PERFORMANCE EVALUATION OF AN ALGORITHM, PROCESSING 0–1 SEQUENCES WITH PRIORITY

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1. Introduction

Iványi and Kátai [2, 3, 4] described an algorithm processing of sequences of input signals. They determined the asymptotic speed of the algorithm for any finite set of possible input signals and for one or two sequences (priority classes), if the input signals are independent random variables with uniform distribution over the set of possible values.

Iványi and Pergel [5] determined the asymptotic speed for any number of priority classes, if the set of possible input signals contains two elements each having the probability 0,5.

Here we consider the problem when the probability of the possible input signals is different from 0,5. Surprisingly we shall find that the limit of the asymptotic speed, when the number of the priority classes tends to infinity, is less than 2.

2. Formulation of the problem

Let $\mathbf{r} \geq 2$ be an integer number,

$$F_1 = f_{11}, f_{12}, \ldots$$
$$\vdots$$
$$F_r = f_{r1}, f_{r2}, \ldots$$

be infinite binary sequences, that is $f_{ij} \in \{0, 1\}$ $(i = 1, 2, \ldots, r; j = 1, 2, \ldots)$.

We process these sequences using the following algorithm A.

Step 1. Processing proceeds in the discrete points of time 1, 2, ... Let $t = 1$.

Step 2. Let $B_t = (f_{11}, f_{21}, \ldots, f_{r1})$.

Step 3. In the moment t algorithm A processes f_{11} and in addition to f_{11} the first element among $f_{12}, f_{21}, f_{31}, \ldots, f_{r1}$, differing from f_{11}. The duration of this processing equals to one time unit.

Step 4. Algorithm A omits the processed elements, reduces the second index of the remaining elements by the number of the omitted (from its sequence in the moment t) elements, in each sequence.

Step 5. A adds 1 to t and continues the processing from Step 2. □

The sequence $B_1, B_2, \ldots$ is called the state sequence of the processing. B_t $(t = 1, 2, \ldots)$ contains the following (first nonprocessed) elements of $F_1, F_2, \ldots, F_r$ in the t-th time unit.

Let now

$$\begin{aligned} \Theta_1 &= \eta_{11}, \eta_{12}, \\ \Theta_2 &= \eta_{21}, \eta_{22}, \ldots \\ &\vdots \\ \Theta_r &= \eta_{r1}, \eta_{r2}, \ldots \end{aligned}$$

be sequences of independent random variables with common distribution $P(\eta_{ij} = 0) = p$ and $P(\eta_{ij} = 1) = q$ $(i = 1, 2, \ldots, r; j = 1, 2, \ldots)$, where $0 < p < 1$ and $q = 1 - p$; obviously we may suppose that $p > \frac{1}{2}$.

The stochastic behaviour of the processing algorithm may be described by $\sigma_1, \sigma_2, \ldots$, where σ_t is determined by the distribution of the possible values of B_t's. We are interested in the characterization of the sequence σ_t $(t = 1, 2, \ldots)$.

3. The ergodic distribution of the cumulative sequence of the first non-processed elements

For any positive integer s $(s \le r)$ the first s components of the σ_t' s (as s-dimensional vectors) form a homogeneous ergodic Markov-chain [2].

Let $\sigma_t = (\nu_{t1}, \nu_{t2}, \ldots, \nu_{tr})$. For our purposes the probability $P(\nu_{t1} = 0, \ldots, \nu_{tr} = 0)$ is of basic importance; to be able to handle it let us define

$$\xi_{tr} = \sum_{i=1}^{r} \nu_{ti} \quad \text{and} \quad \lim_{t \to \infty} P(\xi_{tr} = i) = \pi_{ir} \quad (i = 0, 1, \ldots, r).$$

Theorem 1. *For every natural $r \ge 1$ and real $p > 1/2$ the sequence of random variables ξ_{tr}, $t = 1, 2, \ldots$ is a homogeneous ergodic Markov-chain with the state-space $\{0, 1, \ldots, r\}$ and ergodic probabilitites*

$$\pi_{ir} = \left(\frac{q}{p}\right)^{2i} \left[1 - \left(\frac{q}{p}\right)^2\right] \left[1 - \left(\frac{q}{p}\right)^{2r+2}\right]^{-1}. \qquad \square \tag{1}$$

Proof. It is easy to see the assertion about the state-space and the homogeneity. The transition probabilities are (index r is omitted)

$$\begin{aligned} P(\xi_{t+1} &= \xi_t) = 2pq, \\ P(\xi_{t+1} &= \xi_t + 1) = q^2, \\ P(\xi_{t+1} &= \xi_t - 1) = p^2, \end{aligned}$$

if $0<\xi_t<r$, and

$$P(\xi_{t+1} = 0 | \xi_t = 0) = p+pq = p^2+2pq,$$
$$P(\xi_{t+1} = 1 | \xi_t = 0) = q^2, \quad P(\xi_{t+1} = r | \xi_t = r) = q+pq = q^2+2pq,$$
$$P(\xi_{t+1} = r-1 | \xi_t = r) = p^2,$$

for the boundary states [1].

So for the state probabilities we have the system of linear equations

$$\pi_0 = (p^2+2pq)\pi_0 + p^2\pi_1,$$
$$\vdots$$
$$\pi_i = q^2\pi_{i-1} + 2pq\pi_i + p^2\pi_{i+1},$$
$$\vdots$$
$$\pi_r = q^2\pi_{r-1} + (2pq+q^2)\pi_r,$$
$$\sum_{i=0}^{r} = 1,$$

from where we get (1) [4]. □

Corollary 1. *For every positive integer $r \geq 1$ and real $p > 1/2$*

$$P(\nu_{1r} = 0, \ldots, \nu_{tr} = 0) = \left[1-\left(\frac{q}{p}\right)^2\right]\left[1-\left(\frac{q}{p}\right)^{2r+2}\right]^{-1}. \quad \square$$

Proof. Let us substitute $i = 0$ into (1). □

4. The speed of the processing

Let the random variable χ_{jr} denote the number of the processed elements of the first r sequences in the j-th ($j = 1, 2, \ldots$) point of time. Let the processing speed S_{pr} be defined by

$$S_{pr} = \lim_{T\to\infty} M\left(\frac{1}{T}\sum_{t=1}^{T}\chi_{tr}\right).$$

Then the speed depends on the probability p and the number of processable sequences r according to the following assertion.

Theorem 2. *For every positive integer $r \geq 1$ and real $p \geq 1/2$ we have*

$$S_{pr} = 2 - p\pi_{0r} - q\pi_{rr}. \quad \square \tag{2}$$

Proof. Algorithm A processes one element only if $f_{12} = f_{11} = f_{21} = \ldots = f_{r1}$. Due to the ergodicity of the chain ξ_{tr} ($t = 1, 2, \ldots$) it is enough to consider the ergodic probabilities, i.e. for $p > 1/2$ we have

$$S_{pr} = 2\left(q\pi_{or} + p\pi_{rr} + \sum_{i=1}^{r-1} \pi_{ir}\right) + (p\pi_{or} + q\pi_{rr}),$$

hence we get (2).

In the case $p = 1/2$ the equality follows from the formula (4.7) in [5]. □

Let the limit speed of processing S_p when the number of the processable sequences tends to infinity defined by

$$S_p = \lim_{u \to \infty} S_{pu}.$$

As we have seen in [5], $S_{1/2,\,r} = 2 - 1/(r+1)$, hence we can get the obvious result $S_{1/2} = 2$. The following assertion gives us somewhat surprising result.

Corollary 2. *If* $p > 1/2$, *then*

(3) $$S_p = 2 - \frac{p^2 - q^2}{p} = \frac{1}{p}. \quad □$$

Proof. From (2) we have

$$S_p = 2 - p \lim_{u \to \infty} \pi_{ou} - q \lim_{u \to \infty} \pi_{uu}.$$

In this formula the first limit equals to $1 - \left(\frac{q}{p}\right)^2$, the second one equals to zero, therefore (3) holds. □

Of course, according to (3), $S_p < 2$ for $p > 1/2$.

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