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ON GENERALIZED RELATIVE COMMUTATIVITY DEGREE OF FINITE MOUFANG LOOP

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Abstract. For a given element g of a finite group G, the probablility that the commutator of randomly choosen pair elements in G equals g is the relative commutativity degree of g.

In this paper we are interested in studying the relative commutativity degree of the Dihedral group of order $2n$ and the Quaternion group of order 2^n for any $n \geq 3$ and we examine the relative commutativity degree of infinite class of the Moufang Loops of Chein type, $M(G, 2)$.
Keywords. Relative commutativity degree, Moufang loop.

1. Introduction

Every algebraic structure here is non-commutative. A quasi-group is a non-empty set with a binary operation such that for every three elements x, y and z of that, the equation $xy = z$ has a unique solution in this set, whenever two of the three element are specified. A quasi-group with a neutral element is called a loop and following [2, 6, 7, 8] one may see the definition of Moufang loop satisfying four tantamount relators. These loops are of interest because of their appearance in the projective geometry as planes and even they are non-associative, they retain many properties of the groups. During the study of these loops an interesting class introduced by Chein [3, 4, 5] where, for a finite group G and a new element

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$u, (u \notin G)$, the loop $M(G,2)$ is defined as $M(G,2) = G \cup Gu$ such that the binary operation in $M(G,2)$ is defined by:

$$\begin{array}{ll} goh = gh, & \text{if } g,h \in G, \\ go(hu) = (hg)u, & \text{if } g \in G, \quad hu \in Gu, \\ (gu)oh = (gh^{-1})u, & \text{if } gu \in Gu, \quad h \in G, \\ (gu)o(hu) = h^{-1}g, & \text{if } gu, hu \in Gu. \end{array}$$

These loops are studied for their finiteness property in [1, 2]. It is obvious that $M(G,2)$ is non-associative if and only if the group G is non-abelian. Our next preliminary is the definition of generalized relative commutativity degree. Following [1], for an integer $n \geq 2$, the probability that for two elements x and y of an algebraic structure, $x^n y = yx^n$ holds is called the n^{th}-commutativity degree of the algebraic structure and denoted this probability by $P_n(S)$, for an algebraic structure S.

In what follows we examine $Pr_g(M)$ and $Pr_g(G)$, where for a given group G we give a general relationship between them with $M = M(G,2)$. Since then we give explicit descriptions for $Pr_g(M)$ in two special cases when G is one of the dihedral group of order $2m$ and the quaternion group of order 2^m, for every $m \geq 3$. Note that these groups are non-abelian and then the loop $M = M(G,2)$ is non-associative.

2. Main results

For a given element $g \in G$ we define the g-relative commutativity set of G as

$$C_g(G) = \{(x,y) \mid x,y \in G, \quad xyx^{-1}y^{-1} = g\}.$$

This set will be used in computation of $Pr_g(G)$ and we have

$$Pr_g(G) = \frac{|C_g(G)|}{|G|^2}.$$

Also we use the presentations $< a,b \mid a^n = b^2 = (ab)^2 = 1 >$ and $< a,b \mid a^{2^{n-1}} = 1, \quad b^2 = a^{2^{n-2}}, \quad (ab)^2 = 1 >$ for the groups D_{2n} and Q_{2^n}. Our main results are:

Lemma 2.1. *For even values of $n \geq 4$, if $a, b \in D_{2n}$ then*

(i) $[a^i, b] = g$ *if and only if* $[a^i, a^j b] = g$,

(ii) $[a^i, b] = g$ *if and only if* $[a^{\frac{n}{2}+i}, b] = g$,

(iii) $[b, a^i] = g$ *if and only if* $[a^j b, a^i] = g$,

(iv) $[b, a^i] = g$ *if and only if* $[b, a^{\frac{n}{2}+i}] = g$,

(v) $[b, a^i b] = g$ *if and only if* $[b, a^{\frac{n}{2}+i} b] = g$,

(vi) $[a^i b, a^j b] = g$ *if and only if* $[a^{i+1} b, a^{j+1} b] = g$, *where* $g \in D_{2n}$ *and* $(1 \leq i, j \leq n-1)$.

Proof. Let $n \geq 4$ be an even integer. Then by presentation of the group D_{2n} we get:

(i) :

$$\begin{aligned}
[a^i, b] = g &\iff a^i b a^{-i} b^{-1} = g \\
&\iff a^{-2i} b^2 = g \\
&\iff a^{2i} a^{j-j} b^2 = g \\
&\iff a^{i+j} b a^{-i+j} b = g \\
&\iff a^i a^j b a^{-i} a^j b = g \\
&\iff a^i a^j b a^{-i} b^{-1} a^{-j} = g \\
&\iff [a^i, a^j b] = g.
\end{aligned}$$

(ii) :

$$\begin{aligned}
[a^i, b] = g &\iff a^i b a^{-i} b^{-1} = g \\
&\iff a^{2i} b^2 = g \\
&\iff a^{n+2i} b^2 = g \\
&\iff a^{\frac{n}{2}+i} b a^{-\frac{n}{2}-i} b^{-1} = g \\
&\iff [a^{\frac{n}{2}+i}, b] = g.
\end{aligned}$$

The proof in other cases is similar and we omit it. □

Corollary 2.1. *Let* $n \geq 4$ *be an even integer and* $a, b \in D_{2n}$. *For every integers* $0 \leq i, j \leq n-1$ *if* $[a^i b, a^j b] = g$ *then* $g \in \{1, a^2, a^4, \ldots, a^{n-2}\}$.

Theorem 2.1. *For even values of* $n > 3$ *if* $g \in D_{2n}, (g \neq 1)$ *then*

$$Pr_g(D_{2n}) = \frac{3}{2n}$$

where, $g = a^2, a^4, \ldots, a^{n-2}$.

Proof. Let n be an even integer and $G = D_{2n} = A \cup B$ where, $A = \{1, a, \ldots, a^{n-1}\}$ and $B = \{b, ab, \ldots, a^{n-1}b\}$. Clearly, $[a^i, a^j] = 1$, now if $[a^i, b] = g$ then by using $[i]$ in Lemma 2.1 we get there are n pairs $(x, y) \in A \times B$ such that $[x, y] = g$, also by $[ii]$ in Lemma 2.1 we get there are n pairs $(x, y) \in A \times B$ such that $[x, y] = g$. Also, by $[iii]$ and $[iv]$ in Lemma 2.1 we heve there are $2n$ pairs $(x, y) \in B \times A$ such that $[x, y] = g$ and by $[v]$ and $[vi]$ in Lemma 2.1 there are $2n$ pairs $(x, y) \in B \times B$ such that $[x, y] = g$.

Consequently,

$$|C_g(D_{2n})| = 2n + 2n + 2n = 6n,$$

and

$$Pr_g(D_{2n}) = \frac{|C_g(D_{2n})|}{|D_{2n}|^2} = \frac{6n}{4n^2} = \frac{3}{2n}.$$

Lemma 2.2. *For odd values of $n \geq 3$, if $a, b \in D_{2n}$ then*

(i) $[a^i, b] = g$ *if and only if* $[a^i, a^j b] = g$,

(ii) $[b, a^i] = g$ *if and only if* $[a^j b, a^i] = g$,

(iii) $[b, a^i b] = g$ *if and only if* $[b, a^{\frac{n}{2}+i} b] = g$,

where, $g \in D_{2n}$ *and* $(1 \leq i, j \leq n-1)$.

Proof. The proof is similar to the proof of Lemma 2.1. $\square$

Corollary 2.2. *Let $n \geq 4$ be an odd integer and $a, b \in D_{2n}$. For every integers $0 \leq i, j \leq n-1$ if $[a^i b, a^j b] = g$ then $g \in \{1, a, a^2, \ldots, a^{n-1}\}$.*

Theorem 2.2. *For odd values of $n > 3$ if $g \in D_{2n}, (g \neq 1)$ then*

$$Pr_g(D_{2n}) = \frac{3}{4n}$$

where, $g = a, a^2, \ldots, a^{n-1}$.

Proof. Let n be an odd integer and $G = D_{2n} = A \cup B$ where, $A = \{1, a, \ldots, a^{n-1}\}$ and $B = \{b, ab, \ldots, a^{n-1}b\}$. Clearly, $[a^i, a^j] = 1$, now if $[a^i, b] = g$ then by using $[i]$ in Lemma 2.2 we get there are n pairs $(x, y) \in A \times B$ such that $[x, y] = g$. Also, by $[ii]$ in Lemma 2.2 we heve there are n pairs $(x, y) \in B \times A$ such that $[x, y] = g$ and by $[iii]$ in Lemma 2.2 there are n pairs $(x, y) \in B \times B$ such that $[x, y] = g$.

Consequently,

$$|C_g(D_{2n})| = n + n + n = 3n,$$

and

$$Pr_g(D_{2n}) = \frac{|C_g(D_{2n})|}{|D_{2n}|^2} = \frac{3n}{4n^2} = \frac{3}{4n}.$$

$\square$

Lemma 2.3. *For a given element $g \in Q_{2^n}$ and any values of $n \geq 3$, if $a, b \in Q_{2^n}$ and $(1 \leq i, j \leq n-1)$ then*

(i) $[a^i, b] = g$ *if and only if* $[a^i, a^j b] = g$,

(ii) $[a^i, b] = g$ *if and only if* $[a^{\frac{n}{2}+i}, b] = g$,

(iii) $[b, a^i] = g$ *if and only if* $[a^j b, a^i] = g$,

(iv) $[b, a^i] = g$ *if and only if* $[b, a^{\frac{n}{2}+i}] = g$,

(v) $[b, a^i b] = g$ *if and only if* $[b, a^{\frac{n}{2}+i} b] = g$,

(vi) $[a^i b, a^j b] = g$ *if and only if* $[a^{i+1} b, a^{j+1} b] = g$.

Corollary 2.3. *Let $n \geq 3$ be a positive integer and $a, b \in Q_{2^n}$. For every $0 \leq i, j \leq 2^{n-1} - 1$, if $[a^i b, a^j b] = g$ then $g \in \{1, a^2, a^4, \dots, a^{2^{n-1}-2}\}$.*

Theorem 2.3. *For any values of $n \geq 3$ if $g \in Q_{2^n}, (g \neq 1)$ then*

$$Pr_g(Q_{2^n}) = \frac{3}{2^n}$$

where, $g \in \{1, a^2, a^4, \dots, a^{2^{n-1}-2}\}$.

Proof. Let $n \geq 3$ be an even integer and $G = Q_{2^n} = A \cup B$, where $A = \{1, a, \dots, a^{n-1}\}$ and $B = \{b, ab, \dots, a^{n-1}b\}$. Clearly, $[a^i, a^j] = 1$, now if $[a^i, b] = g$ then by using $[i, ii]$ in Lemma 2.3 we get there are 2^{n-1} pairs $(x, y) \in A \times B$ such that $[x, y] = g$. Also, by $[iii, iv]$ in Lemma 2.3 we heve there are 2^{n-1} pairs $(x, y) \in B \times A$ such that $[x, y] = g$ and by $[v, vi]$ in Lemma 2.3 there are 2^{n-1} pairs $(x, y) \in B \times B$ such that $[x, y] = g$. Consequently,

$$|C_g(Q_{2^n})| = 2(2^{n-1}) + 2(2^{n-1}) + 2(2^{n-1}) = 3(2^n),$$

and

$$Pr_g(Q_{2^n}) = \frac{|C_g(Q_{2^n})|}{|Q_{2^n}|^2} = \frac{3(2^n)}{(2^n)^2} = \frac{3}{2^n}.$$

□

Lemma 2.4. *Let G be a finite group of order n, $g \in G$ and $M(G, 2)$ be a finite Moufang loop of order $2n$. we have for all $x, y \in G$:*

(i) $((xu)oy)o((xu)^{-1}oy^{-1}) = g$ *if and only if* $y^{-2} = g$,

(ii) $((xu)o(yu))o((xu)^{-1}o(yu)^{-1}) = g$ *if and only if* $(x^{-1}y)^{-2} = g$.

Proof. By definition of the multiplication in $M(G, 2)$ clearly:

$$(i)\quad \begin{aligned} ((xu)oy)o((xu)^{-1}oy^{-1}) = g &\iff ((xy^{-1})u)o((xy)u) = g \\ &\iff y^{-1}x^{-1}xy^{-1} = g \\ &\iff y^{-2} = g. \end{aligned}$$

$$(ii)\quad \begin{aligned} ((xu)o(yu))o((xu)^{-1}o(yu)^{-1}) = g &\iff (y^{-1}x)o(y^{-1}x) = g \\ &\iff (y^{-1}x)^2 = g \\ &\iff (x^{-1}y)^{-2} = g. \end{aligned}$$

□

Proposition 2.1. *For a given integer $n \geq 2$ and a non-abelian group G,*

$$Pr_g(M) = \frac{1}{4}(Pr_g(G) + \frac{3N_g}{|G|}),$$

where N_g is the number of elements $y \in G$ such that $y^{-2} = g$.

Proof. Let $g \in M = M(G,2)$ and $C_g(M) = \{(x,y) \mid x,y \in G, \quad xyx^{-1}y^{-1} = g\}$. We first note that the multiplication table of the Moufang loop $M(G,2)$ will be as follows:

o	G	Gu
G	$G*G$	$G*Gu$
Gu	$Gu*G$	$Gu*Gu$

Since $Pr_g(M) = \frac{|C_g(M)|}{|M|^2}$. Thus it is sufficient to enumerate $|C_g(M)|$. For every $(x,y) \in M$ we have the following four cases:
Case1: Both $x,y \in G$. Then there are $|C_g(G)|$ distinct ordered pairs $(x,y) \in C_g(M)$ in this case.
Case2: $x \in Gu$ and $y \in G$. Then $x = x_1u$ where $x_1 \in G$. By (i) of Lemma 2.1 we conclude that $y^{-2} = g$, so there are precisely $N_g|Gu| = N_g|G|$ pairs $(x,y) \in C_g(M)$ of this type.
Case3: $x \in G$ and $y \in Gu$. Then $y = y_1u$ where $y_1 \in G$. By using (i) of Lemma 2.1 we get there are $N_g|G|$ distinct pairs in $C_g(M)$ of this type.
Case4: Both $x,y \in Gu$. Then $x = x_1u$ and $y = y_1u$ where $x_1, y_1 \in G$. Using (ii) of Lemma 2.1 we get there are $N_g|G|$ distinct pairs in $C_g(M)$ such that $(x^{-1}y)^{-2} = g$.

Consequently,

$$|C_g(M)| = |C_g(M)| + 3N_g|G|,$$

and so,

$$Pr_g(M) = \frac{|C_g(M)| + 3N_g|G|}{(2|G|)^2} = \frac{1}{4}(Pr_g(G) + \frac{3N_g}{|G|}).$$

□

Proposition 2.2. *Let $M = M(D_{2n},2)$, $n \geq 3$ is a positive integer. Then,*

$$Pr_g(M) = \begin{cases} \frac{3}{8n}(N_g+1), & n \textit{ is even,} \\ \frac{3}{16n}(2N_g+1), & n \textit{ is odd,} \end{cases}$$

where, N_g is the number of elements $y \in G$ such that $y^{-2} = g$.

Proof. By using Proposition 2.1 and Theorems 2.2 and 2.3 we get

$$Pr_g(M) = \frac{1}{4}(Pr_g(G) + \frac{3N_g}{|G|}) = \begin{cases} \frac{1}{4}(\frac{3}{2n} + \frac{3N_g}{2n}) = \frac{3}{8n}(N_g+1), & n \text{ is even,} \\ \frac{1}{4}(\frac{3}{4n} + \frac{3N_g}{2n}) = \frac{3}{16n}(2N_g+1), & n \text{ is odd,} \end{cases}$$

□

Corollary 2.4. *Let $M = M(D_{2n},2)$, $n \geq 3$ is a positive integer. Then,*

$$Pr_g(M) \leq \frac{15}{48}.$$

Proposition 2.3. *Let $M = M(Q_{2^n}, 2)$, $n \geq$ is an integer . Then*

$$Pr_g(M) = \frac{3}{2^{n+2}}(N_g + 1),$$

where, N_g is the number of elements $y \in G$ such that $y^{-2} = g$.

Proof. The proofs follows by considering the Proposition 2.1:

$$Pr_g(M) = \frac{1}{4}(\frac{3}{2^n} + \frac{3N_g}{2^n}) = \frac{3}{2^{n+2}}(N_g + 1).$$

□

Corollary 2.5. *Let $M = M(Q_{2^n}, 2)$, $n \geq 3$ is an integer . Then*

$$Pr_g(M) \leq \frac{3}{16}.$$

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