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PROPERTIES OF A NEW SUBCLASS OF ANALYTIC FUNCTION ASSOCIATED TO RAFID - OPERATOR AND q-DERIVATIVE

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Abstract. In this article, we introduce a new subclass of analytic functions, using the exponent operators of Rafid and q-derivative. The coefficient estimates, extreme points, convex linear combination, radii of starlikeness, convexity and finally integral have been investigated.

Keywords: Rafid - operator, q-derivative, q-integral, univalent function, coefficient bound, convex set, partial sum.

1. Introduction

The theory of univalent functions can be described by using the theory of the q-calculus. In recent years, such q-calculus as the q-integral and q-derivative have been used to construct several subclasses of analytic functions [1, 6, 11, 12]. The theory of q-analysis has motivated the researchers owing to many branches of mathematics and physics. For example, in the areas of special functions, q-difference, q-integral equations, optimal control problems, q-difference, q-integral equations, q-transform analysis and in quantum physics see for instance, [7, 8, 10, 14].

The main subject of the present paper is to introduce and investigate a new subclass of analytic functions in the open unit disk U by using the operators Rafid and q-derivative. Let $\mathcal{A}$ denote the class of functions $f(z)$ in the form of:

$$f(z) = z + \sum_{k=2}^{+\infty} a_k z^k, \tag{1.1}$$

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which are analytic in the punctured unit disk

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $f(z) \in \mathcal{A}$, the q- derivative, $0 < q < 1$, of $f(z)$ is defined by Gasper and Rahman [5].

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z} & (z \neq 0) \\ f^{'}(0) & (z = 0). \end{cases} \tag{1.2}$$

where $z \in U$ and $0 < q < 1$.
Let $T(p)$ be the class of all p -valent functions of the form

$$f(z) = z^p - \sum_{n=p+1}^{+\infty} a_n z^n \quad a_n \geq 0, \tag{1.3}$$

which are analytic in the punctured unit disk

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

If $f \in T(p)$ is given by Equation (1.3) and $g \in T(p)$ is given by

$$g(z) = z^p - \sum_{n=p+1}^{+\infty} b_n z^n \quad b_n \geq 0, \tag{1.4}$$

then the Hadamard product $f * g$ of f and g is defined by

$$(f * g)(z) = z^p - \sum_{n=p+1}^{+\infty} a_n b_n z^n = (g * f)(z). \tag{1.5}$$

From Equation (1.2) for a function $f(z)$ given by Equation(1.3) we get

$$D_q f(z) = [p]_q z^{p-1} - \sum_{n=p+1}^{\infty} [n]_q a_n z^{p-1} \quad , \quad z \in U, \tag{1.6}$$

where

$$[p]_q := \frac{1-q^p}{1-q} = 1 + q + q^2 + \cdots + q^{p-1},$$

and

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Also $[p]_q \to p$ and $[n]_q \to n$ as $q \to \overline{1}$. So we conclude that

$$\lim_{q \to \overline{1}} D_q f(z) = f^{'}(z) \quad , \quad z \in U,$$

see also [13].

Waggas and Rafid defined the Rafid -operator of a function $f(z)=z-\sum_{n=2}^{+\infty}a_nz^n$ by

$$R_\mu^\theta(f(z))=z-\sum_{n=2}^{+\infty}\frac{(1-\mu)^{n-1}\Gamma(\theta,n)}{\Gamma(\theta+1)}a_nz^n. \tag{1.7}$$

See for instance, [2, 3, 4]).
By using Rafid and q-derivative operators, we define the $R_\mu^\theta D_q(f(z))$ for a function $f\in T(p)$ as follows:

Definition 1.1. The Rafid -operator of $f\in T(p)$, is denoted by $R_\mu^\theta D_q$ and defined as following:

$$R_\mu^\theta D_q(f(z)) = \frac{z}{[p]_q(1-\mu)^{p+\theta+1}\Gamma(p+\theta+1)}\int_0^{+\infty}t^{\theta-1}e^{-(\frac{t}{1-\mu})}D_q(f(zt))dt \tag{1.8}$$

Then it is easy to deduce the series representation of the function $R_\mu^\theta(f(z))$ as following:

$$\begin{aligned}R_\mu^\theta D_q(f(z)) &= z^p-\sum_{n=p+1}^{+\infty}\frac{[n]_q(1-\mu)^{n-p}\Gamma(n+\theta+1)}{[p]_q\Gamma(p+\theta+1)}a_nz^n\\ &= z^p-\sum_{n=p+1}^{+\infty}M(n,p,q,\mu,\theta)a_nz^n\end{aligned} \tag{1.9}$$

where

$$M(n,p,q,\mu,\theta)=\frac{[n]_q(1-\mu)^{n-p}\Gamma(n+\theta+1)}{[p]_q\Gamma(p+\theta+1)}. \tag{1.10}$$

We now define a new subclass $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$ of analytic functions of $T(p)$ by using the operators Rafid and q-derivative. Let $f(z)\in T(p)$ is said to be in the class $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$ if and only if it satisfies the inequality:

$$\left|\frac{\lambda z^2(R_\mu^\theta(D_q(f*g)(z)))''+z(R_\mu^\theta(D_q(f*g)(z)))'}{z(R_\mu^\theta(D_q(f*g)(z)))'+(1-\lambda)(R_\mu^\theta(D_q(f*g)(z)))}-(1-\beta)\right|\le\alpha. \tag{1.11}$$

Here, $0<q<1, 0\le\lambda<1, 0\le\alpha\le1, 0\le\mu<1, 0\le\theta\le1$ and $\beta<1$.

2. Main Results

Unless otherwise mentioned, we suppose throughout this paper that $0<q<1, 0\le\lambda<1, 0\le\alpha\le1, 0\le\mu<1, 0\le\theta\le1$ and $\beta<1$. First we state coefficient estimates on the class $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$.

Theorem 2.1. *Let $f(z) \in T(p)$, then $f(z) \in T_{p,q}R(\lambda, \alpha, \beta, \mu, \theta)$ if and only if*

$$\sum_{n=p+1}^{+\infty} \Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big] M(n,p,q,\mu,\theta)a_n b_n \leq 1-2\lambda. \tag{2.1}$$

Proof. Suppose $f(z)$ difined by Equation(1.3) *and $f(z) \in T_{p,q}R(\lambda, \alpha, \beta, \mu, \theta)$, then Equation* (1.11) *holds true, we have*

$$\Big| \frac{[(2-\beta)\lambda + 2\beta - 1]z^p}{(2-\lambda)z^p - (n-\lambda+1)\sum_{n=p+1}^{+\infty} M(n,p,q,\mu,\theta)a_n b_n z^n}$$
$$- \frac{[n(1-n)+\beta-1]\lambda + 1 - (1+n)\beta]\sum_{n=p+1}^{+\infty} n(n-1)M(n,p,q,\mu,\theta)a_n b_n z^n}{(2-\lambda)z^p - (n-\lambda+1)\sum_{n=p+1}^{+\infty} M(n,p,q,\mu,\theta)a_n b_n z^n} \Big| < \alpha.$$

Since $Re(z) \leq |z|$ for all z,

$$Re\Big\{ \frac{[(2-\beta)\lambda + 2\beta - 1]z^p}{(2-\lambda)z^p - (n-\lambda+1)\sum_{n=p+1}^{+\infty} M(n,p,q,\mu,\theta)a_n b_n z^n}$$
$$- \frac{[n(1-n)+\beta-1]\lambda + 1 - (1+n)\beta]\sum_{n=p+1}^{+\infty} n(n-1)M(n,p,q,\mu,\theta)a_n b_n z^n}{(2-\lambda)z^p - (n-\lambda+1)\sum_{n=p+1}^{+\infty} M(n,p,q,\mu,\theta)a_n b_n z^n} \Big\} < \alpha.$$

By letting $z \to \overline{1}$ through real values, we have

$$\sum_{n=p+1}^{+\infty} \Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big] M(n,p,q,\mu,\theta)a_n b_n \leq 1-2\lambda.$$

Conversely, let Equation (2.1) *holds true, it is enough to show that*

$$X(f) = \Big|\lambda z^2(R_\mu^\theta(D_q(f*g)(z)))'' + z(R_\mu^\theta(D_q(f*g)(z)))'$$
$$-(1-\beta)[z(R_\mu^\theta(D_q(f*g)(z)))' + (1-\lambda)(R_\mu^\theta(D_q(f*g)(z)))]\Big|$$
$$-\alpha\Big|z(R_\mu^\theta(D_q(f*g)(z)))' + (1-\lambda)(R_\mu^\theta(D_q(f*g)(z)))\Big| \leq 0$$

But for $0 < |z| = r < 1$ we have

$$X(f)=\Big|[(2-\beta\lambda[z^p - \textstyle\sum_{n=p+1}^{+\infty} n(n-1)M(n,p,q,\mu,\theta)a_n b_n z^n]$$
$$+z^p - \textstyle\sum_{n=p+1}^{+\infty} nM(n,p,q,\mu,\theta)a_n b_n z^n$$
$$-(1-\beta)([z^p - \textstyle\sum_{n=p+1}^{+\infty} nM(n,p,q,\mu,\theta)a_n b_n z^n]$$
$$+(1-\lambda)[z^p - \textstyle\sum_{n=p+1}^{+\infty} nM(n,p,q,\mu,\theta)a_n b_n z^n])\Big|$$
$$-\alpha([z^p - \textstyle\sum_{n=p+1}^{+\infty} nM(n,p,q,\mu,\theta)a_n b_n z^n]$$
$$+(1-\lambda)[z^p - \textstyle\sum_{n=p+1}^{+\infty} nM(n,p,q,\mu,\theta)a_n b_n z^n])\Big|$$
$$\leq \textstyle\sum_{n=p+1}^{+\infty} \Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big] M(n,p,q,\mu,\theta)|a_n||b_n|r^n$$
$$-(1-2\lambda).$$

Since the above inequality holds for all r $(0 < r < 1)$, *by letting* $r \to \overline{1}$ *and using Equation*(2.1) *we obtain* $X(f) \leq 0$. *This completes the proof.* □

Corollary 2.1. *If function* $f(z)$ *of the form Equation* (1.3) *belongs to* $T_{p,q}R(\lambda, \alpha, \beta, \mu, \theta)$ *then*

$$a_n \leq \frac{1 - 2\lambda}{\Big[(n(n-1) - (\alpha+\beta) + 1)\lambda + (n+1)(\alpha+\beta) - 1\Big] M(n, p, q, \mu, \theta) b_n}$$

where

$$M(n, p, q, \mu, \theta) = \frac{[n]_q (1-\mu)^{n-p} \Gamma(n + \theta + 1)}{[p]_q \Gamma(p + \theta + 1)}, \quad n \geq p + 1.$$

With the equality for the function

$$f(z) = z^p - \frac{1 - 2\lambda}{\Big[(n(n-1) - (\alpha+\beta) + 1)\lambda + (n+1)(\alpha+\beta) - 1\Big] M(n, p, q, \mu, \theta) b_n} z^p$$

Next we obtain extreme points and convex linear combination property for $f(z)$ belongs to $T_{p,q}R(\lambda, \alpha, \beta, \mu, \theta)$.

Theorem 2.2. *The function* $f(z)$ *of the form Equation* (1.3) *belongs to* $T_{p,q}R(\lambda, \alpha, \beta, \mu, \theta)$ *if and only if it can be expressed by*

$$f(z) = \sigma_1 f_1(z) + \sum_{n=p+1}^{\infty} \sigma_n f_n(z) \ , \qquad \sigma_n \geq 1 \ , \qquad \sigma_1 + \sum_{n=p+1}^{\infty} \sigma_n = 1$$

where

$$f_1(z) = z^p,$$

$$f_n(z) = \frac{1 - 2\lambda}{\Big[(n(n-1) - (\alpha+\beta) + 1)\lambda + (n+1)(\alpha+\beta) - 1\Big] M(n, p, q, \mu, \theta) b_n} z^k, \qquad (n \geq p + 1).$$

Proof. Let

$$\begin{aligned} f(z) &= \sigma_1 f_1(z) + \textstyle\sum_{n=p+1}^{\infty} \sigma_n f_n(z) \\ &= \sigma_1 f_1(z) + \textstyle\sum_{n=p+1}^{\infty} \sigma_n \left[z^n - \frac{1-2\lambda}{\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big] M(n,p,q,\mu,\theta) b_n} \right] z^n \\ &= z^p - \textstyle\sum_{n=p+1}^{\infty} \frac{1-2\lambda}{\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big] M(n,p,q,\mu,\theta) b_n} \sigma_n z^n. \end{aligned}$$

Now apply Theorem 2.1 to conclude that $f(z) \in T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$. Conversely, if $f(z)$ given by Equation (1.3) belongs to $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$, by letting

$$\sigma_1 = 1 - \sum_{n=p+1}^{+\infty} \sigma_n,$$

where

$$\sigma_k = \frac{\Big[(n(n-1)-(\alpha+\beta)+1)\lambda + (n+1)(\alpha+\beta) - 1\Big] M(n,p,q,\mu,\theta) b_n}{1-2\lambda} a_n \qquad (n \geq p+1).$$

we conclude the required result. □

Theorem 2.3. *Let for $t = 1, 2, \cdots, k$, $f_t(z) = z^p - \sum_{n=p+1}^{+\infty} a_{n,t} z^n$ belongs to $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$, then $F(z) = \sum_{t=1}^{k} \sigma_t f_t(z)$ is also in the same class, where $\sum_{t=1}^{k} \sigma_t = 1$. Hence $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$ is a convex set.*

Proof. *According to Theorem 2.1 for every $t = 1, 2, \cdots, k$ we have*

$$\sum_{n=p+1}^{+\infty} \Big[(n(n-1)-(\alpha+\beta)+1)\lambda + (n+1)(\alpha+\beta) - 1\Big] M(n,p,q,\mu,\theta) a_{n,t} b_n \leq 1 - 2\lambda.$$

But

$$\begin{aligned} F(z) &= \sum_{t=1}^{k} \sigma_t f_t(z) \\ &= \sum_{t=1}^{k} \sigma_t \left(z^p - \sum_{n=p+1}^{\infty} a_{n,t} z^n \right) \\ &= z^p \sum_{t=1}^{k} \sigma_t - \sum_{n=p+1}^{\infty} \left(\sum_{t=1}^{k} \sigma_t a_{n,t} \right) z^n \\ &= z^p - \sum_{n=p+1}^{\infty} \Big(\sum_{t=1}^{k} \sigma_t a_{n,t} \Big) z^n. \end{aligned}$$

Since

$$\begin{aligned}
&\sum_{n=p+1}^{+\infty}\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]\\
&\times M(n,p,q,\mu,\theta)a_{n,t}b_n\big(\sum_{n=1}^{m}\sigma_n a_{k,n}\big)\\
=\;&\sum_{n=1}^{m}\sigma_n\Big(\sum_{n=p+1}^{+\infty}\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]\\
&\times M(n,p,q,\mu,\theta)a_{n,t}b_n\\
\le\;&\sum_{t=1}^{k}\sigma_t\big(1-2\lambda\big)=(1-2\lambda)\sum_{t=1}^{k}\sigma_t=1-2\lambda,
\end{aligned}$$

by Theorem 2.1 the proof is complete. □

3. Radii of close−to−convexity, starlikeness and convexity

In this section we obtain radii of close−to−convexity, starlikeness , convexity and investigate about partial sum property.

In the proof of next theorem, we need the Alexander's Theorem. This theorem states that if f is an analytic function in the unit disk and normalized by $f(0)=f^{'}(0)-1=0$, then $f(z)$ is convex if and only if $zf^{'}(z)$ is starlike.

Theorem 3.1. *Let $f(z)$ of the form Equation (1.3) belongs to $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$ then*

(i) $f(z)$ is p−valently close−to−convex of order γ in $|z|<R_1$, where $0\le\gamma<p$ and

$$R_1=\inf_n\Big\{\frac{(p-\gamma)\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_nb_n}{n(1-2\lambda)}\Big\}^{\frac{1}{n-p}},$$

(ii) $f(z)$ is p−valently starlike of order γ in $|z|<R_2$, where $0\le\gamma<p$ and

$$R_2=\inf_n\Big\{\frac{(p-\gamma)\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_nb_n}{(n-\gamma)(1-2\lambda)}\Big\}^{\frac{1}{n-p}},$$

(iii) $f(z)$ is p−valently convex of order γ in $|z|<R_3$, where $0\le\gamma<p$ and

$$R_3=\inf_n\Big\{\frac{p(p-\gamma)\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_nb_n}{n(n-\gamma)(1-2\lambda)}\Big\}^{\frac{1}{n-p}}.$$

Proof. *(i) For close−to−convexity it is enough to show that*

$$\Big|\tfrac{zf^{'}}{z^{p-1}}-p\Big|<p-\gamma,$$

but

$$\left|\frac{zf'}{z^{p-1}} - p\right| = \left|\frac{pz^{p-1} - \sum_{n=p+1}^{+\infty} na_n|z|^n - pz^{p-1}}{z^{p-1}}\right| \leq \sum_{n=p+1}^{+\infty} na_n|z|^{n-p} \leq p-\gamma,$$

or $\sum_{n=p+1}^{+\infty} \frac{n}{p-\gamma} a_n|z|^{n-p} \leq 1$. *By using Equation* (2.1) *we obtain*

$$\begin{aligned}&\sum_{n=p+1}^{+\infty} \frac{n}{p-\gamma} a_n|z|^{n-p}\\ &\leq \sum_{k=1}^{+\infty} \frac{n(1-2\lambda)|z|^{n-p}}{(p-\gamma)\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]}\\ &\times \frac{1}{M(n,p,q,\mu,\theta)a_nb_n} \leq 1.\end{aligned}$$

So, it is enough to suppose

$$\begin{aligned}&|z|^{n-p}\\ &\leq \frac{(p-\gamma)\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_nb_n}{n(1-2\lambda)},\end{aligned}$$

which completes the case (i).

(ii) *For starlikeness it is enough to show that* $\left|\frac{zf'}{f} - p\right| < p-\gamma$.
But

$$\left|\frac{zf'}{f} - p\right| = \left|\frac{\sum_{n=p+1}^{+\infty}(n-p)a_nz^n}{z^p - \sum_{n=p+1}^{+\infty} a_nz^n}\right| \leq \frac{\sum_{n=p+1}^{+\infty}(n-p)a_n|z|^{n-p}}{1-\sum_{n=p+1}^{+\infty} a_n|z|^{n-p}} \leq p-\gamma.$$

Therefore,

$$\sum_{n=p+1}^{+\infty}(n-p)a_n|z|^{n-p} \leq (p-\gamma)(1-\sum_{n=p+1}^{+\infty} a_n|z|^{n-p}),$$

or

$$\sum_{n=p+1}^{+\infty}\frac{n-\gamma}{p-\gamma}a_n|z|^{n-p} \leq 1.$$

Now by Equation (2.1), *we obtain*

$$\begin{aligned}&\sum_{n=p+1}^{+\infty}\frac{n-\gamma}{p-\gamma}a_n|z|^{n-p}\\ &\leq \sum_{n=p+1}^{+\infty} \frac{(n-\gamma)(1-2\lambda)|z|^{n-p}}{(p-\gamma)\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]}\\ &\times \frac{1}{M(n,p,q,\mu,\theta)a_nb_n} \leq 1.\end{aligned}$$

So, it is enough to suppose

$$|z|^{n-p} \le \frac{(p-\gamma)\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_nb_n}{(n-\gamma)(1-2\lambda)}.$$

Hence we get the required result.

(iii) For convexity, by Alexander's Theorem and by applying an easy calculation, we reach the required result. Hence the result. □

Theorem 3.2. *The class $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$ is a convex set.*

Proof. Let $f(z) = z^p - \sum_{n=p+1}^{+\infty} a_nz^n$ and $g(z) = z^p - \sum_{n=p+1}^{+\infty} b_nz^n$, be in the class $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$. For $t \in (0,1)$, it is enough to show that the function $h(z) = (1-t)f(z) + tg(z)$ is in the class $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$. Since $h(z) = z^p - \sum_{n=p+1}^{+\infty}((1-t)a_n + tb_n)z^n$,

$$\sum_{n=p+1}^{+\infty}\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)((1-t)a_n+tb_n)b_n \le (1-2\lambda)$$

and so $h(z) \in T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$. □

Corollary 3.1. *Let $f_k(z)$, $1 \le k \le m$, defined by $f_k(z) = z^p - \sum_{n=p+1}^{+\infty} a_{n,k}z^n$ be in the class $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$, then the function $F(z) = \sum_{k=1}^{m} c_kf_k(z)$ is also in $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$, where $\sum_{k=1}^{m} c_k = 1$.*

4. Integral operators on $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$

In this section we investigate properties of functions in the class $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$, involving the familiar operator $F_c(z)$.

Theorem 4.1. *If $f(z) = z^p - \sum_{n=p+1}^{+\infty} a_nz^n$ belongs to $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$, then the function $F_c(z)$ defined by $F_c(z) = \frac{c+p}{z^c}\int_0^1 t^cf(tz)dt$, $c \ge 1$, is also in $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$.*

Proof. Since $f(z)$ belong to $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$,

$$\begin{aligned} F_c(z) &= \frac{p+c}{z^c}\int_0^z t^{c-1}[z^p - \sum_{n=p+1}^{+\infty} a_n t^n]dt \quad , \quad c>1, \\ &= \frac{p+c}{z^c}\int_0^z [t^{p+c-1} - \sum_{n=p+1}^{+\infty} a_n t^{n+c-1}]dt \\ &= \frac{p+c}{z^c}[\frac{1}{p+c}t^{p+c} - \sum_{n=p+1}^{+\infty} a_n \frac{1}{n+c}t^{n+c}]_0^z \\ &= \frac{p+c}{z^c}[\frac{1}{p+c}z^{p+c} - \sum_{n=p+1}^{+\infty} a_n \frac{1}{n+c}z^{n+c}] \\ &= z^p - \sum_{n=p+1}^{+\infty} \frac{p+c}{n+c} a_n z^n \end{aligned}$$

Since $\frac{p+c}{n+c} < 1$,

$$\begin{aligned} &\sum_{n=p+1}^{+\infty} \frac{p+c}{n+c}\Big[(n(n-1)-(\alpha+\beta)+1)\lambda + (n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_n b_n \\ &\le \sum_{n=p+1}^{+\infty} \Big[(n(n-1)-(\alpha+\beta)+1)\lambda + (n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_n b_n \\ &\le (1-2\lambda). \end{aligned}$$

Hence the result. □

Corollary 4.1. *If $f(z) \in T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$ and $F_c(z)$ is defined as $F_c(z) = c\int_0^1 v^c f(vz)\,dv$, $c \ge 1$. Then $F_c(z) \in T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$.*

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