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ON BINOMIAL SUMS WITH THE TERMS OF SEQUENCES $\{g_{kn}\}$ AND $\{h_{kn}\}$

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Abstract. In this paper, we derive sums and alternating sums of products of terms of the sequences $\{g_{kn}\}$ and $\{h_{kn}\}$ with binomial coefficients. For example,

$$\sum_{i=0}^n \binom{n}{i} i^m g_{k(n-ti)} h_{kti} \\
= 2^{n-m} n^m g_{kn} - n^m \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{n(1-t)} h_{kt}^{n-m} g_{k(tm+tn-n)},$$

where c is a nonzero real number, t is any integer and m is a nonnegative integer.

Keywords: Binomial sums, alternating sums, generalized Fibonacci numbers, recurrence relation.

1. Introduction

Define the second order linear recursive sequences $\{u_n\}$ and $\{v_n\}$ for $n \geq 1$ and nonzero integers p, q , by

$$u_{n+1} = pu_n + qu_{n-1} \text{ and } v_{n+1} = pv_n + qv_{n-1}$$

with initials $u_0 = 0, u_1 = 1$ and $v_0 = 2, v_1 = p$, respectively.

When $q = 1$, $u_n = U_n$ (the n th generalized Fibonacci number) and $v_n = V_n$ (the n th generalized Lucas number). Also, when $p = q = 1$, $u_n = F_n$ (the n th Fibonacci number) and $v_n = L_n$ (the n th Lucas number). If α and β are the roots of the

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equation $x^2 - px - q = 0$, the Binet formulae of the sequences $\{u_n\}$ and $\{v_n\}$ have the forms

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } v_n = \alpha^n + \beta^n,$$

respectively, where $\alpha, \beta = (p \pm \sqrt{\Delta})/2$ and $\Delta = p^2 + 4q$. From [3], Kılıç and Stanica derived the following recurrence relations for the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ for $k \geq 0, n > 0$. It is clear that

$$u_{k(n+1)} = v_k u_{kn} + (-1)^{k+1} q^k u_{k(n-1)} \text{ and } v_{k(n+1)} = v_k v_{kn} + (-1)^{k+1} q^k v_{k(n-1)},$$

where the initial conditions are 0, u_k , and 2, v_k , respectively. The Binet formulae of the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ are given by

$$u_{kn} = \frac{\alpha^{kn} - \beta^{kn}}{\alpha - \beta} \text{ and } v_{kn} = \alpha^{kn} + \beta^{kn},$$

respectively. It is clearly seen that $u_{-kn} = (-1)^{kn+1} u_{kn}$ and $u_{2kn} = u_{kn} v_{kn}$.

In [9], Komatsu obtained the two binomial sums of the generalized Fibonacci numbers as follows:

$$\sum_{i=0}^n \binom{n}{i} c^i u_i = g_n \quad (n \geq 0)$$

which satisfies the recurrence relation

$$g_{n+1} = (pc + 2) g_n + (qc^2 - pc - 1) g_{n-1} \quad (n \geq 1)$$

with $g_0 = 0, g_1 = c$ and

$$\sum_{i=0}^n \binom{n}{i} c^{n-i} d^i u_i = h_n \quad (n \geq 0)$$

which satisfies the recurrence relation

$$h_{n+1} = (pd + 2c) h_n + (qd^2 - pcd - c^2) h_{n-1} \quad (n \geq 1)$$

with $h_0 = 0$ and $h_1 = d$, where c, d are nonzero real numbers. Also he gave several Fibonacci identities including binomial coefficients by using the method of ordinary power series generating functions.

In [1], Cook et al. obtained some binomial summation identities including the terms of the sequence $\{g_n\}$ in [9]. For example,

$$\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i (qc^2 - pc - 1)^{2n-i} g_{2i+1} = (pc + 2)^{2n} g_{2n+1},$$

and

$$\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i (qc^2 - pc - 1)^{2(2n-i)} g_{4i} = c^{2n} (pc + 2)^{2n} (p^2 + 4q)^n g_{4n}.$$

In [10], Ömür et al. defined the subsequences $\{g_{kn}\}$ and $\{h_{kn}\}$ with binomial sums $g_{kn} = \sum_{i=0}^n \binom{n}{i} c^{ki} u_{ki}$ and $h_{kn} = \sum_{i=0}^n \binom{n}{i} c^{ki} v_{ki}$. These subsequences satisfy the following relations

$$g_{k(n+1)} = (c^k v_k + 2) g_{kn} - (c^{2k} (-q)^k + c^k v_k + 1) g_{k(n-1)}$$

and

$$h_{k(n+1)} = (c^k v_k + 2) h_{kn} - (c^{2k} (-q)^k + c^k v_k + 1) h_{k(n-1)}$$

in which $g_0 = 0$, $g_k = c^k u_k$ and $h_0 = 2$, $h_k = 2 + c^k v_k$, respectively. The Binet formulae of the sequences $\{g_{kn}\}$ and $\{h_{kn}\}$ are

$$g_{kn} = \frac{(c^k \alpha^k + 1)^n - (c^k \beta^k + 1)^n}{\alpha - \beta} \text{ and } h_{kn} = (c^k \alpha^k + 1)^n + (c^k \beta^k + 1)^n,$$

respectively. The authors obtained some binomial summation identities of sequence $\{g_{kn}\}$. For example, for $n > 0$,

$$\sum_{i=0}^{2n} \binom{2n}{i} (c^{2k} (-q)^k + c^k v_k + 1)^{2n-i} g_{k(2i+1)} = (c^k v_k + 2)^{2n} g_{k(2n+1)},$$

and

$$\begin{aligned} \sum_{i=0}^{2n} \binom{2n}{i} (-1)^i (c^{2k} (-q)^k + c^k v_k + 1)^{2n-i} g_{k(2i+1)} \\ = c^{2kn} (v_k^2 + 4q^k (-1)^{k+1})^n g_{k(2n+1)}. \end{aligned}$$

In [7], Kılıç et al. introduced sums and alternating sums of products of terms of sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ as follows: for odd number n ,

$$\begin{aligned} \sum_{i=0}^n \binom{n}{i} U_{k(a+bi)} U_{k(e+fi)} = D^{(n-1)/2} \left(U_{k(b+f)/2}^n U_{k(n(b+f)/2+a+e)} \right. \\ \left. + (-1)^{e+(b-f)/2} U_{k(b-f)/2}^n U_{k(n(b-f)/2+a-e)} \right), \end{aligned}$$

and

$$\begin{aligned} \sum_{i=0}^n \binom{n}{i} (-1)^i U_{k(a+bi)} U_{k(e+fi)} = \frac{1}{D} \left((-1)^n V_{k(b+f)/2}^n V_{k(n(b+f)/2+a+e)} \right. \\ \left. - (-1)^{e+n(b-f)/2} V_{k(b-f)/2}^n V_{k(n(b-f)/2+a-e)} \right), \end{aligned}$$

where a, b, e, f are any integers, $b + f \equiv 2 \pmod{4}$ and $D = p^2 + 4$.

In [5, 6], Kılıç considered and computed the alternating binomial sums of the forms

$$\sum_{i=0}^n \binom{n}{i} (-1)^i f(n, i, k, t) \text{ and } \sum_{i=0}^n \binom{n}{i} g(n, i, k, t),$$

where $f(n, i, k, t)$ is $U_{kti}V_{k(n-ti)}$ and $U_{kti}V_{(k+1)tn-(k+2)ti}$, and, $g(n, i, k, t)$ is $U_{ki}U_{k(tn+i)}$, $U_{ki}V_{k(tn+i)}$, $V_{ki}V_{k(tn+i)}$ and $V_{ki}U_{k(tn+i)}$ for positive integers t and n . For example, for odd k ,

$$\sum_{i=0}^n \binom{n}{i} (-1)^i U_{kti}V_{k(n-ti)} = U_{kt}^n \begin{cases} (-1)^t V_{kn(t-1)} D^{(n-1)/2} & \text{if } n \text{ is odd,} \\ U_{kn(t-1)} D^{n/2} & \text{if } n \text{ is even,} \end{cases}$$

where D is defined as before.

In [4], inspired by the works of [5, 6, 8], Kılıç et al. gave rising factorial of the summation index instead of its powers. Clearly, they considered and computed the generalized alternating weighted binomial sums :

$$\sum_{i=0}^n \binom{n}{i} i^m (-1)^i f(n, i, k, t),$$

where $f(n, i, k, t)$ as before and m is a nonnegative integer and x^m stands for the falling factorial defined by $x^m = x(x-1)\dots(x-m+1)$. These kinds of binomial sums (except some special cases of k and t) have not been considered according to our best literature acknowledgement. For example, for any integers k and t ,

$$\begin{aligned} & \sum_{i=0}^n \binom{n}{i} i^m (-1)^i U_{kti}V_{k(n-ti)} = (-1)^{kn(t+1)+m} n^m U_{kt}^{n-m} \\ & \times \begin{cases} U_{k(tn+tm-n)} D^{(n-m)/2} & \text{if } n \equiv m \pmod{2}, \\ -V_{k(tn+tm-n)} D^{(n-m-1)/2} & \text{if } n \equiv m+1 \pmod{2}, \end{cases} \end{aligned}$$

where m is a nonnegative integer.

2. Sums of certain products with the terms of $\{g_{kn}\}$ and $\{h_{kn}\}$

In this section, firstly, we will start with some lemmas for further use.

Lemma 2.1. *For any integers m and n , we have*

$$g_{k(m+n)} + g_{k(m-n)} \left(q^k (-1)^{k+1} c^{2k} - c^k v_k - 1 \right)^n = \begin{cases} h_{km} g_{kn} & \text{if } n \text{ is odd,} \\ g_{km} h_{kn} & \text{if } n \text{ is even,} \end{cases}$$

$$\begin{aligned}
g_{k(m+n)} - g_{k(m-n)} \left(q^k (-1)^{k+1} c^{2k} - c^k v_k - 1 \right)^n &= \begin{cases} g_{km} h_{kn} & \text{if } n \text{ is odd,} \\ h_{km} g_{kn} & \text{if } n \text{ is even,} \end{cases} \\
h_{k(m+n)} - h_{k(m-n)} \left(q^k (-1)^{k+1} c^{2k} - c^k v_k - 1 \right)^n &= \begin{cases} h_{km} h_{kn} & \text{if } n \text{ is odd,} \\ \Delta g_{km} g_{kn} & \text{if } n \text{ is even,} \end{cases} \\
h_{k(m+n)} + h_{k(m-n)} \left(q^k (-1)^{k+1} c^{2k} - c^k v_k - 1 \right)^n &= \begin{cases} \Delta g_{km} g_{kn} & \text{if } n \text{ is odd,} \\ h_{km} h_{kn} & \text{if } n \text{ is even,} \end{cases}
\end{aligned}$$

where c is a nonzero real number.

Proof. By the Binet formulae of $\{g_{kn}\}$ and $\{h_{kn}\}$, the claimed equalities are obtained. \square

We recall some facts for the readers convenience: For any real numbers m and n ,

$$(2.1) \quad (m+n)^t = \begin{cases} \sum_{i=0}^{(t-1)/2} \binom{t}{i} (mn)^i (m^{t-2i} + n^{t-2i}) & \text{if } t \text{ is odd,} \\ \sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^i (m^{t-2i} + n^{t-2i}) \\ \quad + \binom{t}{t/2} (mn)^{t/2} & \text{if } t \text{ is even,} \end{cases}$$

and

$$(2.2) \quad (m-n)^t = \begin{cases} \sum_{i=0}^{(t-1)/2} \binom{t}{i} (mn)^i (-1)^i (m^{t-2i} - n^{t-2i}) & \text{if } t \text{ is odd,} \\ \sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^i (-1)^i (m^{t-2i} + n^{t-2i}) \\ \quad + \binom{t}{t/2} (-1)^{t/2} (mn)^{t/2} & \text{if } t \text{ is even,} \end{cases}$$

where t is a positive integer.

Lemma 2.2. For any integers r and s , we have

$$\begin{aligned}
&\sum_{i=0}^n \binom{n}{i} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{r(n-i)} h_{k(2ri+s)} = h_{k(rn+s)} h_{kr}^n, \\
&\sum_{i=0}^n \binom{n}{i} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{r(n-i)} g_{k(2ri+s)} = g_{k(rn+s)} h_{kr}^n, \\
&\sum_{i=0}^n \binom{n}{i} (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{r(n-i)} g_{k(2ri+s)} \\
&= \begin{cases} -\Delta^{(n-1)/2} g_{kr}^n h_{k(rn+s)} & \text{if } n \text{ is odd,} \\ \Delta^{n/2} g_{kr}^n g_{k(rn+s)} & \text{if } n \text{ is even,} \end{cases}
\end{aligned}$$

and

$$(2.3) \quad \sum_{i=0}^n \binom{n}{i} (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{r(n-i)} h_{k(2ri+s)}$$

$$= \begin{cases} -\Delta^{(n+1)/2} g_{kr}^n g_{k(rn+s)} & \text{if } n \text{ is odd,} \\ \Delta^{n/2} g_{kr}^n h_{k(rn+s)} & \text{if } n \text{ is even,} \end{cases}$$

where c is a nonzero real number.

Proof. From (2.1), (2.2) and Lemma 2.1, the proof is obtained. \square

Lemma 2.3. [2] Let n and m be integers such that $0 \leq m < n$. For $z \neq -1$,

$$\sum_{k=0}^n \binom{n}{k} k^m z^k = z^m n^{\underline{m}} (1+z)^{n-m}.$$

Theorem 2.1. Let a, b and e be any integers. Then

$$\sum_{i=0}^n \binom{n}{i} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} g_{k(ai+b)} g_{k(ai+e)}$$

$$= \frac{1}{\Delta} \left(h_{k(an+b+e)} h_{ka}^n \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an} \right. \\ \left. - 2^n h_{k(b-e)} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^e \right),$$

$$\sum_{i=0}^n \binom{n}{i} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} h_{k(ai+b)} h_{k(ai+e)}$$

$$= h_{k(an+b+e)} h_{ka}^n \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an} \\ + 2^n h_{k(b-e)} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^e,$$

and

$$\sum_{i=0}^n \binom{n}{i} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} g_{k(ai+b)} h_{k(ai+e)}$$

$$= g_{k(an+b+e)} h_{ka}^n \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an} \\ + 2^n g_{k(b-e)} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^e,$$

where c is a nonzero real number.

Proof. Consider that

$$\begin{aligned}
& \sum_{i=0}^n \binom{n}{i} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} g_{k(ai+b)} g_{k(ai+e)} \\
&= \sum_{i=0}^n \binom{n}{i} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} \\
& \quad \times \frac{\left[(c^k \alpha^k + 1)^{ai+b} - (c^k \beta^k + 1)^{ai+b} \right] \left[(c^k \alpha^k + 1)^{ai+e} - (c^k \beta^k + 1)^{ai+e} \right]}{(\alpha - \beta)^2} \\
&= \frac{1}{(\alpha - \beta)^2} \sum_{i=0}^n \binom{n}{i} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} \\
& \quad \times \left((c^k \alpha^k + 1)^{2ai+b+e} - (c^k \alpha^k + 1)^{ai+b} (c^k \beta^k + 1)^{ai+e} \right. \\
& \quad \quad \left. + (c^k \beta^k + 1)^{2ai+b+e} - (c^k \alpha^k + 1)^{ai+e} (c^k \beta^k + 1)^{ai+b} \right) \\
&= \frac{1}{\Delta} \sum_{i=0}^n \binom{n}{i} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} h_{k(2ai+b+e)} \\
& \quad - \frac{1}{\Delta} \sum_{i=0}^n \binom{n}{i} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^e h_{k(b-e)}.
\end{aligned}$$

From Lemma 2.2, the desired result is obtained. Similarly, the other cases are given. \square

Theorem 2.2. *Let a, b and e be any integers. Then*

$$\begin{aligned}
& \sum_{i=0}^n \binom{n}{i} (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} g_{k(ai+b)} g_{k(ai+e)} \\
&= \begin{cases} -\Delta^{(n-1)/2} g_{k(an+b+e)} g_{ka}^n \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an} & \text{if } n \text{ is odd,} \\ \Delta^{(n-2)/2} h_{k(an+b+e)} g_{ka}^n \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an} & \text{if } n \text{ is even,} \end{cases}
\end{aligned}$$

$$\begin{aligned}
& \sum_{i=0}^n \binom{n}{i} (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} h_{k(ai+b)} h_{k(ai+e)} \\
&= \begin{cases} -\Delta^{(n+1)/2} g_{k(an+b+e)} g_{ka}^n \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an} & \text{if } n \text{ is odd,} \\ \Delta^{n/2} h_{k(an+b+e)} g_{ka}^n \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an} & \text{if } n \text{ is even,} \end{cases}
\end{aligned}$$

and

$$\begin{aligned} & \sum_{i=0}^n \binom{n}{i} (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} g_{k(ai+b)} h_{k(ai+d)} \\ &= \begin{cases} -\Delta^{(n-1)/2} h_{k(an+b+e)} g_{ka}^n \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an} & \text{if } n \text{ is odd,} \\ \Delta^{n/2} g_{k(an+b+e)} g_{ka}^n \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an} & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

where c is a nonzero real number.

Proof. Consider that

$$\begin{aligned} & \sum_{i=0}^n \binom{n}{i} (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} g_{k(ai+b)} g_{k(ai+e)} \\ &= \sum_{i=0}^n \binom{n}{i} (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} \\ & \quad \times \frac{\left[(c^k \alpha^k + 1)^{ai+b} - (c^k \beta^k + 1)^{ai+b} \right] \left[(c^k \alpha^k + 1)^{ai+e} - (c^k \beta^k + 1)^{ai+e} \right]}{(\alpha - \beta)^2} \\ &= \frac{1}{(\alpha - \beta)^2} \sum_{i=0}^n \binom{n}{i} (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} \\ & \quad \times \left((c^k \alpha^k + 1)^{2ai+b+e} - (c^k \alpha^k + 1)^{ai+b} (c^k \beta^k + 1)^{ai+e} \right. \\ & \quad \left. + (c^k \beta^k + 1)^{2ai+b+e} - (c^k \alpha^k + 1)^{ai+e} (c^k \beta^k + 1)^{ai+b} \right) \\ &= \frac{1}{\Delta} \sum_{i=0}^n \binom{n}{i} (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} h_{k(2ai+b+e)} \\ & \quad - \frac{1}{\Delta} \sum_{i=0}^n \binom{n}{i} (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^e h_{k(b-e)}. \end{aligned}$$

From (2.3), the desired result is obtained. Similarly, using Lemma 2.2, the other cases can be obtained. \square

Theorem 2.3. *Let k and t be any integers. Then*

$$\begin{aligned} & \sum_{i=0}^n \binom{n}{i} i^{\underline{m}} (-1)^i g_{k(n-ti)} h_{kti} = (-1)^m n^{\underline{m}} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{n(1-t)} g_{kt}^{n-m} \\ & \quad \times \begin{cases} -\Delta^{(n-m)/2} g_{k(tn+tm-n)} & \text{if } n \equiv m \pmod{2}, \\ \Delta^{(n-m-1)/2} h_{k(tn+tm-n)} & \text{if } n \equiv m+1 \pmod{2}, \end{cases} \end{aligned}$$

and

$$\sum_{i=0}^n \binom{n}{i} i^m (-1)^i g_{kti} h_{k(n-ti)} = (-1)^m n^m \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{n(1-t)} g_{kt}^{n-m} \\ \times \begin{cases} \Delta^{(n-m)/2} g_{k(tn+tm-n)} & \text{if } n \equiv m \pmod{2}, \\ -\Delta^{(n-m-1)/2} h_{k(tn+tm-n)} & \text{if } n \equiv m+1 \pmod{2}, \end{cases}$$

where c is a nonzero real number and m is a nonnegative integer.

Proof. Observe that

$$\sum_{i=0}^n \binom{n}{i} i^m (-1)^i g_{k(n-ti)} h_{kti} \\ = \frac{1}{\alpha - \beta} \sum_{i=0}^n \binom{n}{i} i^m (-1)^i \left((c^k \alpha^k + 1)^{n-ti} - (c^k \beta^k + 1)^{n-ti} \right) \\ \times \left((c^k \alpha^k + 1)^{ti} + (c^k \beta^k + 1)^{ti} \right) \\ = \frac{(c^k \alpha^k + 1)^n}{\alpha - \beta} \left(\sum_{i=0}^n \binom{n}{i} i^m (-1)^i + \sum_{i=0}^n \binom{n}{i} i^m (-1)^i \left(\frac{c^k \beta^k + 1}{c^k \alpha^k + 1} \right)^{ti} \right) \\ - \frac{(c^k \beta^k + 1)^n}{\alpha - \beta} \left(\sum_{i=0}^n \binom{n}{i} i^m (-1)^i + \sum_{i=0}^n \binom{n}{i} i^m (-1)^i \left(\frac{c^k \alpha^k + 1}{c^k \beta^k + 1} \right)^{ti} \right),$$

which by Lemma 2.3, equals

$$\frac{(c^k \alpha^k + 1)^n}{\alpha - \beta} \left\{ (-1)^m \left(\frac{c^k \beta^k + 1}{c^k \alpha^k + 1} \right)^{tm} n^m \left(1 - \left(\frac{c^k \beta^k + 1}{c^k \alpha^k + 1} \right)^t \right)^{n-m} \right\} \\ - \frac{(c^k \beta^k + 1)^n}{\alpha - \beta} \left\{ (-1)^m \left(\frac{c^k \alpha^k + 1}{c^k \beta^k + 1} \right)^{tm} n^m \left(1 - \left(\frac{c^k \alpha^k + 1}{c^k \beta^k + 1} \right)^t \right)^{n-m} \right\} \\ = (-1)^m n^m \frac{1}{\alpha - \beta} \left\{ (c^k \alpha^k + 1)^n \left(\frac{c^k \beta^k + 1}{c^k \alpha^k + 1} \right)^{tm} \left(1 - \left(\frac{c^k \beta^k + 1}{c^k \alpha^k + 1} \right)^t \right)^{n-m} \right. \\ \left. - (c^k \beta^k + 1)^n \left(\frac{c^k \alpha^k + 1}{c^k \beta^k + 1} \right)^{tm} \left(1 - \left(\frac{c^k \alpha^k + 1}{c^k \beta^k + 1} \right)^t \right)^{n-m} \right\} \\ = (-1)^m n^m (\alpha - \beta)^{n-m-1} \\ \times \left\{ (c^k \alpha^k + 1)^n \left(\frac{c^k \beta^k + 1}{c^k \alpha^k + 1} \right)^{tm} \left(\frac{(c^k \alpha^k + 1)^t - (c^k \beta^k + 1)^t}{(\alpha - \beta)(c^k \alpha^k + 1)^t} \right)^{n-m} \right. \\ \left. - (-1)^{n-m} (c^k \beta^k + 1)^n \left(\frac{c^k \alpha^k + 1}{c^k \beta^k + 1} \right)^{tm} \left(\frac{(c^k \alpha^k + 1)^t - (c^k \beta^k + 1)^t}{(\alpha - \beta)(c^k \beta^k + 1)^t} \right)^{n-m} \right\},$$

which by the Binet formulae, gives us the claimed result. \square

Similar to the proof method of Theorem just above, we have the following results without proof.

Theorem 2.4. *Let k and t be any integers. Then*

$$\begin{aligned} & \sum_{i=0}^n \binom{n}{i} i^m g_{k(n-ti)} h_{kti} \\ &= 2^{n-m} n^m g_{kn} - n^m \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{n(1-t)} h_{kt}^{n-m} g_{k(tm+tn-n)}, \end{aligned}$$

and

$$\begin{aligned} & \sum_{i=0}^n \binom{n}{i} i^m g_{kti} h_{k(n-ti)} \\ &= 2^{n-m} n^m g_{kn} + n^m \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{n(1-t)} h_{kt}^{n-m} g_{k(tm+tn-n)}, \end{aligned}$$

where c is a nonzero real number and m is a nonnegative integer.

Theorem 2.5. *Let k and t be any integers. Then*

$$\begin{aligned} & \sum_{i=0}^n \binom{n}{i} i^m \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kti} h_{kn(t+1)-ki(t+2)} \\ &= n^m \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^m h_k^{n-m} g_{k(nt-m)} + n^m h_{k(t+1)}^{n-m} g_{km(t+1)}, \end{aligned}$$

and

$$\begin{aligned} & \sum_{i=0}^n \binom{n}{i} i^m \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kn(t+1)-ki(t+2)} h_{kti} \\ &= n^m \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^m h_k^{n-m} g_{k(nt-m)} - n^m h_{k(t+1)}^{n-m} g_{km(t+1)}, \end{aligned}$$

where c is a nonzero real number and m is a nonnegative integer.

Theorem 2.6. *Let k and t be any integers. Then*

$$\begin{aligned} & \sum_{i=0}^n \binom{n}{i} i^m (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kti} h_{kn(t+1)-ki(t+2)} \\ &= (-1)^m n^m \Delta^{(n-m)/2} \\ & \times \begin{cases} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^m & \text{if } n \equiv m \pmod{2}, \\ \times g_k^{n-m} g_{k(tn-m)} + g_{k(t+1)}^{n-m} g_{km(t+1)} & \\ \Delta^{-1/2} \left\{ \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^m & \text{if } n \equiv m+1 \pmod{2}, \right. \\ \times g_k^{n-m} h_{k(tn-m)} - g_{k(t+1)}^{n-m} h_{km(t+1)} \left. \right\} & \end{cases} \end{aligned}$$

and

$$\begin{aligned}
& \sum_{i=0}^n \binom{n}{i} i^m (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kn(t+1)-ki(t+2)} h_{kti} \\
&= (-1)^m n^m \Delta^{(n-m)/2} \\
& \times \begin{cases} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^m \\ \quad \times g_k^{n-m} g_k(tn-m) - g_{k(t+1)}^{n-m} g_{km(t+1)} & \text{if } n \equiv m \pmod{2}, \\ \Delta^{-1/2} \left\{ \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^m \right. \\ \quad \left. \times g_k^{n-m} h_k(tn-m) + g_{k(t+1)}^{n-m} h_{km(t+1)} \right\} & \text{if } n \equiv m+1 \pmod{2}, \end{cases}
\end{aligned}$$

where c is a nonzero real number and m is a nonnegative integer.

Theorem 2.7. Let k and t be any integers. Then

$$\begin{aligned}
& \sum_{i=0}^n \binom{n}{i} i^m \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kti} h_{kn-ki(t+2)} \\
&= n^m \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-nt} h_{k(t+1)}^{n-m} g_{k(mt+nt+m)} - n^m h_k^{n-m} g_{km},
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{i=0}^n \binom{n}{i} i^m \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kn-ki(t+2)} h_{kti} \\
&= -n^m \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-nt} h_{k(t+1)}^{n-m} g_{k(mt+nt+m)} - n^m h_k^{n-m} g_{km},
\end{aligned}$$

where c is a nonzero real number and m is a nonnegative integer.

Theorem 2.8. Let k and t be any integers. Then

$$\begin{aligned}
& \sum_{i=0}^n \binom{n}{i} i^m (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kti} h_{kn-ki(t+2)} \\
&= (-1)^m n^m \Delta^{(n-m)/2} \\
& \times \begin{cases} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-nt} \\ \quad \times g_{k(t+1)}^{n-m} g_{k(mt+nt+t)} - g_k^{n-m} g_{km} & \text{if } n \equiv m \pmod{2}, \\ -\Delta^{-1/2} \left\{ \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-nt} \right. \\ \quad \left. \times g_{k(t+1)}^{n-m} h_{k(mt+nt+t)} - g_k^{n-m} h_{km} \right\} & \text{if } n \equiv m+1 \pmod{2}, \end{cases}
\end{aligned}$$

and

$$\begin{aligned} & \sum_{i=0}^n \binom{n}{i} i^m (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kn-ki(t+2)} h_{kti} \\ &= (-1)^m n^m \Delta^{(n-m)/2} \\ & \times \begin{cases} \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-nt} & \text{if } n \equiv m \pmod{2}, \\ \times g_{k(t+1)}^{n-m} g_{k(mt+nt+t)} - g_k^{n-m} g_{km} & \\ \Delta^{-1/2} \left\{ \left(c^{2k} (-q)^k + c^k v_k + 1 \right)^{-nt} & \text{if } n \equiv m+1 \pmod{2}, \right. \\ \left. \times g_{k(t+1)}^{n-m} h_{k(mt+nt+t)} + g_k^{n-m} h_{km} \right\} & \end{cases} \end{aligned}$$

where c is a nonzero real number and m is a nonnegative integer.

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