}$. Let $I \in \mathbb{I}(R)^*$ be such that $I \subseteq \mathfrak{p}$. Note that $\mathfrak{p}^{n-1} \neq (0 + TX^n)$ and $I\mathfrak{p}^{n-1} = (0 + TX^n)$. Hence, $I \in \mathbb{A}(R)^*$. Let $A \in \mathbb{A}(R)^*$. As any annihilating ideal of a ring is contained in its set of zero-divisors, we get that $A \subseteq Z(R) = \mathfrak{p}$. This proves that $\mathbb{A}(R)^* = \{I \in \mathbb{I}(R)^* \mid I \subseteq \mathfrak{p}\}$. Observe that $I = R(2X + TX^n) \subseteq \mathfrak{p}$, $I \neq (0 + TX^n)$, and $I \notin \{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\}$. Hence, $I \in \mathbb{A}(R)^* \backslash \mathbb{EA}(R)^*$. Therefore, $\mathbb{A}(R)^* \neq \mathbb{EA}(R)^*$. □

In Example 2.13, we illustrate that (2) $\Rightarrow$ (1) of Theorem 2.9 can fail to hold if the assumption that $I^2 = (0)$ is omitted.

Example 2.13. *Let R be as in Example 2.3. In the notation of Example 2.3, R is a local Artinian ring with unique maximal ideal $\mathfrak{m} = Rx + Ry + Rz$, $\mathfrak{m}^3 = (0)$, and $Z(R) = \mathfrak{m}$. It is already verified in the verification of Example 2.3 that R has 16 non-zero proper ideals and $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$.*

In Example 2.14, we illustrate that (2) $\Rightarrow$ (1) of Theorem 2.9 can fail to hold if the assumption that $Z(R) = I$ is omitted.

Example 2.14. *Let $T = K[X, Y]$ be the polynomial ring in two variables X, Y over a field K. Let $A = TX^2 + TXY$. Let $R = \frac{T}{A}$. Let $\mathfrak{p} = \frac{TX}{A}$. Then $\mathfrak{p} = R(X + A)$ is principal, $\mathfrak{p}^2 = (0 + A)$, and $|\mathbb{EA}(R)^*| = 2$.*

Proof. As X is a prime element of T, it follows that $TX \in Spec(T)$ and so, $\mathfrak{p} = \frac{TX}{A} \in Spec(\frac{T}{A} = R)$. It is clear that $\mathfrak{p} = R(X + A)$ is principal and from $X^2 \in A$, we obtain that $\mathfrak{p}^2 = (0 + A)$. Observe that $\frac{T}{TX+TY} \cong K$ as rings. From K is a field, it follows that $TX + TY \in Max(T)$ and so, $\mathfrak{m} = \frac{TX+TY}{A} \in Max(R)$. It is convenient to denote $X + A$ by x and $Y + A$ by y. Note that $\mathfrak{m} = Rx + Ry$. Observe that $A = TX \cap (TX^2 + TY)$. As $\sqrt{TX^2 + TY} = TX + TY \in Max(T)$, we obtain from [4, Proposition 4.2] that $TX^2 + TY$ is a $TX + TY$-primary ideal of T. Hence, $A = TX \cap (TX^2 + TY)$ is a minimal primary decomposition of A with TX is a TX-primary ideal of T and $TX^2 + TY$ is a $TX + TY$-primary ideal of T. Therefore, $(0 + A) = \frac{TX}{A} \cap \frac{TX^2+TY}{A}$ is a minimal primary decomposition of the zero ideal of R with $\frac{TX}{A}$ is a $\mathfrak{p}$-primary ideal of R and $\frac{TX^2+TY}{A}$ is a $\mathfrak{m}$-primary ideal of R. Hence, it follows from [4, Proposition 4.7] that $Z(R) = \mathfrak{p} \cup \mathfrak{m} = \mathfrak{m}$, since $\mathfrak{p} \subset \mathfrak{m}$. Note that $\mathfrak{m}\mathfrak{p} = (0 + A)$ and so, $\mathfrak{m} \subseteq Ann(\mathfrak{p})$. From $\mathfrak{m} \in Max(R)$ and $Ann(\mathfrak{p}) \neq R$, it follows that $Ann(\mathfrak{p}) = \mathfrak{m}$. From $\mathfrak{p}\mathfrak{m} = (0 + A)$, we get that

$\mathfrak{p} \subseteq Ann(\mathfrak{m})$. Let $r \in Ann(\mathfrak{m})$. Then $r\mathfrak{m} = (0 + A) \subset \mathfrak{p}$. As $\mathfrak{p} \in Spec(R)$ and $\mathfrak{m} \not\subseteq \mathfrak{p}$, we obtain that $r \in \mathfrak{p}$. This proves that $Ann(\mathfrak{m}) \subseteq \mathfrak{p}$ and so, $Ann(\mathfrak{m}) = \mathfrak{p}$. Thus the non-zero ideals $\mathfrak{m}, \mathfrak{p}$ of R are such that $Ann(\mathfrak{p}) = \mathfrak{m}$ and $Ann(\mathfrak{m}) = \mathfrak{p}$. Hence, $\mathfrak{p}, \mathfrak{m} \in \mathbb{EA}(R)^*$ and so, $|\mathbb{EA}(R)^*| \geq 2$. Let $C \in \mathbb{EA}(R)^*$. We claim that $C \in \{\mathfrak{p}, \mathfrak{m}\}$. As $C \in \mathbb{EA}(R)^*$, there exists a non-zero ideal D of R such that $Ann(C) = D$ and $Ann(D) = C$. Observe that $CD = (0 + A) \subset \mathfrak{p}$. From $\mathfrak{p} \in Spec(R)$, it follows that either $C \subseteq \mathfrak{p}$ or $D \subseteq \mathfrak{p}$. Suppose that $C \subseteq \mathfrak{p}$. Then $\mathfrak{m} = Ann(\mathfrak{p}) \subseteq Ann(C) = D$. From $D \neq R$, we get that $D = \mathfrak{m}$ and so, $C = Ann(D) = \mathfrak{p}$. If $D \subseteq \mathfrak{p}$, then it follows similarly that $C = \mathfrak{m}$. Therefore, $C \in \{\mathfrak{p}, \mathfrak{m}\}$. This proves that $\mathbb{EA}(R)^* = \{\mathfrak{p}, \mathfrak{m}\}$ and so, $|\mathbb{EA}(R)^*| = 2$. □

Let R be a ring such that $\mathbb{EA}(R)^* \neq \emptyset$. Inspired by Theorem 2.9 and Example 2.14, we try to characterize ideals I, J of a ring R such that $\mathbb{EA}(R)^* = \{I, J\}$. In Theorem 2.16, we are able to characterize ideals I, J of a reduced ring R such that $\mathbb{EA}(R)^* = \{I, J\}$. We use Proposition 2.15 in the proof of $(3) \Rightarrow (1)$ of Theorem 2.16.

Let R be a reduced ring which is not an integral domain. Suppose that $|\mathbb{EA}(R)^*| < \infty$. Let $A \subseteq Z(R)^*$ be such that $xy = 0$ for all distinct $x, y \in A$. Note that for each $x \in A$, $x \neq 0$, $Ann(x) \in \mathbb{A}(R)^*$, and it follows from $(2) \Rightarrow (1)$ of Lemma 2.1 that $Ann(x) \in \mathbb{EA}(R)^*$. Let $x, y \in A$ be such that $x \neq y$. Observe that $y \in Ann(x)$. Since R is reduced and $y \neq 0$, it follows that $y \notin Ann(y)$. Hence, $Ann(x) \neq Ann(y)$. From the assumption that $|\mathbb{EA}(R)^*| < \infty$, we get that A is finite. Hence, we obtain from $(4) \Rightarrow (3)$ of [6, Theorem 3.7] that $Min(R)$ is finite. This shows that if $|\mathbb{EA}(R)^*| < \infty$ for a reduced ring R, then $|Min(R)| < \infty$. Let R be a reduced ring with $|Min(R)| = n \geq 2$. Then we prove in Proposition 2.15 that $|\mathbb{EA}(R)^*| = 2^n - 2$.

Proposition 2.15. *Let R be a reduced ring which is not an integral domain. Let $|Min(R)| = n$ and let $Min(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Then $|\mathbb{EA}(R)^*| = 2^n - 2$. Moreover, $\mathbb{A}(R)^* = \mathbb{EA}(R)^*$ if and only if $\mathfrak{p}_i \in Max(R)$ for each $i \in \{1, 2, \ldots, n\}$.*

Proof. It is known that any prime ideal $\mathfrak{p}$ of a ring T contains a minimal prime ideal of T [14, Theorem 10]. Since R is reduced, $nil(R) = (0)$. We know from [4, Proposition 1.8] that $(0) = nil(R) = \bigcap_{\mathfrak{p} \in Spec(R)} \mathfrak{p}$. Since any prime ideal of R contains at least one minimal prime ideal of R, we obtain that $\bigcap_{\mathfrak{p} \in Min(R)} \mathfrak{p} = (0)$. As $Min(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, we obtain that $\bigcap_{i=1}^{n} \mathfrak{p}_i = (0)$. It is clear that $n \geq 2$, since R is not an integral domain. Note that distinct minimal prime ideals of a ring R are not comparable under the inclusion relation and hence, it follows from [4, Proposition 1.11(ii)] that for any proper non-empty subset A of $\{1, 2, \ldots, n\}$, $\bigcap_{i \in A} \mathfrak{p}_i \neq (0)$. Let $A \subset \{1, 2, \ldots, n\}$ with $A \neq \emptyset$. Let us denote $\bigcap_{i \in A} \mathfrak{p}_i$ by I_A. Observe that for any $A \subset \{1, 2, \ldots, n\}$ with $A \neq \emptyset$, $A^c \subset \{1, 2, \ldots, n\}$ and $A^c \neq \emptyset$, where $A^c = \{1, 2, \ldots, n\} \backslash A$ and it is easy to verify that $Ann(I_A) = I_{A^c}$. Hence, $I_A \in \mathbb{A}(R)^*$ and note that $Ann(Ann(I_A)) = I_A$. Therefore, we obtain from $(3) \Rightarrow (1)$ of Lemma 2.1 that $I_A \in \mathbb{EA}(R)^*$. This proves that $\{I_A \mid A \subset \{1, 2, \ldots, n\}, A \neq \emptyset\} \subseteq \mathbb{EA}(R)^*$. Let $I \in \mathbb{EA}(R)^*$. As $I \in \mathbb{A}(R)^*$, $Ir = (0)$ for some $r \in R \backslash \{0\}$. Since $\bigcap_{i=1}^{n} \mathfrak{p}_i = (0)$, $r \notin \mathfrak{p}_i$ for at least one $i \in \{1, 2, \ldots, n\}$. From $Ir = (0) \subset \mathfrak{p}_i \in Spec(R)$, we get that $I \subseteq \mathfrak{p}_i$. Since $I \neq (0)$, there exists at least one $j \in \{1, 2, \ldots, n\}$ such that $I \not\subseteq \mathfrak{p}_j$. Thus there exists $A \subset \{1, 2, \ldots, n\}, A \neq \emptyset$ such that $I \subseteq \mathfrak{p}_i$ for each $i \in A$ and $I \not\subseteq \mathfrak{p}_j$ for any $j \in \{1, 2, \ldots, n\} \backslash A$. From $IAnn(I) = (0) \subseteq \mathfrak{p}_j$ for any $j \in A^c$, we obtain that $Ann(I) \subseteq \mathfrak{p}_j$ for each $j \in A^c$. Thus $I \subseteq I_A$ and $Ann(I) \subseteq I_{A^c}$. From $Ann(I) \subseteq I_{A^c}$, it follows that $I_A = Ann(I_{A^c}) \subseteq Ann(Ann(I))$. Since $I \in \mathbb{EA}(R)^*$, we obtain from $(1) \Rightarrow (3)$ of Lemma 2.1 that $Ann(Ann(I)) = I$ and so, $I_A \subseteq I$. Hence, $I = I_A$ for some $A \subset \{1, 2, \ldots, n\}$ with $A \neq \emptyset$. This proves that $\mathbb{EA}(R)^* \subseteq \{I_A \mid A \subset \{1, 2, \ldots, n\}, A \neq \emptyset\}$ and so, $\mathbb{EA}(R)^* = \{I_A \mid A \subset \{1, 2, \ldots, n\}, A \neq \emptyset\}$. If A_1, A_2 are distinct non-empty proper subsets of $\{1, 2, \ldots, n\}$, then it is clear that $I_{A_1} \neq I_{A_2}$. Since $|\{A \subset \{1, 2, \ldots, n\}, A \neq \emptyset\}| = 2^n - 2$, it follows that $|\mathbb{EA}(R)^*| = 2^n - 2$.

We next verify the moreover part of this proposition. Assume that $\mathbb{A}(R)^* = \mathbb{EA}(R)^*$. Hence, $|\mathbb{A}(R)^*| = 2^n - 2 < \infty$. Therefore, R satisfies d.c.c. on $\mathbb{A}(R)^*$ and so, we obtain from [7, Theorem 1.1] that R is Artinian. We know from [4, Proposition 8.1] that $Spec(R) = Max(R)$. Therefore, $\mathfrak{p}_i \in Max(R)$

for each $i \in \{1, 2, \ldots, n\}$. We also include a direct argument to show that $\mathfrak{p}_i \in Max(R)$ for each $i \in \{1, 2, \ldots, n\}$. First, we show that $\mathfrak{p}_1 \in Max(R)$. Let $\mathfrak{m} \in Max(R)$ be such that $\mathfrak{p}_1 \subseteq \mathfrak{m}$. Since distinct minimal prime ideals of a ring are not comparable under the inclusion relation, it follows from [4, Proposition 1.11(*ii*)] that there exists $x \in (\bigcap_{j=2}^{n} \mathfrak{p}_j) \backslash \mathfrak{p}_1$. Let $m \in \mathfrak{m}$. Suppose that $xm \neq 0$. As $x \in Z(R)^*$, it follows that $xm \in Z(R)^*$, and so, $Rxm \in \mathbb{A}(R)^*$. It is shown in the previous paragraph that $\mathbb{EA}(R)^* = \{I_A \mid A \subset \{1, 2, \ldots, n\}, A \neq \emptyset\}$, where for a non-empty proper subset A of $\{1, 2, \ldots, n\}$, $I_A = \bigcap_{i \in A} \mathfrak{p}_i$. From the assumption $\mathbb{A}(R)^* = \mathbb{EA}(R)^*$, it follows that $Rxm = I_A$ for some non-empty proper subset A of $\{1, 2, \ldots, n\}$. From $xm \in (\bigcap_{j=2}^{n} \mathfrak{p}_j) \backslash \{0\}$ and $\bigcap_{i=1}^{n} \mathfrak{p}_i = (0)$, we get that $1 \notin A$. Hence, $A \subseteq \{1, 2, \ldots, n\} \backslash \{1\}$. It follows from the choice of x that $x \in I_A$. Therefore, $x \in Rxm$. This implies that $x(1 - rm) = 0$ for some $r \in R$. As $x \notin \mathfrak{p}_1$, we obtain that $1 - rm \in \mathfrak{p}_1 \subseteq \mathfrak{m}$. This implies that $1 \in \mathfrak{m}$ and this is impossible. Therefore, $xm = 0$. Hence, $m \in \mathfrak{p}_1$, since $\mathfrak{p}_1 \in Spec(R)$ and $x \notin \mathfrak{p}_1$. This proves that $\mathfrak{m} \subseteq \mathfrak{p}_1$. Therefore, $\mathfrak{p}_1 = \mathfrak{m} \in Max(R)$. Similarly, it can be shown that $\mathfrak{p}_j \in Max(R)$ for each $j \in \{2, \ldots, n\}$.

Conversely, assume that $\mathfrak{p}_i \in Max(R)$ for each $i \in \{1, 2, \ldots, n\}$. Note that $\mathfrak{p}_i + \mathfrak{p}_j = R$ for all distinct $i, j \in \{1, 2, \ldots, n\}$ and $\bigcap_{i=1}^{n} \mathfrak{p}_i = (0)$. Hence, we obtain from [4, Proposition 1.10(*ii*) and (*iii*)] that the mapping $f : R \to \frac{R}{\mathfrak{p}_1} \times \frac{R}{\mathfrak{p}_2} \times \cdots \times \frac{R}{\mathfrak{p}_n}$ defined by $f(r) = (r + \mathfrak{p}_1, r + \mathfrak{p}_2, \ldots, r + \mathfrak{p}_n)$ is an isomorphism of rings. Let $i \in \{1, 2, \ldots, n\}$. Since $\mathfrak{p}_i \in Max(R)$, it follows that $\frac{R}{\mathfrak{p}_i}$ is a field. Let us denote the ring $\frac{R}{\mathfrak{p}_1} \times \frac{R}{\mathfrak{p}_2} \times \cdots \times \frac{R}{\mathfrak{p}_n}$ by T. It follows from Example 2.8(1) that $\mathbb{I}(T)^* = \mathbb{A}(T)^* = \mathbb{EA}(T)^*$. Since $R \cong T$ as rings, we obtain that $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$. □

Theorem 2.16. *Let R be a reduced ring which is not an integral domain. The following statements are equivalent:*

(1) $\mathbb{EA}(R)^* = \{I, J\}$.

(2) $J = Ann(I)$ *and* $I, J \in Spec(R)$.

(3) $Min(R) = \{I, J\}$.

Proof. (1) ⇒ (2) As $I \in \mathbb{EA}(R)^*$, it follows that $I \in \mathbb{A}(R)^*$ and so, $Ann(I) \neq (0)$. It is clear that $Ann(I) \in \mathbb{A}(R)^*$. Observe that we obtain from (2) ⇒ (1) of Lemma 2.1 that $Ann(I) \in \mathbb{EA}(R)^* = \{I, J\}$. Since R is reduced, $I^2 \neq (0)$ and so, $Ann(I) \neq I$ and therefore, $Ann(I) = J$. Let $B \in \mathbb{A}(R)^*$. Then $Ann(B) \in \mathbb{A}(R)^*$. Therefore, we obtain from (2) ⇒ (1) of Lemma 2.1 that $Ann(B) \in \mathbb{EA}(R)^*$. From the hypothesis $\mathbb{EA}(R)^* = \{I, Ann(I)\}$, it follows that if $B \in \mathbb{A}(R)^*$, then either $Ann(B) = I$ or $Ann(B) = Ann(I)$. We next verify that $I, J = Ann(I) \in Spec(R)$. Let $a, b \in R$ be such that $ab \in I$. Then $abAnn(I) = (0)$. We know from (1) ⇒ (3) of Lemma 2.1 that $Ann(Ann(I)) = I$. If $aAnn(I) = (0)$, then $a \in Ann(Ann(I)) = I$. Similarly, if $bAnn(I) = (0)$, then $b \in I$. Hence, we can assume that $aAnn(I) \neq (0)$ and $bAnn(I) \neq (0)$. Now, $aAnn(I) \neq (0)$, $Ann(aAnn(I)) \neq (0)$, $bAnn(I) \neq (0)$, and $Ann(bAnn(I)) \neq (0)$. Therefore, $Ann(aAnn(I)), Ann(bAnn(I)) \in \mathbb{EA}(R)^* = \{I, Ann(I)\}$. Observe that $Ann(aAnn(I)) \neq Ann(bAnn(I))$. For if $Ann(aAnn(I)) = Ann(bAnn(I))$, then from $abAnn(I) = (0)$, it follows that $b^2Ann(I) = (0)$. Since R is reduced, we get that $bAnn(I) = (0)$ and this contradicts our assumption. Hence, $Ann(aAnn(I)) \neq Ann(bAnn(I))$. Therefore, either $Ann(aAnn(I)) = I$ or $Ann(bAnn(I)) = I$. If $Ann(aAnn(I)) = I$, then $b \in I$. If $Ann(bAnn(I)) = I$, then $a \in I$. This proves that $I \in Spec(R)$. Similarly, it can be shown that $Ann(I) \in Spec(R)$.

(2) ⇒ (3) We are assuming that the ideals I, J of R are such that $J = Ann(I)$, and $I, J \in Spec(R)$. By hypothesis, R is not an integral domain. Hence, $I \neq (0)$ and $J \neq (0)$. As R is reduced, $I^2 \neq (0)$, and so, it follows that $I \neq Ann(I) = J$. Note that $IJ = (0)$. If $r \in I \cap J$, then $r^2 \in IJ = (0)$ and since R is reduced, we obtain that $r = 0$ and so, $I \cap J = (0)$. It is convenient to denote I by $\mathfrak{p}_1$ and $Ann(I)$ by $\mathfrak{p}_2$. Note that $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$. We claim that $Min(R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. If $\mathfrak{p} \in Spec(R)$, then from $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$, it follows that $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some $i \in \{1, 2\}$. Since R is not an integral domain, we obtain that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are not comparable under the inclusion relation. The above arguments imply that $Min(R) = \{\mathfrak{p}_1 = I, \mathfrak{p}_2 = J\}$.

(3) $\Rightarrow$ (1) We are assuming that $Min(R) = \{I, J\}$. It now follows from the proof of Proposition 2.15 that $\mathbb{EA}(R)^* = \{I, J\}$. □

Let T be a UFD. If $\mathbb{A}(\frac{T}{Tp^2})^* = \mathbb{EA}(\frac{T}{Tp^2})^*$ for every prime element p of T, then we prove in Theorem 2.17 that T is a PID.

Theorem 2.17. *Let T be a UFD which is not a field. The following statements are equivalent:*

(1) *For any prime element p of T, $\mathbb{A}(\frac{T}{Tp^2})^* = \mathbb{EA}(\frac{T}{Tp^2})^*$.*

(2) *T is a PID.*

(3) *For any non-zero proper ideal I of T with $I \notin Max(T)$, $\mathbb{I}(\frac{T}{I})^* = \mathbb{A}(\frac{T}{I})^* = \mathbb{EA}(\frac{T}{I})^*$.*

Proof. (1) $\Rightarrow$ (2) Let p be a prime element of T. We claim that $Tp \in Max(T)$. For the sake of convenience, let us denote $\frac{T}{Tp^2}$ by R. Observe that $Tp \in Spec(T)$ and let us denote $\frac{Tp}{Tp^2}$ by $\mathfrak{p}$. Note that $\mathfrak{p} \in Spec(R)$, $\mathfrak{p} = R(p + Tp^2)$ is principal, $\mathfrak{p} \neq (0 + Tp^2)$ but $\mathfrak{p}^2 = (0 + Tp^2)$ and we know from the proof of Lemma 2.11 that $Z(R) = \mathfrak{p}$. We are assuming that that $\mathbb{A}(R)^* = \mathbb{EA}(R)^*$. Therefore, we obtain from the moreover part of Proposition 2.10 that $\mathfrak{p} \in Max(R)$. As $\mathfrak{p} = \frac{Tp}{Tp^2}$, we get that $Tp \in Max(T)$. This is true for any prime element p of T. Let $\mathfrak{P} \in Spec(T)\backslash\{(0)\}$. Since any non-zero non-unit of T can be expressed as the product of a finite number of prime elements of T, it follows that $\mathfrak{P} \supseteq Tp$ for some prime element p of T. As $Tp \in Max(T)$, we obtain that $\mathfrak{P} = Tp \in Max(T)$. This shows that $dim T = 1$. Hence, any prime ideal of T is principal. Therefore, we obtain from [14, Exercise 10, page 8] that any ideal of T is principal. Therefore, T is a PID.

(2) $\Rightarrow$ (3) This follows from Example 2.8(3).

(3) $\Rightarrow$ (1) This is clear, since for any prime element p of T, $Tp^2 \notin Max(T)$. □

Let T be a UFD which is not a field. If for every pair of non-associate prime elements p_1, p_2 of T, $\mathbb{EA}(\frac{T}{Tp_1p_2})^* = \mathbb{A}(\frac{T}{Tp_1p_2})^*$, then we prove in Theorem 2.18 that T is a PID. Suppose that T has a prime element p such that any prime element of T is an associate of p in T. Let a be any non-zero non-unit of T. Then $a = up^n$ for some $u \in U(T)$ and $n \geq 1$. Hence, $Ta \subseteq Tp$. Therefore, $Max(T) = Spec(T)\backslash\{(0)\} = \{Tp\}$. Let I be any non-zero proper ideal of T. Then $I \subseteq Tp$. From $\bigcap_{n=1}^{\infty} Tp^n = (0)$, we get that there exists $n \in \mathbb{N}$ such that $I \subseteq Tp^n$ but $I \nsubseteq Tp^{n+1}$. Let $x \in I\backslash Tp^{n+1}$. Then $x = up^n$ for some $u \in U(T)$. This implies that $p^n = u^{-1}x \in I$. This proves that $Tp^n \subseteq I$ and so, $I = Tp^n$. Thus any ideal of T is principal and so, T is a PID. Hence, in proving Theorem 2.18, we assume that T has at least two non-associate prime elements.

Theorem 2.18. *Let T be a UFD such that T has at least two non-associate prime elements. The following statements are equivalent:*

(1) *For any non-associate prime elements p_1, p_2 of T, $\mathbb{EA}(\frac{T}{Tp_1p_2})^* = \mathbb{A}(\frac{T}{Tp_1p_2})^*$.*

(2) *T is a PID.*

(3) *For any non-zero proper ideal I of T with $I \notin Max(T)$, $\mathbb{EA}(\frac{T}{I})^* = \mathbb{A}(\frac{T}{I})^*$.*

Proof. (1) $\Rightarrow$ (2) We are assuming that for any two non-associate prime elements p_1, p_2 of T, $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$ with $R = \frac{T}{Tp_1p_2}$. Let p be any prime element of T. By assumption, T has at least two non-associate prime elements. Let q be a prime element of T such p and q are non-associates in T. By (1), $\mathbb{EA}(\frac{T}{Tpq})^* = \mathbb{A}(\frac{T}{Tpq})^*$. Observe that $\frac{T}{Tpq}$ is a reduced ring with $Min(\frac{T}{Tpq}) = \{\frac{Tp}{Tpq}, \frac{Tq}{Tpq}\}$. From $\mathbb{EA}(\frac{T}{Tpq})^* = \mathbb{A}(\frac{T}{Tpq})^*$, we obtain from the moreover part of Proposition 2.15 that $\frac{Tp}{Tpq}, \frac{Tq}{Tpq} \in Max(R)$

and so, $Tp, Tq \in Max(T)$. Thus for any prime element p of T, $Tp \in Max(T)$. Now, it follows as in the proof of (1) $\Rightarrow$ (2) of Theorem 2.17 that T is a PID.

(2) $\Rightarrow$ (3) This follows from Example 2.8(3).

(3) $\Rightarrow$ (1) This is clear, since for any non-associate prime elements p_1, p_2 of T, $Tp_1p_2 \notin Max(T)$. □

Recall from [10, Exercise 16, page 111] that a ring T is *von Neumann regular* if given $a \in T$, there exists $b \in T$ such that $a = a^2b$. If a is a non-zero non-unit of a von Neumann regular ring T, then from $a = a^2b$, it follows that $e = ab = a^2b^2 = e^2$. Hence, e is an idempotent element of T with $e \notin \{0, 1\}$. It is known from $(a) \Leftrightarrow (d)$ of [10, Exercise 16, page 111] that a ring T is von Neumann regular if and only if $dimT = 0$ and T is reduced. An idempotent element e of R with $e \notin \{0, 1\}$ is referred to as a non-trivial idempotent element. Let R be a von Neumann regular ring which is not a field. We verify in Corollary 2.19 that $|\mathbb{EA}(R)^*| < \infty$ if and only if there exist $n \geq 2$ and fields $F_1, F_2, \ldots, F_n$ such that $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings.

Corollary 2.19. *Let R be a von Neumann regular ring which is not a field. The following statements are equivalent:*

(1) $|\mathbb{EA}(R)^*| < \infty$.

(2) *There exist $n \geq 2$ and fields $F_1, F_2, \ldots, F_n$ such that $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings.*

Proof. (1) $\Rightarrow$ (2) Since R is von Neumann regular, we obtain that $Spec(R) = Max(R) = Min(R)$. Since R is reduced, we get that $\bigcap_{\mathfrak{m} \in Max(R)} \mathfrak{m} = (0)$. From R is not a field, it follows that $|Max(R)| \geq 2$. We are assuming that $|\mathbb{EA}(R)^*| < \infty$. Hence, we obtain from the remark which appears just preceding the statement of Proposition 2.15 that $|Min(R) = Max(R)| < \infty$. Let $Max(R) = \{\mathfrak{m}_i | i \in \{1, 2, \ldots, n\}\}$. Now, it follows as in the proof of the moreover part of Proposition 2.15 that $R \cong \prod_{i=1}^{n} \frac{R}{\mathfrak{m}_i}$ as rings. Let $i \in \{1, 2, \ldots, n\}$ and let us denote the field $\frac{R}{\mathfrak{m}_i}$ by F_i. Thus there exist $n \geq 2$ and fields $F_1, F_2, \ldots, F_n$ such that $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings.

(2) $\Rightarrow$ (1) We are assuming that there exist $n \geq 2$ and fields $F_1, F_2, \ldots, F_n$ such that $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings. Let us denote the ring $\prod_{i=1}^{n} F_i$ by T. We know from Example 2.8(1) that $\mathbb{I}(T)^* = \mathbb{A}(T)^* = \mathbb{EA}(T)^*$. Therefore, $|\mathbb{EA}(T)^*| = |\mathbb{I}(T)^*| = 2^n - 2$. Hence, $|\mathbb{EA}(R)^*| = 2^n - 2 < \infty$. □

3. Some results on $\mathbb{EAG}(R)$

Let $G = (V, E)$ be a graph. G is said to be *connected* if for distinct vertices $a, b \in V$, there exists at least one path in G between a and b. Let $G = (V, E)$ be a connected graph. Let $a, b \in V$ with $a \neq b$. Recall from [5] that the *distance between a and b*, denoted by $d(a, b)$ is defined as the length of a shortest path in G between a and b. We define $d(a, a) = 0$ and define the *diameter of G*, denoted by $diam(G)$ as $diam(G) = sup\{d(a, b) \mid a, b \in V\}$. A simple graph G is said to be *complete* if every pair of distinct vertices of G are adjacent in G. Let $n \geq 1$. A complete graph with n vertices is denoted by K_n [5, Definition 1.1.11].

Let R be a ring such that $\mathbb{EA}(R)^* \neq \emptyset$. The aim of this section is to discuss some results on $\mathbb{EAG}(R)$. First, we prove some results regarding the connectedness of $\mathbb{EAG}(R)$.

Proposition 3.1. *Let R be a ring such that $\mathbb{EA}(R)^* \neq \emptyset$. Let $I, J \in \mathbb{EA}(R)^*$ be such that there is a path in $\mathbb{EAG}(R)$ between I and J. Then I and J are adjacent in $\mathbb{EAG}(R)$. In particular, if $\mathbb{EAG}(R)$ is connected and if $|\mathbb{EA}(R)^*| \geq 2$, then $diam(\mathbb{EAG}(R)) = 1$.*

Proof. Let $I, J \in \mathbb{EA}(R)^*$ be such that there is a path in $\mathbb{EAG}(R)$ between I and J. We claim that$I$ and J are adjacent in $\mathbb{EAG}(R)$. Suppose that I and J are not adjacent in $\mathbb{EAG}(R)$. Let $I_0 = I - I_1 - \cdots - I_n = J$ be a shortest path in $\mathbb{EAG}(R)$ between I and J. It is clear that $n \geq 2$. Note that for all $i \in \{0, 1, \ldots, n-1\}$, I_i and I_{i+1} are adjacent in $\mathbb{EAG}(R)$. Hence, $Ann(I_i) = I_{i+1}$ and $Ann(I_{i+1}) = I_i$. If $A \in \mathbb{EA}(R)^*$, then we know from (1) $\Rightarrow$ (3) of Lemma 2.1 that $A = Ann(Ann(A))$. Therefore, $I = I_0 = Ann(Ann(I_0)) = Ann(I_1) = I_2$. This is a contradiction. Therefore, I and J are adjacent in $\mathbb{EAG}(R)$.

We now verify the in particular statement of this proposition. Suppose that $\mathbb{EAG}(R)$ is connected and $|\mathbb{EA}(R)^*| \geq 2$. Let $I, J \in \mathbb{EA}(R)^*$ be such that $I \neq J$. Since $\mathbb{EAG}(R)$ is connected, there exists a path in $\mathbb{EAG}(R)$ between I and J. Hence, we obtain from what is shown in the previous paragraph that I and J are adjacent in $\mathbb{EAG}(R)$. Therefore, it follows that $diam(\mathbb{EAG}(R)) = 1$. □

Let $G = (V, E)$ be a graph. Recall from [9, page 21] that a maximal connected subgraph of G is called a *component* of G. Let R be a ring such that $\mathbb{EA}(R)^* \neq \emptyset$. We prove in Corollary 3.3 that each component of $\mathbb{EAG}(R)$ is a complete graph with at most two vertices. We use Lemma 3.2 in the proof of Corollary 3.3.

Lemma 3.2. *Let R be a ring such that $\mathbb{EA}(R)^* \neq \emptyset$. Let $I - J$ be an edge of $\mathbb{EAG}(R)$. Let $A \in \mathbb{EA}(R)^* \backslash \{I, J\}$. Then I and A are not adjacent in $\mathbb{EAG}(R)$ and J and A are not adjacent in $\mathbb{EAG}(R)$.*

Proof. Since $I - J$ is an edge of $\mathbb{EAG}(R)$, we obtain that $Ann(I) = J$ and $Ann(J) = I$. As $A \in \mathbb{EA}(R)^*$, we know from (1) $\Rightarrow$ (3) of Lemma 2.1 that $Ann(Ann(A)) = A$. As $A \notin \{I, J\}$, it follows that $Ann(A) \notin \{I, J\}$. Therefore, we obtain that I and A are not adjacent in $\mathbb{EAG}(R)$ and J and A are not adjacent in $\mathbb{EAG}(R)$. □

Corollary 3.3. *Let R be a ring such that $\mathbb{EA}(R)^* \neq \emptyset$. If g is any component of $\mathbb{EAG}(R)$, then g is a complete graph with at most two vertices. In particular, if $\mathbb{EAG}(R)$ is connected, then $\mathbb{EAG}(R)$ is a complete graph with at most two vertices.*

Proof. Let g be any component of $\mathbb{EAG}(R)$. Suppose that $|V(g)| \geq 2$. Let $I, J \in V(g)$ with $I \neq J$. Then there exists a path in $\mathbb{EAG}(R)$ between I and J. Hence, we obtain from Proposition 3.1 that I and J are adjacent in $\mathbb{EAG}(R)$ and so, they are adjacent in g. Let $A \in \mathbb{EA}(R)^* \backslash \{I, J\}$. We know from Lemma 3.2 that I and A are not adjacent in $\mathbb{EAG}(R)$ and J and A are not adjacent in $\mathbb{EAG}(R)$. Therefore, $A \notin V(g)$ and so, $V(g) = \{I, J\}$. This proves that any component g of $\mathbb{EAG}(R)$ is a complete graph with at most two vertices.

We next verify the in particular statement of this corollary. Suppose that $\mathbb{EAG}(R)$ is connected. Then $\mathbb{EAG}(R)$ is the only component of $\mathbb{EAG}(R)$ and so, $\mathbb{EAG}(R)$ is a complete graph with at most two vertices. □

Next, we assume that $(R, \mathfrak{m})$ is a SPIR and try to determine the structure of $\mathbb{EAG}(R)$.

Proposition 3.4. *Let R be a ring. Let $\mathfrak{p} \in Spec(R)$ be such that $\mathfrak{p} \neq (0)$ but $\mathfrak{p}^2 = (0)$. If $\mathfrak{p} = Z(R)$, then $\mathbb{EAG}(R)$ is a graph with $V(\mathbb{EAG}(R)) = \{\mathfrak{p}\}$. In particular, if $(R, \mathfrak{m})$ is a SPIR with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$, then $\mathbb{EAG}(R)$ is a graph with $V(\mathbb{EAG}(R)) = \{\mathfrak{m}\}$.*

Proof. We know from (2) $\Rightarrow$ (1) of Theorem 2.9 that $\mathbb{EA}(R)^* = \{\mathfrak{p}\}$. As $V(\mathbb{EAG}(R)) = \mathbb{EA}(R)^*$, we obtain that $V(\mathbb{EAG}(R)) = \{\mathfrak{p}\}$.

We next verify the in particular statement of this proposition. Let $(R, \mathfrak{m})$ be a SPIR with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$. As $Z(R) = \mathfrak{m}$, it follows that $V(\mathbb{EAG}(R)) = \{\mathfrak{m}\}$. □

Proposition 3.5. *Let R be a ring. Let $\mathfrak{p} \in Spec(R)$ be such that $\mathfrak{p} = Rp$ is principal, $n \geq 3$ is least with the property that $\mathfrak{p}^n = (0)$, and $Z(R) = \mathfrak{p}$. Then the following statements hold:*

(1) *If n is odd, then $\mathbb{EAG}(R)$ has exactly $[\frac{n}{2}]$ components and each component is a complete graph with two vertices.*

(2) *If n is even, then $\mathbb{EAG}(R)$ has exactly $\frac{n}{2}$ components $g_1, g_2, \ldots, g_{\frac{n}{2}}$ such that g_j is a complete graph with two vertices for each $j \in \{1, \ldots, \frac{n}{2} - 1\}$ and $g_{\frac{n}{2}}$ is a complete graph on a single vertex.*

In particular, if $(R, \mathfrak{m})$ is a SPIR and $n \geq 3$ is least with the property that $\mathfrak{m}^n = (0)$, then the statements (1) and (2) hold for $\mathbb{EAG}(R)$.

Proof. Note that $V(\mathbb{EAG}(R)) = \mathbb{EA}(R)^*$ and we know from Proposition 2.10 that $\mathbb{EA}(R)^* = \{\mathfrak{p}^i \mid i \in \{1, 2, \ldots, n-1\}\}$. We know from the proof of Proposition 2.10 that for each $i \in \{1, 2, \ldots, n-1\}$, $Ann(\mathfrak{p}^i) = \mathfrak{p}^{n-i}$ and $Ann(\mathfrak{p}^{n-i}) = \mathfrak{p}^i$. Suppose that $n \geq 4$. Let $j \in \{1, \ldots, [\frac{n}{2}] - 1\}$. As $2j < n$ and n is least with the property that $\mathfrak{p}^n = (0)$, it follows that $\mathfrak{p}^j \neq \mathfrak{p}^{n-j}$. Observe that $\mathfrak{p}^j$ and $\mathfrak{p}^{n-j}$ are adjacent in $\mathbb{EAG}(R)$. Let g_j be the subgraph of $\mathbb{EAG}(R)$ induced by $\{\mathfrak{p}^j, \mathfrak{p}^{n-j}\}$. Then g_j is a complete graph with two vertices and it follows from Corollary 3.3 that g_j is necessarily a component of $\mathbb{EAG}(R)$.

(1) Assume that n is odd. If $n = 3$, then $V(\mathbb{EAG}(R)) = \{\mathfrak{p}, \mathfrak{p}^2\}$ and $\mathbb{EAG}(R)$ is a complete graph with two vertices. Let $n \geq 5$. Note that $\mathfrak{p}^{\frac{n-1}{2}} \neq \mathfrak{p}^{\frac{n+1}{2}}$. Observe that $Ann(\mathfrak{p}^{\frac{n-1}{2}}) = \mathfrak{p}^{\frac{n+1}{2}}$ and $Ann(\mathfrak{p}^{\frac{n+1}{2}}) = \mathfrak{p}^{\frac{n-1}{2}}$. Hence, $\mathfrak{p}^{\frac{n-1}{2}}$ and $\mathfrak{p}^{\frac{n+1}{2}}$ are adjacent in $\mathbb{EAG}(R)$. Let $g_{[\frac{n}{2}]}$ be the subgraph of $\mathbb{EAG}(R)$ induced by $\{\mathfrak{p}^{\frac{n-1}{2}}, \mathfrak{p}^{\frac{n+1}{2}}\}$. Note that $g_{[\frac{n}{2}]}$ is a complete graph with two vertices and it follows from Corollary 3.3 that $g_{[\frac{n}{2}]}$ is necessarily a component of $\mathbb{EAG}(R)$. Observe that $V(\mathbb{EAG}(R)) = \mathbb{EA}(R)^* = \bigcup_{j=1}^{[\frac{n}{2}]} \{\mathfrak{p}^j, \mathfrak{p}^{n-j}\} = \bigcup_{j=1}^{[\frac{n}{2}]} V(g_j)$. It is clear that for any distinct $j_1, j_2 \in \{1, 2, \ldots, [\frac{n}{2}]\}$, $V(g_{j_1}) \cap V(g_{j_2}) = \emptyset$. From the above arguments, it is clear that $\mathbb{EAG}(R)$ has exactly $[\frac{n}{2}]$ components and each component is a complete graph with two vertices.

(2) Assume that n is even. It is clear that $n \geq 4$. Observe that $V(\mathbb{EAG}(R)) = \mathbb{EA}(R)^* = (\bigcup_{j=1}^{\frac{n}{2}-1} \{\mathfrak{p}^j, \mathfrak{p}^{n-j}\}) \cup \{\mathfrak{p}^{\frac{n}{2}}\} = (\bigcup_{j=1}^{\frac{n}{2}-1} V(g_j)) \cup \{\mathfrak{p}^{\frac{n}{2}}\}$. Let $g_{\frac{n}{2}}$ be the subgraph of $\mathbb{EAG}(R)$ induced by $\{\mathfrak{p}^{\frac{n}{2}}\}$. It is clear that for all distinct $j_1, j_2 \in \{1, 2, \ldots, \frac{n}{2}\}$, $V(g_{j_1}) \cap V(g_{j_2}) = \emptyset$. From the above given arguments, it follows that $\mathbb{EAG}(R)$ has exactly $\frac{n}{2}$ components $g_1, g_2, \ldots, g_{\frac{n}{2}}$ such that g_j is a complete graph with two vertices for each $j \in \{1, \ldots, \frac{n}{2} - 1\}$ and $g_{\frac{n}{2}}$ is a complete graph on a single vertex.

We next verify the in particular statement of this proposition. Now, $(R, \mathfrak{m})$ is a SPIR and $n \geq 3$ is least with the property that $\mathfrak{m}^n = (0)$. Let $m \in \mathfrak{m}$ be such that $\mathfrak{m} = Rm$. Observe that $Z(R) = \mathfrak{m}$. Hence, the hypotheses of this proposition are satisfied and therefore, the statements (1) and (2) hold for $\mathbb{EAG}(R)$. □

Remark 3.6. *Let R be a ring. Let $\mathfrak{p} \in Spec(R)$ be such that $\mathfrak{p} = Rp$ is principal, $\mathfrak{p} \neq (0)$ but $\mathfrak{p}$ is nilpotent. Let $n \geq 2$ be least with the property that $\mathfrak{p}^n = (0)$. Then the following hold:*

(1) *$(R, \mathfrak{p})$ is a SPIR if and only if $\mathfrak{p} \in Max(R)$.*

(2) *Suppose that $Z(R) = \mathfrak{p}$. Then $\mathbb{EAG}(R)$ is connected if and only if $n \in \{2, 3\}$.*

Proof. (1) Assume that $(R, \mathfrak{p})$ is a SPIR. Then $\mathfrak{p} \in Max(R)$ and indeed, it is the only prime ideal of R. Conversely, assume that $\mathfrak{p} \in Max(R)$. Then it is shown in the proof of the moreover part of Proposition 2.10 that $(R, \mathfrak{p})$ is a SPIR.

(2) If $n \geq 4$, then $[\frac{n}{2}] \geq 2$. We know from Proposition 3.5 that $\mathbb{EAG}(R)$ has exactly $[\frac{n}{2}]$ components. Thus if $\mathbb{EAG}(R)$ is connected, then $n \in \{2, 3\}$. Assume that $n \in \{2, 3\}$. If $n = 2$, then we know from Proposition 3.4 that $V(\mathbb{EAG}(R)) = \{\mathfrak{p}\}$. If $n = 3$, then we know from the proof of Proposition 3.5(1) that $\mathbb{EAG}(R)$ is a complete graph with two vertices. Therefore, if $n \in \{2, 3\}$, then $\mathbb{EAG}(R)$ is connected. □

Let R be a ring. Let $\mathfrak{p} \in Spec(R)$ be such that $\mathfrak{p}$ satisfies the hypotheses mentioned in the statement of Remark 3.6. If $Z(R) = \mathfrak{p}$, then in Theorem 3.7, we characterize R such that $\mathbb{EAG}(R) = \mathbb{AG}(R)$.

Theorem 3.7. *Let R be a ring. Let $\mathfrak{p} \in Spec(R)$ be such that $\mathfrak{p} = Rp$ is principal. Let $n \geq 2$ be least with the property that $\mathfrak{p}^n = (0)$. If $Z(R) = \mathfrak{p}$, then the following statements are equivalent:*

(1) $\mathbb{EAG}(R) = \mathbb{AG}(R)$.

(2) $(R, \mathfrak{p})$ *is a SPIR and* $n \in \{2, 3\}$.

Proof. (1) $\Rightarrow$ (2) From the assumption $\mathbb{EAG}(R) = \mathbb{AG}(R)$, we get that $\mathbb{EA}(R)^* = V(\mathbb{EAG}(R)) = V(\mathbb{AG}(R)) = \mathbb{A}(R)^*$. We know from Proposition 2.10 that $\mathbb{EA}(R)^* = \{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\}$. Hence, $\mathbb{A}(R)^* = \{\mathfrak{p}^i \mid i \in \{1, \ldots, n-1\}\}$. We first verify that $(R, \mathfrak{p})$ is a SPIR. In view of the statement (1) of Remark 3.6, it is enough to prove that $\mathfrak{p} \in Max(R)$. As $\mathbb{A}(R)^* = \mathbb{EA}(R)^*$, we obtain from the moreover part of Proposition 2.10 that $\mathfrak{p} \in Max(R)$. Therefore, $(R, \mathfrak{p})$ is a SPIR. It is known that $\mathbb{AG}(R)$ is connected and $diam(\mathbb{AG}(R)) \leq 3$ [7, Theorem 2.1]. Therefore, from $\mathbb{EAG}(R) = \mathbb{AG}(R)$, we get that $\mathbb{EAG}(R)$ is connected. Hence, we obtain from Remark 3.6(2) that $n \in \{2, 3\}$.

(2) $\Rightarrow$ (1) We are assuming that $(R, \mathfrak{p})$ is a SPIR and $n \in \{2, 3\}$. If $n = 2$, then we know from the proof of Lemma 2.4 that $\mathbb{EA}(R)^* = \mathbb{A}(R)^* = \{\mathfrak{p}\}$. Hence, $\mathbb{EAG}(R) = \mathbb{AG}(R)$ in this case. If $n = 3$, then we know from the proof of Lemma 2.4 that $\mathbb{EA}(R)^* = \mathbb{A}(R)^* = \{\mathfrak{p}, \mathfrak{p}^2\}$. Thus $V(\mathbb{EAG}(R)) = V(\mathbb{AG}(R)) = \{\mathfrak{p}, \mathfrak{p}^2\}$. We know from the proof of the statement (1) of Proposition 3.5 that $\mathbb{EAG}(R)$ is a complete graph with vertex set $\{\mathfrak{p}, \mathfrak{p}^2\}$. For any ring T, $\mathbb{EAG}(T)$ is a subgraph of $\mathbb{AG}(T)$. Hence, $\mathbb{AG}(R)$ is a complete graph with vertex set $\{\mathfrak{p}, \mathfrak{p}^2\}$. Therefore, $\mathbb{EAG}(R) = \mathbb{AG}(R)$ in this case also. Therefore, if $(R, \mathfrak{p})$ is a SPIR and $n \in \{2, 3\}$, then $\mathbb{EAG}(R) = \mathbb{AG}(R)$. □

Let R be a ring. Let $\mathfrak{p} \in Spec(R)$ be such that $\mathfrak{p} = Rp$ is principal, $\mathfrak{p}^2 \neq (0)$ but $\mathfrak{p}^3 = (0)$, and $Z(R) = \mathfrak{p}$. Then we know from the proof of Proposition 3.5(1) that $\mathbb{EAG}(R)$ is a complete graph with $V(\mathbb{EAG}(R)) = \{\mathfrak{p}, \mathfrak{p}^2\}$. We provide Example 3.8 to illustrate that in the above result, if the hypothesis that $\mathfrak{p}$ is principal is omitted, then the conclusion can fail to hold.

Example 3.8. *Let R be the ring considered by D.D. Anderson and M. Naseer in [3, page 501]. Then $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$, $\mathbb{AG}(R)$ is connected with $diam(\mathbb{AG}(R)) = 2$, and $\mathbb{EAG}(R)$ has exactly eight components and each component is a complete graph with two vertices.*

Proof. The ring R is also considered in Example 2.3 of this article. In the notation of Example 2.3, $R = \frac{T}{I}$, where $T = \mathbb{Z}_4[X, Y, Z]$, the polynomial ring in three variables X, Y, Z over $\mathbb{Z}_4$, and I is the ideal of T generated by $\{X^2 - 2, Y^2 - 2, Z^2, XY, XZ, YZ - 2, 2X, 2Y, 2Z\}$. Let us denote $X + I$ by x, $Y + I$ by y, and $Z + I$ by z. Observe that R is a local Artinian ring with $\mathfrak{m} = Rx + Ry + Rz$ as its unique maximal ideal, $\mathfrak{m}^2 = \{0 + I, 2 + I\}$, $\mathfrak{m}^3 = (0 + I)$, and $|R| = 32$. It is already noted in Example 2.3 that $\mathbb{I}(R)^* = \{\mathfrak{m}, \mathfrak{m}^2, Rx, Ry, Rz, R(x + y), R(y + z), R(x + z), R(x + y + z), Rx + Ry, Ry + Rz, Rx + Rz, Rx + R(y + z), Ry + R(z + x), Rz + R(x + y), R(x + y) + R(y + z)\}$. It is verified in Example 2.3 that $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$. Let $A, B \in \mathbb{A}(R)^*$ with $A \neq B$. Suppose that A and B are not adjacent in $\mathbb{AG}(R)$. From $\mathfrak{m}^3 = (0)$, we obtain that $A - \mathfrak{m}^2 - B$ is a path of length two between A and B in $\mathbb{AG}(R)$. This proves that $diam(\mathbb{AG}(R)) \leq 2$. Observe that $(Ry)(Rz) \neq (0)$ and so, Ry and Rz are not adjacent in $\mathbb{AG}(R)$. This shows that $diam(\mathbb{AG}(R)) \geq 2$ and so, $diam(\mathbb{AG}(R)) = 2$. We next verify that $\mathbb{EAG}(R)$ has exactly eight components and each component is a complete graph with two vertices. Let $A_1 = \{A_{11} = \mathfrak{m}, A_{12} = \mathfrak{m}^2\}$, $A_2 = \{A_{21} = Rx, A_{22} = Ry + Rz\}$, $A_3 = \{A_{31} = Ry, A_{32} = Rx + R(y + z)\}$, $A_4 = \{A_{41} = Rz, A_{42} = Rx + Rz\}$, $A_5 = \{A_{51} = R(x + y), A_{52} = R(y + z) + R(z + x)\}$, $A_6 = \{A_{61} = R(y + z), A_{62} = Rx + Ry\}$, $A_7 = \{A_{71} = R(z + x), A_{72} = Rz + R(x + y)\}$, and $A_8 = \{A_{81} = R(x + y + z), A_{82} = Ry + R(x + z)\}$. Let g_i be the subgraph of $\mathbb{EAG}(R)$ induced by A_i for each $i \in \{1, 2, \ldots, 8\}$. We know from the proof of Example 2.3, that $Ann(A_{i1}) = A_{i2}$ and $Ann(A_{i2}) = A_{i1}$ for each $i \in \{1, 2, 3, \ldots, 8\}$. It is clear that g_i is a complete graph with two vertices for each $i \in \{1, 2, 3, \ldots, 8\}$ and it follows from Corollary 3.3 that each g_i is a component of $\mathbb{EAG}(R)$. As

$A_i = V(g_i)$ for each $i \in \{1, 2, 3, \ldots, 8\}$, $\mathbb{EA}(R)^* = \bigcup_{i=1}^{8} A_i$, $A_i \cap A_j = \emptyset$ for all distinct $i, j \in \{1, 2, 3, \ldots, 8\}$, it follows that $\{g_i \mid i \in \{1, 2, 3, \ldots, 8\}\}$ is the set of all components of $\mathbb{EAG}(R)$. □

Let R be a reduced ring which is not an integral domain. If T is a ring which is not an integral domain, then it is already noted in the paragraph which appears just preceding the statement of Corollary 2.2 that $\mathbb{EA}(T)^* \neq \emptyset$. Hence, $\mathbb{EA}(R)^* \neq \emptyset$. In Corollary 3.10, we answer when $\mathbb{EAG}(R)$ is connected. In Corollary 3.11, we prove that $\mathbb{EAG}(R) = \mathbb{AG}(R)$ if and only if $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. We use Lemma 3.9 in the proof of Corollary 3.10.

Lemma 3.9. *Let R be a reduced ring which is not an integral domain. Then $\mathbb{EA}(R)^* \neq \emptyset$ and any component g of $\mathbb{EAG}(R)$ is a K_2.*

Proof. If R is not an integral domain (whether it is reduced or not), then it is already noted that $\mathbb{EA}(R)^* \neq \emptyset$.

Let g be a component of $\mathbb{EAG}(R)$. Let $I \in V(g)$. It follows from (1) $\Rightarrow$ (3) of Lemma 2.1 that $Ann(Ann(I)) = I$. Since R is reduced, $I^2 \neq (0)$ and so, $I \neq Ann(I)$. With $J = Ann(I)$, it follows that $Ann(J) = I$. Hence, I and J are adjacent in $\mathbb{EAG}(R)$. Therefore, $J \in V(g)$. Also, I and J are adjacent in g. It follows from Corollary 3.3 that g is a complete graph with two vertices. □

Corollary 3.10. *Let R be a reduced ring which is not an integral domain. The following statements are equivalent:*

(1) $\mathbb{EAG}(R)$ *is connected.*

(2) $|Min(R)| = 2$.

Proof. (1) $\Rightarrow$ (2) Assume that $\mathbb{EAG}(R)$ is connected. We know from Lemma 3.9 that $\mathbb{EAG}(R)$ is a complete graph with two vertices. Hence, $|\mathbb{EAG}(R)| = 2$. As $V(\mathbb{EAG}(R)) = \mathbb{EA}(R)^*$, we get that $|\mathbb{EA}(R)^*| = 2$. Hence, it follows from (1) $\Rightarrow$ (3) of Theorem 2.16 that $|Min(R)| = 2$.

(2) $\Rightarrow$ (1) Let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. We know from the proof of Proposition 2.15 that $\mathbb{EA}(R)^* = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$, $Ann(\mathfrak{p}_1) = \mathfrak{p}_2$, and $Ann(\mathfrak{p}_2) = \mathfrak{p}_1$. Therefore, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are adjacent in $\mathbb{EAG}(R)$. This shows that $\mathbb{EAG}(R)$ is a complete graph with two vertices and so, we obtain that $\mathbb{EAG}(R)$ is connected. □

Corollary 3.11. *Let R be a reduced ring which is not an integral domain. The following statements are equivalent:*

(1) $\mathbb{EAG}(R) = \mathbb{AG}(R)$.

(2) $R \cong F_1 \times F_2$ *as rings, where F_i is a field for each $i \in \{1, 2\}$.*

Proof. (1) $\Rightarrow$ (2) We are assuming that $\mathbb{EAG}(R) = \mathbb{AG}(R)$. We know from [7, Theorem 2.1] that $\mathbb{AG}(R)$ is connected. Therefore, $\mathbb{EAG}(R)$ is connected. Hence, we obtain from the proof of (1) $\Rightarrow$ (2) of Corollary 3.10 that $|Min(R)| = 2$ and $\mathbb{EA}(R)^* = Min(R)$. Let $Min(R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Now, $\mathbb{A}(R)^* = V(\mathbb{AG}(R)) = V(\mathbb{EAG}(R)) = \mathbb{EA}(R)^*$. In such a case, we obtain from the proof of moreover part of Proposition 2.15 that $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$.

(2) $\Rightarrow$ (1) Assume that $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. Let us denote the ring $F_1 \times F_2$ by T. We know from Example 2.8(1) that $\mathbb{I}(T)^* = \mathbb{A}(T)^* = \mathbb{EA}(T)^*$. Note that $\mathbb{I}(T)^* = \{\mathfrak{m}_1 = (0) \times F_2, \mathfrak{m}_2 = F_1 \times (0)\}$. Observe that $Min(T) = \{\mathfrak{m}_i \mid i \in \{1, 2\}\}$. Now, it follows from the proof of (2) $\Rightarrow$ (1) of Corollary 3.10 that $\mathbb{EAG}(T)$ is a complete graph with two vertices. Since $\mathbb{EAG}(T)$ is a subgraph of $\mathbb{AG}(T)$, we get that $\mathbb{EAG}(T) = \mathbb{AG}(T)$ is a complete graph with two vertices. Since $R \cong T$ as rings, we obtain that $\mathbb{EAG}(R) = \mathbb{AG}(R)$. □

Corollary 3.12. *Let R be a reduced ring with $|Min(R)| = n \geq 2$. Then $\mathbb{EAG}(R)$ has exactly $2^{n-1} - 1$ components and each component is a K_2.*

Proof. We know from Proposition 2.15 that $|V(\mathbb{EAG}(R)) = \mathbb{EA}(R)^*| = 2^n - 2$. Let t be the number of components of $\mathbb{EAG}(R)$. Let $\{g_i \mid i \in \{1, \ldots, t\}\}$ be the set of all components of $\mathbb{EAG}(R)$. We know from Lemma 3.9 that g_i is a K_2 for each $i \in \{1, \ldots, t\}$. Now, $\mathbb{EA}(R)^* = \bigcup_{i=1}^{t} V(g_i)$, $|V(g_i)| = 2$ for each $i \in \{1, \ldots, t\}$, $V(g_i) \cap V(g_j) = \emptyset$ for all distinct $i, j \in \{1, \ldots, t\}$. Therefore, $2^n - 2 = |\mathbb{EA}(R)^*| = 2t$ and so, $t = 2^{n-1} - 1$. This proves that $\mathbb{EAG}(R)$ has exactly $2^{n-1} - 1$ components and each component is a K_2. □

Corollary 3.13. *Let $n \geq 2$ and let R_i be an integral domain for each $i \in \{1, 2, \ldots, n\}$. Let $R = R_1 \times R_2 \times \cdots \times R_n$. Then $\mathbb{EAG}(R)$ has exactly $2^{n-1} - 1$ components and each component is a K_2.*

Proof. Note that R is a reduced ring and $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ is the set of all minimal prime ideals of R, where for each $i \in \{1, 2, \ldots, n\}$, $\mathfrak{p}_i = I_1 \times \cdots \times I_i \times \cdots \times I_n$ with $I_i = (0)$ and $I_j = R_j$ for all $j \in \{1, 2, \ldots, n\} \backslash \{i\}$. Thus $|Min(R)| = n$ and so, we obtain from Corollary 3.12 that $\mathbb{EAG}(R)$ has exactly $2^{n-1} - 1$ components and each component is a complete graph with two vertices. □

Let R be a ring which is not reduced. We are not able to determine $I, J \in \mathbb{EA}(R)^*$ such that $\mathbb{EA}(R)^* = \{I, J\}$. However, as a consequence of Corollary 3.3 and [7, Theorem 2.7], we characterize in Theorem 3.14 rings R with $\mathbb{EA}(R)^* \neq \emptyset$ such that $\mathbb{EAG}(R) = \mathbb{AG}(R)$.

Theorem 3.14. *Let R be a ring such that $\mathbb{EA}(R)^* \neq \emptyset$. The following statements are equivalent:*

(1) $\mathbb{EAG}(R) = \mathbb{AG}(R)$.

(2) *Either $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$ or $(R, \mathfrak{m})$ is a SPIR satisfying the property that if $n \in \mathbb{N}$ is least such that $\mathfrak{m}^n = (0)$, then $n \in \{2, 3\}$.*

Proof. (1) ⇒ (2) We are assuming that $\mathbb{EAG}(R) = \mathbb{AG}(R)$. We know from [7, Theorem 2.1] that $\mathbb{AG}(R)$ is connected. Therefore, $\mathbb{EAG}(R)$ is connected. Hence, we obtain from Corollary 3.3 that $\mathbb{EAG}(R)$ is complete. Therefore, $\mathbb{AG}(R)$ is complete and so, it follows from [7, Theorem 2.7] that one of the following holds: (a) $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$; (b) $Z(R)$ is an ideal of R with $(Z(R))^2 = (0)$; (c) $(R, \mathfrak{m})$ is a SPIR with $\mathfrak{m}^3 = (0)$ but $\mathfrak{m}^2 \neq (0)$. Assume that (b) holds. As $Z(R)$ is an ideal of R, $Z(R)$ is necessarily a prime ideal of R. Let us denote $Z(R)$ by $\mathfrak{p}$. Now, $\mathfrak{p} \neq (0)$, $\mathfrak{p}^2 = (0)$, and $\mathfrak{p} = Z(R)$. Therefore, we obtain from (2) ⇒ (1) of Theorem 2.9 that $\mathbb{EA}(R)^* = \{\mathfrak{p}\}$. From $\mathbb{EAG}(R) = \mathbb{AG}(R)$, it follows that $\mathbb{A}(R)^* = \mathbb{EA}(R)^* = \{\mathfrak{p}\}$. Hence, we obtain from [7, Corollary 2.9(a)] that $(R, \mathfrak{m})$ (with $\mathfrak{m} = \mathfrak{p}$) is a SPIR with $\mathfrak{m}^2 = (0)$. Therefore, we obtain that either $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$ or $(R, \mathfrak{m})$ is a SPIR satisfying the property that if $n \in \mathbb{N}$ is least such that $\mathfrak{m}^n = (0)$, then $n \in \{2, 3\}$.

(2) ⇒ (1) Suppose that $R \cong F_1 \times F_2$ as rings. Then we obtain from (2) ⇒ (1) of Corollary 3.11 that $\mathbb{EAG}(R) = \mathbb{AG}(R)$. Suppose that $(R, \mathfrak{m})$ is a SPIR satisfying the property that if $n \in \mathbb{N}$ is least such that $\mathfrak{m}^n = (0)$, then $n \in \{2, 3\}$. Then we know from (2) ⇒ (1) of Theorem 3.7 that $\mathbb{EAG}(R) = \mathbb{AG}(R)$. □

Let $n \geq 2$ and let $T = K[X_1, X_2, \ldots, X_n]$ be the polynomial ring in n variables $X_1, X_2, \ldots, X_n$ over a field K. Let $R = \frac{T}{TX_1X_2}$. Observe that R is a reduced ring with $Min(R) = \{\frac{TX_1}{TX_1X_2}, \frac{TX_2}{TX_1X_2}\}$ and thus $|Min(R)| = 2$. Hence, we obtain from (2) ⇒ (1) of Corollary 3.10 that $\mathbb{EAG}(R)$ is connected. We know from [18, Theorem 3, page 281] that each maximal ideal of T is of height n and hence, it follows that $dim R = n - 1$. It follows from [18, Corollary 1, page 279] that TX_1 is the intersection of all maximal ideals of T that contain TX_1. Therefore, it follows that $Max(R)$ is infinite. If a ring is zero-dimensional which admits at least one non-zero exact annihilating ideal and if its exact annihilating-ideal graph is connected, then we prove in Proposition 3.16 that there is a bound on the number of its maximal ideals. We use Lemma 3.15 in the proof of Proposition 3.16.

Lemma 3.15. *Let R be a ring such that $dim R = 0$. Let $n \geq 2$. If $|Max(R)| \geq n$, then there exist zero-dimensional rings $R_1, R_2, \ldots, R_n$ such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ as rings.*

Proof. We prove this lemma using induction on n. Suppose that $|Max(R)| \geq 2$. Let us denote the ring $\frac{R}{nil(R)}$ by T. From $|Max(R)| \geq 2$, it follows that $|Max(T)| \geq 2$. Note that $dimT = 0$ and we know from [4, Proposition 1.7] that T is reduced. Therefore, T is a von Neumann regular ring which is not a field. Hence, T admits a non-trivial idempotent element $a + nil(R)$. Since $nil(R)$ is a nil ideal of R, we obtain from [13, Proposition 7.14, page 405] that there exists an idempotent element e of R such that $a+nil(R) = e+nil(R)$. It is clear that $e \notin \{0, 1\}$. Note that the mapping $f : R \to Re \times R(1-e)$ defined by $f(r) = (re, r(1-e))$ is an isomorphism of rings. Let us denote the ring Re by R_1 and $R(1-e)$ by R_2. Let $i \in \{1, 2\}$. Since R_i is a homomorphic image of R, it follows that $dimR_i = 0$. Thus there exist zero-dimensional rings R_1, R_2 such that $R \cong R_1 \times R_2$ as rings. Let $n \geq 3$ and assume by induction that the lemma is true for $n-1$. Now, $|Max(R)| \geq n > n-1$. By induction hypothesis, there exist zero-dimensional rings $R'_1, \ldots, R'_{n-1}$ such that $R \cong R'_1 \times \cdots \times R'_{n-1}$ as rings. Since $|Max(R)| \geq n$, $|Max(R'_i)| > 1$ for at least one $i \in \{1, \ldots, n-1\}$. Without loss of generality, we can assume that $|Max(R'_1)| > 1$. Hence, by the case $n = 2$, there exist zero-dimensional rings R'_{11}, R'_{12} such that $R'_1 \cong R'_{11} \times R'_{12}$ as rings. Therefore, $R \cong R'_{11} \times R'_{12} \times R'_2 \times \cdots \times R'_{n-1}$ as rings. Let $R_1 = R'_{11}, R_2 = R'_{12}, R_3 = R'_2, \ldots, R_n = R'_{n-1}$. Then $dimR_i = 0$ for each $i \in \{1, 2, \ldots, n\}$ and $R \cong R_1 \times R_2 \times R_3 \times \cdots \times R_n$ as rings. □

Proposition 3.16. *Let R be a zero-dimensional ring such that $\mathbb{EA}(R)^* \neq \emptyset$. If $\mathbb{EAG}(R)$ is connected, then $|Max(R)| \leq 2$.*

Proof. We are assuming that $dimR = 0$, $\mathbb{EA}(R)^* \neq \emptyset$, and $\mathbb{EAG}(R)$ is connected. Suppose that $|Max(R)| \geq 3$. Then it follows from Lemma 3.15 that there exist zero-dimensional rings R_1, R_2, and R_3 such that $R \cong R_1 \times R_2 \times R_3$ as rings. Let us denote the ring $R_1 \times R_2 \times R_3$ by T. Since $R \cong T$ as rings, we obtain that $\mathbb{EAG}(T)$ is connected. Hence, we obtain from Proposition 3.1 that for any $I.J \in \mathbb{EA}(T)^*$ with $I \neq J$, I and J are adjacent in $\mathbb{EAG}(T)$. Let $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. Observe that for all distinct $i, j \in \{1, 2, 3\}$, $e_ie_j = (0, 0, 0)$ and so, $Te_i \in \mathbb{A}(T)^*$. Moreover, for all $i \in \{1, 2, 3\}$, $Ann(Ann(Te_i)) = Te_i$ and so, we obtain from (3) ⇒ (1) of Lemma 2.1 that $Te_i \in \mathbb{EA}(T)^*$. Observe that $Ann(Te_1) = Te_2 + Te_3 \neq Te_2$ and so, Te_1 and Te_2 are not adjacent in $\mathbb{EAG}(T)$. This is a contradiction. Therefore, $|Max(R)| \leq 2$. □

Let R be a ring such that $dimR = 0$ and $|Max(R)| = 2$. In Corollary 3.17, we characterize R such that $\mathbb{EAG}(R)$ is connected.

Corollary 3.17. *Let R be a ring such that $dimR = 0$ and $|Max(R)| = 2$. The following statements are equivalent:*

(1) *$\mathbb{EAG}(R)$ is connected.*

(2) *$R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$.*

Proof. (1) ⇒ (2) By hypothesis, $dimR = 0$ and $|Max(R)| = 2$. We know from Lemma 3.15 that there exist rings R_1, R_2 such that $dimR_i = 0$ for each $i \in \{1, 2\}$ and $R \cong R_1 \times R_2$ as rings. Since $|Max(R)| = 2$, it follows that $|Max(R_i)| = 1$ for each $i \in \{1, 2\}$. Let $\mathfrak{m}_i$ denote the unique maximal ideal of R_i for each $i \in \{1, 2\}$. Let us denote the ring $R_1 \times R_2$ by T. As $\mathbb{EAG}(R)$ is connected, it follows that $\mathbb{EAG}(T)$ is connected. Note that $\{(0) \times R_2, R_1 \times (0)\} \subseteq \mathbb{EA}(T)^*$. We know from Corollary 3.3 that $\mathbb{EAG}(T)$ is a complete graph with two vertices. Therefore, $\mathbb{EA}(T)^* = \{(0) \times R_2, R_1 \times (0)\}$. We next verify that R_i is a field for each $i \in \{1, 2\}$. Suppose that R_1 is not a field. Then $\mathfrak{m}_1 \neq (0)$. Since $Spec(R_1) = \{\mathfrak{m}_1\}$, it follows from [4, Proposition 1.8] that $nil(R_1) = \mathfrak{m}_1$. Let $x \in \mathfrak{m}_1, x \neq 0$. Let $n \geq 2$ be least with the property that $x^n = 0$. Then $x^{n-1} \in Ann(x)$ and $x^{n-1} \neq 0$. Observe that $Ann(x) \times (0) \in \mathbb{EA}(T)^* = \{(0) \times R_2, R_1 \times (0)\}$. This is impossible. Therefore, R_1 is a field. Similarly, it can be shown that R_2 is a field. Hence, $R \cong F_1 \times F_2$ as rings, where $F_i = R_i$ is a field for each $i \in \{1, 2\}$.

(2) ⇒ (1) We are assuming that $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. Note that R is reduced and $|Min(R)| = 2$. Hence, we obtain from (2) ⇒ (1) of Corollary 3.10 that $\mathbb{EAG}(R)$ is connected. □

Let T be a Dedekind domain. Let I be a non-zero proper ideal of T $I \notin Max(T)$ and let $R = \frac{T}{I}$. It is shown in Example 2.8(2) that $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \mathbb{EA}(R)^*$. In Corollary 3.18, we characterize R such that $\mathbb{EAG}(R)$ is connected.

Corollary 3.18. *Let I be a non-zero proper ideal of a Dedekind domain T such that $I \notin Max(T)$. Let $R = \frac{T}{I}$. The following statements are equivalent:*

(1) *$\mathbb{EAG}(R)$ is connected.*

(2) *$|Max(R)| \leq 2$ and $\mathbb{EAG}(R)$ is complete.*

(3) *Either $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$ or $(R, \mathfrak{M})$ is a SPIR and if $k \geq 2$ is least with the property that $\mathfrak{M}^k = (0 + I)$, then $k \in \{2, 3\}$.*

Proof. (1) $\Rightarrow$ (2) Since $dim T = 1$, it follows that $dim R = 0$. Hence, we obtain from Proposition 3.16 that $|Max(R)| \leq 2$ and we obtain from Corollary 3.3 that $\mathbb{EAG}(R)$ is a complete graph with at most two vertices.

(2) $\Rightarrow$ (3) Suppose that $|Max(R)| = 2$. Since $dim R = 0$ and $\mathbb{EAG}(R)$ is connected, we obtain from (1) $\Rightarrow$ (2) of Corollary 3.17 that $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$ and in this case, $I = \mathfrak{m}_1\mathfrak{m}_2$ for some distinct $\mathfrak{m}_1, \mathfrak{m}_2 \in Max(T)$. Suppose that $|Max(R)| = 1$. We know from the proof of Example 2.8(2) that $I = \mathfrak{m}^k$ for some $k \geq 2$ and $(R, \mathfrak{M} = \frac{\mathfrak{m}}{\mathfrak{m}^k})$ is a SPIR. It follows from [4, Corollary 9.4] that $\mathfrak{m}^i \neq \mathfrak{m}^j$ for all distinct $i, j \in \mathbb{N}$. Hence, k is least with the property that $\mathfrak{M}^k = (0+I)$. From the assumption that $\mathbb{EAG}(R)$ is connected, we obtain from Remark 3.6(2) that $k \in \{2, 3\}$.

(3) $\Rightarrow$ (1) If $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$, then it follows from the proof of (2) $\Rightarrow$ (1) of Corollary 3.11 that $\mathbb{EAG}(R)$ is a complete graph with two vertices. Suppose that $(R, \mathfrak{M})$ is a SPIR and if $k \geq 2$ is least with the property that $\mathfrak{M}^k = (0 + I)$, then $k \in \{2, 3\}$. Then it follows from the proof of Remark 3.6(2) that $\mathbb{EAG}(R)$ is a complete graph with at most two vertices. Therefore, $\mathbb{EAG}(R)$ is connected. □

Let $n \geq 2$ be such that n is not a prime number. Since $\mathbb{Z}$ is a PID and hence, a Dedekind domain, and $\mathbb{Z}_n \cong \frac{\mathbb{Z}}{n\mathbb{Z}}$ as rings, the following corollary is an immediate consequence of Corollary 3.18.

Corollary 3.19. *Let $n \geq 2$ be not a prime number. Let $R = \mathbb{Z}_n$. Then $\mathbb{EAG}(R)$ is connected if and only if either $n = p_1p_2$ for some distinct prime numbers p_1, p_2 or $n \in \{p^2, p^3\}$ for some prime number p.*

Let $G = (V, E)$ be a graph. Suppose that G admits a cycle. Recall from [5, page 159] that the *girth of G*, denoted by $girth(G)$ is defined as the length of a shortest cycle in G. If G does not contain any cycle, then we define $girth(G) = \infty$. Recall from [5, Definition 1.2.2] that a *clique of G* is a complete subgraph of G. Suppose that there exists $k \in \mathbb{N}$ such that any clique of G is a clique on at most k vertices. Then the *clique number of G*, denoted by $\omega(G)$ is defined as the largest integer $n \geq 1$ such that G contains a clique on n vertices [5, Definition, page 185]. We set $\omega(G) = \infty$ if G contains a clique on n vertices for all $n \geq 1$.

Let $G = (V, E)$ be a graph. Recall from [5, page 129] that a *vertex coloring of G* is a map $f : V \to S$, where S is a set of distinct colors. A vertex coloring $f : V \to S$ is said to be *proper* if adjacent vertices of G receive distinct colors of S; that is, if a and b are adjacent in G, then $f(a) \neq f(b)$. The *chromatic number of G*, denoted by $\chi(G)$ is the minimum number of colors needed for a proper vertex coloring of G [5, Definition 7.1.2]. It is well-known that for any graph G, $\omega(G) \leq \chi(G)$. Recall from [5] that a graph G is said to be *weakly perfect* if $\chi(G) = \omega(G)$. A graph G is said to be *perfect* if any induced subgraph H of G is weakly perfect; that is, for any induced subgraph H of G, $\chi(H) = \omega(H)$.

Corollary 3.20. *Let R be a ring such that $\mathbb{EA}(R)^* \neq \emptyset$. Then the following hold:*

(1) *$girth(\mathbb{EAG}(R)) = \infty$.*

(2) *$\mathbb{EAG}(R)$ is perfect.*

Proof. (1) If g is any component of $\mathbb{EAG}(R)$, then we know from Corollary 3.3 then g is a complete graph with at most two vertices. Therefore, $\mathbb{EAG}(R)$ does not contain any cycle and so, $girth(\mathbb{EAG}(R)) = \infty$.

(2) Let H be any induced subgraph of $\mathbb{EAG}(R)$. Let h be any component of H. Suppose that $|V(h)| \geq 2$. Then it follows from Proposition 3.1 and Lemma 3.2 that h is a complete graph with two vertices. Hence, it follows that $\chi(H) = \omega(H) \in \{1, 2\}$. Therefore, we obtain that $\mathbb{EAG}(R)$ is perfect. ☐

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