PROCESSING OF INDEPENDENT MARKOV-CHAINS

by

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Introduction

This paper is devoted to study processing of independent homogeneous Markov-chains with priority.

In § 1. we formulate the way of processing.

In § 2. we prove the following assertion: if the given sequences form Markov-chains, then the sequence of the processed blocks can be considered as a homogeneous, ergodic Markov-chain.

We deal with characterizing the performance of the processing algorithm (§ 3.). We determine the stationary initial distribution for the sequences defined in § 2. in some special cases (§ 4.).

In § 5. the speed of processing is defined and computed for the previous special cases.

Our work has some practical importance e.g. in computer performance analysis, more precisely in modelling of multiprogrammed computers with one processor and interleaved memory [1, 2, 3]. In this case the programs are modelled by sequences (the program with the greatest priority by the first sequence etc.), the chosen performance measure corresponds to the average number of executed operations in a time unit, and the number of states of the Markov-chains is equal to the number of interleaved memory moduls. The processing algorithm characterizes some features of processing of programs with fixed priority-list.

§ 1. Processing algorithm

We consider some sequences consisting of positive integers and process these sequences. The sequences have priority: the first sequence the highest one etc.

The processing proceeds in discrete points of time. The processing algorithm in every point of time processes the maximal possible number of the first elements from the first sequence, and also so many of the first elements of the second, ..., last sequence, as possible.

The processing is controlled by the so called stopping sequences. If the algorithm finds a stopping sequence at the beginning of the first sequence, then it stops the processing for the given point of time: the elements of the stopping sequence without its last element will be processed from the first sequence. After finding a stopping sequence its completion is permitted to an other stopping sequence using the beginning elements of the further sequences. At the completion the order-preserving is requested for every sequence separately.

Now we give the definition of the set of stopping sequences, the definition of the order-preserving completion of a sequence by an other sequence and the one of the processing algorithm.

Let $\mathcal{H}$ denote the set $\{1, \ldots, N\}$ $(N \ge 2)$ and let

$$f_1^{(1)}, f_2^{(1)}, \ldots$$

(1.1)

$$\vdots$$

$$f_1^{(r)}, f_2^{(r)}, \ldots$$

be r $(r \ge 1)$ infinite sequences consisting of the elements of $\mathcal{H}$.

Definition 1. Definition of the stopping set

Let S be a set of sequences consisting of the elements of $\mathcal{H}$. If the following conditions hold, then S is called stopping set:

a) there exists such a positive integer $k \ge 3$ that every sequence belonging to S contains at least two and at most k elements;

b) let S_j be an ordered subset of S containing all the elements with length j of S. S_k is nonempty (S_j may be empty for $j = 2, \ldots, k-1$);

c) if $(e_1, e_2, \ldots, e_l) \in S_l$, then $(e_1, e_2, \ldots, e_h) \notin S_h$ for $h = 1, 2, \ldots, l-1$;

d) for every sequence $(g_1, g_2, \ldots, g_k)$ of elements of $\mathcal{H}$ there exists a $j(\le k)$ for which $(g_1, \ldots, g_j) \in S_j$;

e) for every pair of different elements i, j of $\mathcal{H}$ holds: if $(i, \ldots, i) \in S_m$, then $(j, i, \ldots, i) \in S_{m+1}$. ⊠

Definition 2. Definition of the order-preserving union of two sequences

Let $\alpha = (a_1, \ldots, a_p)$ and $\beta = (b_1, \ldots, b_q)$ be sequences of elements $\mathcal{H}$ with $p \ge 1$, $q \ge 0$. The sequence $\gamma = (c_1, \ldots, c_s)$ is called an order-preserving union of α and β (or an order-preserving addition of β to α) if the following four conditions hold (where $i(z)$ denotes the index of z in the sequence γ):

a) $\{c_1, \ldots, c_s\} = \{a_1, \ldots, a_p\} \cup \{b_1, \ldots, b_q\}$;

b) $c_s = a_p$;

c) if $u < v$, $a_u \in \alpha$, $a_v \in \alpha$, then $i(a_u) < i(a_v)$;

d) if $x < y$, $b_x \in \beta$, $b_y \in \beta$, then $i(b_x) < i(b_y)$. ⊠

Definition 3. Definition of the processing algorithm

Step 1. Processing proceeds in the points of time 1, 2, ... Let i be equal to 1. Go to Step 2.

Step 2. Let $k_i^{(1)}$ denote the greatest positive integer k_1 for which

(1.2) $$(f_1^{(1)}, f_2^{(1)}, \ldots, f_{k_1+1}^{(1)}) \in S.$$

Let s_1 denote the sequence $f_1^{(1)}, f_2^{(1)}, \ldots, f_{k_i^{(1)}+1}^{(1)}$. Go to Step 3.

Step 3. If $k_i^{(1)}, k_i^{(2)}, \ldots, k_i^{(t-1)}$ and s_{t-1} have been defined but $k_i^{(t)}$ has not been defined, then let $k_i^{(t)}$ be the greatest nonnegative integer k_i for which there exists an order-preserving union $\gamma \in S$ of the sequences s_{t-1} and $f_1^{(t)}, \ldots, f_{k_i}^{(t)}$. Let $s_t = s_{t-1}$, if $k_i^{(t)} = 0$, and let $s_t = \gamma$, if $k_i^{(t)} > 0$ (if there exist several such γ, then we choose the first occuring of them in the corresponding ordered S_j).

If $t < r$ then go to Step 3.

Step 4. In the i-th point of time we process the first $k_i^{(t)}$ $(t = 1, \ldots, r)$ elements of the t-th sequence. We omit these processed elements and decrease the index of the remaining elements by $k_i^{(t)}$ in the t-th sequence. Go to Step 5.

Step 5. We add 1 to i. Go to Step 2. ⊠

In connection with the definition we remark

a) in Step 2 due to the property *d)* of S there exists a positive integer $k_i^{(1)}$, and $2 \le k_i^{(1)} \le k$.

b) in Step 2 due to the property *c)* of S there exists only one $k_i^{(1)}$.

c) in Step 3 due to the definition of the order-preserving union there exists a nonnegative k_i — for example $k_i = 0$. Of course $k_i^{(t)} < k - \sum_{j=1}^{t-1} k_i^{(j)}$.

For characterizing the processing we register the processed and the first nonprocessed elements for every sequence and for every point of time. Therefore the processing in the i-th point of time is characterized by the array

$$
f_{1,\,i}^{(1)}, | \ldots, f_{k_i^{(1)},\,i}^{(1)}, \|f_{k_i^{(1)}+1,\,i}^{(1)}
$$

(1.3)
$$
\vdots
$$

$$
f_{1,\,i}^{(r)}, | \ldots, f_{k_i^{(r)},\,i}^{(r)}, \|f_{k_i^{(1)}+1,\,i}^{(r)} .
$$

If $k_i^{(t)} = 0$ $(t = 2, \ldots, r; i = 1, 2, \ldots)$ holds for given t and i, then we write $*$, $\|f_{1,\,i}^{(t)}$ in the t-th line of (1.3). For the sake of brevity let

(1.4) $$A_i^{(t)} = \langle f_{1,\,i}^{(t)}, | \ldots, f_{k_i^{(t)},\,i}^{(t)}, \|f_{k_i^{(t)}+1,\,i}^{(t)} \rangle$$

or

(1.5) $$A_i^{(t)} = \langle *, |f_{k_i^{(t)}+1,\,i}^{(t)} \rangle .$$

Using this notation the processing in the s-th point of time is characterized by

(1.6) $$\vartheta_s = (A_s^{(1)}, \ldots, A_s^{(r)}) .$$

Let $\mathcal{D}_r$ denote the set of all possible ϑ-s. Then the processing of a given array of type (1.1) can be described by the state sequence

(1.7) $$\vartheta_1, \vartheta_2, \ldots, \vartheta_s, \ldots \quad (\vartheta_s \in \mathcal{D}_r, s = 1, 2, \ldots) .$$

Let ϑ_s have the form (1.6), where

$$(1.8) \qquad A_s^{(t)} = \langle i_{1,s}^{(t)}, \, | \dots, i_{k_s^{(t)},s}^{(t)}, \, \| j_s^{(t)} \rangle$$

or

$$(1.9) \qquad A_s^{(t)} = \langle *, \, \| j_s^{(t)} \rangle.$$

Of course, if $\vartheta_s \in \mathcal{D}_r$, then

a) at least one order-preserving union of $(i_{1,s}^{(1)}, \, i_{2,s}^{(1)}, \dots, i_{k_s^{(1)},s}^{(1)}, \, j_s^{(1)})$

and $(i_{1,s}^{(2)}, \dots, i_{k_s^{(2)},s}^{(2)})$belongs to S_{n_t}, where $n_2 = k_s^{(1)} + k_s^{(2)}$. If there are several unions, then let δ_2 be the first occuring of them in S_{n_t};

b) at least one order-preserving union of δ_{t-1} and $(i_{1,s}^{(t)}, \dots, i_{k_s^{(t)},s}^{(t)})$ belongs to S_{n_t}, where $n_t = n_{t-1} + k_s^{(t)}$ (for $t = 3, \dots, r$). If there are several unions, then let δ_t be the first occuring of them in S_{n_t}.

There are such pairs $D_1, D_2 \in \mathcal{D}_r$, that cannot occur as consecutive states, i.e. for which $\vartheta_s = D_1$, $\vartheta_{s+1} = D_2$ is not possible.

Let $in \, \vartheta_s$ and $fin \, \vartheta_s$ denote the initial and final elements of ϑ_s, i.e.

$$(1.10) \qquad in \, \vartheta_s = (i_{1,s}^{(1)}, \, i_{1,s}^{(2)}, \, \dots, i_{1,s}^{(r)})$$

and

$$(1.11) \qquad fin \, \vartheta_s = (j_s^{(1)}, \, j_s^{(2)}, \, \dots, j_s^{(r)})$$

remarking that if $A_s^{(t)} = \langle *, \, \| j_s^{(t)} \rangle$, then in $in \, \vartheta_s$ we put $* j_t^{(t)}$ instead of $i_{1,s}^{(t)}$. It is clear, that ϑ_s and ϑ_{s+1} can occur as consecutive states (with other words: the transition $\vartheta_s \to \vartheta_{s+1}$ is realizable) if and only if $fin \, \vartheta_s = in \, \vartheta_{s+1}$. Deciding about this equality we do not take into account whether the components of $in \, \vartheta_{s+1}$ contain asterisk or not.

§ 2. Processing of independent Markov-chains

Let $\xi_i^{(l)}$ $(l = 1, \dots, r; \, i = 1, 2, \dots)$ be random variables with values from $\mathcal{H}$, for which the following conditions hold.

a) The sequences $\xi_i^{(l)}$ $(i = 1, 2, \dots)$ for every l form homogeneous Markov-chains with an initial distribution π_l and transition probability matrix Π_l, i.e.

$$\pi_l = (p(1, l), \dots, p(N, l)), \quad \text{where} \quad p(k, l) = P(\xi_1^{(l)} = k) \quad (k = 1, \dots, N)$$

and

$$\pi_l = \begin{bmatrix} p(1, 1, l) \dots p(1, N, l) \\ \vdots \\ p(N, 1, l) \dots p(N, N, l) \end{bmatrix}$$

where

$$p(x, y, l) = P(\xi_{r+1}^{(l)} = y \mid \xi_r^{(l)} = x) \quad (x, y = 1, \dots, N; \, \nu = 1, 2, \dots).$$

b) The sequences $\xi_i^{(t)}$ ($i = 1, 2, \ldots$) are mutually independent.

c) There exists such a j ($1 \leq j \leq N$) for which all of the conditional probabilities $p(x, j, l)$ ($x = 1, \ldots, N; l = 1, \ldots, r$) are positive.

Let $\min\limits_{x, l} p(x, j, l)$ be denoted by ε. Of course $\varepsilon > 0$.

Our aim is to characterize the processing of the array of random variables

(2.1)
$$\begin{array}{c} \xi_1^{(1)}, \; \xi_2^{(1)}, \; \ldots \\ \vdots \\ \xi_1^{(r)}, \; \xi_2^{(r)}, \; \ldots \end{array}$$

Let

(2.2)
$$\mathcal{B}_1, \, \mathcal{B}_2, \, \ldots, \, \mathcal{B}_s, \, \ldots$$

be random variables with values from $\mathcal{D}_r$. Let us suppose that this sequence is the state sequence of type (1.6) which describes the processing of the array (2.1).

Now we are going to show the homogeneity and ergodicity of this sequence.

Theorem 1. If the sequences $\xi_i^{(t)}$ ($t = 1, \ldots, r; i = 1, 2, \ldots$) form for every t mutually independent, homogeneous Markov-chains (that is under the conditions *a)* and *b)*), then the corresponding state sequence of type (2.2) represents a homogeneous Markov-chain. ⊠

Proof. Let us compute the probabilities

(2.3)
$$P(\mathcal{B}_1 = \vartheta) = q(\vartheta)$$

and

(2.4)
$$P(\mathcal{B}_{s+1} = \vartheta_{s+1} | \mathcal{B}_1 = \vartheta_1, \, \ldots, \, \mathcal{B}_s = \vartheta_s)$$

for every possible state ϑ and state-sequence $\vartheta_1, \, \ldots, \, \vartheta_{s+1}$.

We shall use the notations (1.8) and (1.9).

Let

(2.5)
$$\tau(A_1^{(t)}) = p(i_{1,1}^{(t)}, t) \, p(i_{1,1}^{(t)}, i_{2,1}^{(t)}, t) \, \cdots \, p(i_{k_1^{(t)}, t}^{(t)}, j_1^{(t)}, t)$$

if $A_1^{(t)}$ has the form (1.8) and let

(2.6)
$$\tau(A_1^{(t)}) = p(j^{(t)}, t)$$

if $A_1^{(t)}$ has the form (1.9).

It is clear that

(2.7)
$$q(\vartheta) = \prod_{t=1}^{r} \tau_t(A_1^{(t)}).$$

Let

(2.8)
$$\lambda^{(t)}(A_s^{(t)}) = p(i_{1,s}^{(t)}, i_{2,s}^{(t)}, t) \, \cdots \, p(i_{k_t, s}^{(t)}, j_s^{(t)} \, t),$$

if $A_s^{(t)}$ has the form (1.8) and let

$$(2.9) \qquad \qquad \lambda^{(t)}(A_s^{(t)}) = 1$$

if $A_s^{(t)}$ has the form (1.9).

Further let

$$(2.10) \qquad \qquad Q(\vartheta_s) = \prod_{t=1}^{r} \lambda^{(t)}(A_s^{(t)}) \, .$$

Since the sequences $\xi_i^{(t)}$ form homogeneous Markov-chains, therefore

$$(2.11) \qquad P(\mathcal{B}_1 = \vartheta_1, \ldots, \mathcal{B}_{s+1} = \vartheta_{s+1}) = q(\vartheta_1)\, Q(\vartheta_2) \cdots Q(\vartheta_{s+1})$$

if all of the transitions $\vartheta_1 \to \vartheta_2, \ldots, \vartheta_s \to \vartheta_{s+1}$ are realizable, and

$$(2.12) \qquad P(\mathcal{B}_1 = \vartheta_1, \ldots, \mathcal{B}_{s+1} = \vartheta_{s+1}) = 0$$

otherwise.

So we have proved that the sequence (2.2) is a homogeneous Markov-chain with initial distribution (2.7) and with the following transition probabilities:

$$(2.13) \qquad P(\mathcal{B}_{s+1} = \vartheta_{s+1} | \mathcal{B}_1 = \vartheta_1, \ldots, \mathcal{B}_s = \vartheta_s) = \begin{cases} Q(\vartheta_{s+1}), \text{ if } in\ \vartheta_{s+1} = fin\ \vartheta_{s+1} \\ \text{otherwise. } \boxed{\times} \end{cases}$$

After this we prove the ergodicity of (2.2) using the following assertion due to Markov.

Let (Ω, A, P) be a probability space, $\varrho_1, \varrho_2, \ldots$ a homogeneous Markov-chain with a finite set of possible states $\{1, \ldots, n\}$. Let Π denote the matrix of transition probabilities, i.e. $\Pi = [p_{ij}]_{i,j=1,\ldots,n}$ and $p_{ij}^{(m)}$ denote the m-step transition probabilities, i.e. $p_{ij}^{(m)} = P(\varrho_{s+m} = j | \varrho_s = i)$ $(i, j = 1, \ldots, N; s = 1, 2, \ldots)$.

Lemma 1. Let us suppose that there exist j and m so that $p_{ij}^{(m)} > 0$ for $i = 1, \ldots, n$. Then

$$(2.14) \qquad \lim_{q \to \infty} p_{ij}^{(q)} = x_j, \quad \sum_{j=1}^{n} x_j = 1 \, ,$$

further

$$(2.15) \qquad |p_{ij}^{(q)} - x_j| \le C \cdot \varphi^q \, ,$$

where $C > 0$ and φ $(0 < \varphi < 1)$ are suitable constants. $\boxed{\times}$

Theorem 2. Under the conditions a), b) and c) the sequence (2.2) is an ergodic Markov-chain. $\boxed{\times}$

Proof. We shall show that there exist a state $C^* \in \mathcal{D}_r$ and a positive integer z for which all the conditional probabilities

$$(2.16) \qquad P(\mathcal{B}^{s+z} = C^* | \mathcal{B}^s = C)$$

are positive for every $C \in \mathcal{D}_r$.

Hence by Lemma 1 we shall get our theorem immediately.

Let

$$(2.17) \quad C(A_s^{(1)}, \ldots, A_s^{(r)}) = \begin{bmatrix} i_{1,\,s}^{(1)}, & i_{2,\,s}^{(1)}, & \ldots, & i_{k_s^{(1)},\,s}^{(1)}, & \| f_s^{(1)} \\ \vdots & & & & \\ i_{1,\,s}^{(r)}, & i_{2,\,s}^{(r)}, & \ldots, & i_{k_s^{(r)}}^{(r)}, & \| f_s^{(r}} \end{bmatrix},$$

where each but the first rows may have the form $*$, $\| f_s^{(t)}$, too.

Let $j \in \mathcal{R}$ denote the index and state belonging to the positive columns of matrices Π_t and let

$$(2.18) \qquad C^* = \begin{bmatrix} j, & j, & \ldots, & j, & \| j \\ *, & & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ *, & & \ldots, & & \| j \end{bmatrix}$$

where $p(x, j, t) \geq \varepsilon > 0$ holds for every x and t due to the condition $c)$.

Let P_t denote the probability of the event that $f_s^{(t)}$ will be followed by j at least $(r+1)(k-1)$ times, that is that the t-th sequence has the form

$$(2.19) \qquad \ldots, f_s^{(t)}, \overset{1}{j}, \ldots, \overset{(r+1)(k-1)}{\overbrace{j}} \ .$$

Then

$$(2.20) \qquad P_t \geq \varepsilon^{(r+1)(k-1)},$$

and therefore

$$(2.21) \qquad \prod_{t=1}^{r} P_t \geq \varepsilon^{r(r+1)(k-1)} > 0 \ .$$

In the case of such continuation of the sequences from the first sequence our algorithm processes $f_s^{(1)}$ (and at most $(k-1)$ j' s) in the $(s+1)$-th point of time, and at most $m \leq k-1$ j's in the $(s+2)$-th, $\ldots$, $(s+r)$-th points of time. Therefore

$$(2.22) \qquad A_{s+2}^{(1)}, \ldots, A_{s+r+1}^{(1)} = (\overset{1}{j}, \ldots, \overset{m}{j}, \| j) \ .$$

From the t-th $(t = 2, \ldots, r)$ sequence in the $(s+1)$-th, $\ldots$, $(s+t-1)$-th points of time our algorithm may process some beginning elements (but not more than $k-2$ elements in one point of time). If $f_s^{(t)}$ is not processed earlier, then it will be processed in the $(s+t)$-th point of time, and therefore

$$(2.23) \qquad A_{s+t+1}^{(t)} = \ldots = A_{s+r+1}^{(t)} = (*, \| j) \quad (t = 2, \ldots, r) \ .$$

We see that choosing $z = r+1$ all of the z-step transition probabilities of type $C \rightarrow C^*$ are positive, and so the proof is finished. $\boxtimes$

§ 3. Characterization of the performance of the processing algorithm

Let $R(\vartheta)$ be a real valued function defined for every ϑ belonging to $\mathcal{D}_r$.

Since any given array of type (1.1) determines uniquely the sequence (1.6), therefore the sequence

$$(3.1) \qquad\qquad R(\vartheta_1),\, R(\vartheta_2),\, \ldots$$

is determined too. We are interested in such functions R that characterize the performance of the processing. Assuming that the conditions stated for $\xi_i^{(l)}$ in § 2. are satisfied, we shall show that the mean values of the random variables

$$(3.2) \qquad\qquad \eta_t(l) = \sum_{j=1}^{t} R(\mathcal{B}_j)$$

can be computed using Lemma 1.

Let (Ω, A, P) be a probability space, $\varrho_1, \varrho_2, \ldots$ a homogeneous ergodic Markov-chain with a finite set of possible states $\mathcal{H} = \{1, \ldots, n\}$.
Let

$$(3.3) \qquad\qquad \pi = (p_1, \ldots, p_n)$$

denote the initial distribution and

$$(3.4) \qquad\qquad \Pi = [p_{ij}]_{i,\, j\, =\, 1, \ldots, n}$$

the matrix of transition probabilities.

Let f be a function having real values and defined on the set $\{1, 2, \ldots, n\}$. Let $M_\pi f(\varrho_i)$ denote the mean value of $f(\varrho_i)$ supposing that ϱ_i has an initial distribution π. Let $\Theta_1, \Theta_2, \ldots$ be a stationary Markov-chain on the set $\{1, \ldots, n\}$ with a transition probability matrix (3.4). Therefore the Markov-chain $\Theta_1, \Theta_2, \ldots$ has an initial distribution $x = \{x_1, \ldots, x_n\}$. As an immediate consequence of Lemma 1 we get

$$(3.5) \qquad\qquad |M_\pi f(\varrho_i) - M_x f(\Theta_i)| \leq C_1 \cdot q^i,$$

where $C_1 > 0$, φ $(0 < \varphi < 1)$ are constants. Since $\Theta_1, \Theta_2, \ldots$ is stationary, therefore

$$(3.6) \qquad\qquad M_x f(\Theta_i) = M_x f(\Theta_1),$$

and from (3.6) it follows that

$$(3.7) \qquad\qquad M_\pi\left(\sum_{j=1}^{i} f(\varrho_j)\right) = i\, M_x f(\Theta_1) + 0(1).$$

Theorem 2 guarantees the fulfilment of the conditions of Lemma 1 for the sequence (2.2). The approximate determination of $M\, \eta_t(l)$ is simple, if the stationary initial distribution belonging to the chain (2.2) is known.

The explicite calculation of the stationary values is in general case a cumbersome matter, since the number of elements in $\mathcal{D}_r$ is about n^3 even for the minimal values $r = 1$, $k = 3$.

Now we compute it in some special cases.

§. 4. Computation of the stationary distribution

Let us suppose that in any point of time any element of $\mathcal{H}$ is processable (independently) at most b $(b \geq 1)$ times. Therefore we construct the special stopping set $S^{(b)}$ as follows. Let

$$(4.1) \qquad S^{(b)} = \bigcup_{j=b+1}^{Nb+1} S_j^{(b)}.$$

For a sequence $r_1, \ldots, r_k$ $(r_i \in \mathcal{H}, i = 1, \ldots, k)$ let v_i denote the number of occurences of i in the sequence. Using this notation let $S_j^{(b)}$ $(j = b+1, \ldots, Nb+1)$ be the set of sequences $r_1, r_2, \ldots, r_j$ for which the following conditions hold:

a) $r_i \in \mathcal{H}$ $(i = 1, \ldots, j)$;
b) $0 \leq v_1 \leq \ldots \leq v_N \leq b$ hold for the sequence $r_1 \ldots, r_{j-1}$;
c) $v_{r_j} = b$ for $r_1, \ldots, r_{j-1}$.

For example $S^{(1)} = \{(1, 1), (2, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2)$.

Let $\xi_i^{(l)}$ $(i = 1, 2, \ldots; l = 1, \ldots, r; r \geq 1)$ be random variables with values belonging to $\mathcal{H}$ and having the following properties.

The sequences $\xi_i^{(l)}$ for every l form homogeneous Markov-chains with a common initial distribution $\pi = (1, 0, \ldots, 0)$ and transition probability matrix

$$(4.2) \qquad \Pi = \begin{bmatrix} \alpha & \beta & 0 & \ldots & 0 \\ \alpha & 0 & \beta & \ldots & 0 \\ \vdots & & & & \\ \alpha & 0 & 0 & \ldots & \beta \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix},$$

that is

$$(4.3) \qquad p_{ij} = \begin{cases} 1, & \text{if } j = 1, \ i = N; \\ \alpha, & \text{if } j = 1, \ i = 1, \ldots, N-1; \\ \beta, & \text{if } j = 2, \ldots, N-1; \ i = j+1; \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$, $\beta > 0$, $\alpha + \beta > 1$.

Using this model of program behaviour (transition matrix) we suppose: all of the entry-points and jumps in the programs concern to the memory-modul 1.

In our case the conditions of the Theorem 1 and Theorem 2 hold, therefore the state sequence

(4.4) $$\mathcal{B}_1, \mathcal{B}_2, \ldots (\mathcal{B}_s \in \mathcal{D}_r, \; s = 1, 2, \ldots)$$

belonging to the given sequences $\xi_i^{(l)}$ is a homogeneous ergodic Markov-chain.

Let us observe, that for the given π and II only the states of form

(4.5)
$$\begin{bmatrix} 1, & \ldots, & 1 \\ *, & & 1 \\ \vdots & & \vdots \\ *, & & 1 \end{bmatrix}$$

can occur in (4.4). Namely only such elements of $S^{(b)}$ are used for stopping of the processing, for which (neglecting the last, „stopping" elements) hold the inequalities

(4.6) $$\nu_1 \geq \nu_2 \geq \ldots \geq \nu_N$$

(the value 2 can occur only after the value 1, 3 only after 2, ..., N only after $N-1$), and therefore $\nu_1 = b$, and the stopping element is 1.

Hence we get: no elements from the second, ..., r-th sequences will be processed. Further, we prove the following

Theorem 3. For the mutually independent Markov-chains $\xi_i^{(l)}$ with initial distribution $\pi = (1, 0, \ldots, 0)$ and transition probability matrix (4.2) using the processing algorithm defined in § 1. with stopping set $S^{(1)}$ we have the following stationary distribution of (4.4):

(4.7)
$$x(D) = \begin{cases} \beta^{N-1}, & \text{if } D \text{ has a form of } D_1, \\ \alpha \cdot \beta^{i-1}, & \text{if } D \text{ has a form of } D_2, \\ 0 & \text{otherwise,} \end{cases}$$

where

(4.8)
$$D_1 = \begin{bmatrix} 1, & 2, \; 3, \; \ldots, \; N, & 1 \\ *, & & 1 \\ \vdots & & \vdots \\ * & & 1 \end{bmatrix},$$

(4.9)
$$D_2 = \begin{bmatrix} 1, & 2, \; 3, \; \ldots, \; i, & 1 \\ *, & & 1 \\ \vdots & & \vdots \\ *, & & 1 \end{bmatrix} \qquad (i = 1, \ldots, N-1). \; \boxtimes$$

Proof. At first we show that (4.7) gives the initial distribution, i.e.

$$P(\mathcal{B}_1 = D_1) = \beta^{N-1},$$

(4.10) $$P(\mathcal{B}_1 = D_2) = \alpha \cdot \beta^{i-1},$$

$$P(\mathcal{B}_1 = D_3) = 0 \quad (D_3 \neq D_1, \, D_3 \neq D_2).$$

In D_1 there are $N-1$ transitions of type $i \to i+1$ ($i = 1, \ldots, N-1$; all with probability β) and one transition $N \to 1$ (with probability 1).

In D_2 there are $i-1$ transitions of type $i \to i+1$ (so called step — with probability β) and one transition of type $(N >) \, i \to 1$ (so called jump — with probability α).

As other states can not occur, therefore for other states $P(\mathcal{B}_1 = D_3) = 0$. As both possible states are stopped by 1, (4.7) gives the stationary distribution. ⊠

Now extending the stopping set and restricting $\mathcal{H}$ to $\mathcal{H} = \{1, 2\}$ we get the following

Theorem 4. Let $\mathcal{H} = \{1, 2\}$ and the stopping set $S = S^{(b)}$. Under the further conditions of Theorem 3 the stationary distribution has the form

(4.11) $$x(D) = \begin{cases} \alpha^{b-i} \beta^i, & \text{if } D \text{ has the form } D_1, \\ 0 & \text{otherwise,} \end{cases}$$

where

(4.12) $$D_1 = \begin{bmatrix} 1, & \| & \cdots, & \| & 1 \\ *, & & & & 1 \\ \vdots & & & & \vdots \\ *, & \| & & \| & 1 \end{bmatrix},$$

and the first line of D_1 consists of $(b+1+i)$ elements ($i = 0, \ldots, N$): at the places $2, \ldots, b+i$ the 1's occur exactly $(b-1)$ times (and the 2's exactly i-times), in any ordering for which 2 is followed by 1. ⊠

The proof of Theorem 4 is omitted, because it is similar to the proof of Theorem 3.

§ 5. Computation of the speed

For the characterization of the performance of the processing algorithm we shall use some abstraction of the average number of the executed operations in a time unit, the so called speed.

Let the function $R(\vartheta)$ introduced in § 3. have the form

(5.1) $$R(\vartheta) = \sum_{j=1}^{r} k_j(\vartheta),$$

where k_j denotes the number of processed elements of the j-th sequence.

For the algorithm defined in § 1 with $S = \bigcup\limits_{j=2}^{k} S_j$ we have

(5.2) $1 \le R(\vartheta) \le k - 1$

for every $\vartheta \in \mathcal{D}_r$, where k denotes (according to the definition) the length of the longest sequence in S.

We use the following

Definition 4. Definition of the speed

Let the sequences $\xi_i^{(l)}$ $(l = 1, \ldots, r;\ i = 1, 2, \ldots)$ and the processing algorithm satisfying the conditions stated in § 1. be given. We shall say that the quantity

(5.3) $$V = \lim_{t \to \infty} \frac{M\left[\sum\limits_{i=1}^{t} \sum\limits_{j=1}^{r} k_j(\vartheta_i)\right]}{t}$$

is the speed of processing. ⊠

For our algorithm V exists for any sequences, and it holds due to (5.2)

(5.4) $1 \le V \le k - 1$.

Under the conditions of Theorems 1 and 2 the sequence $\vartheta_1, \vartheta_2, \ldots$ forms an ergodic Markov-chain, therefore from (3.7) we get

(5.5) $$V = M_x \sum_{j=1}^{r} k_j(\vartheta_1),$$

where x denotes the stationary distribution.

As we have seen, under the conditions of Theorems 3 or 4

(5.6) $$\sum_{j=2}^{r} k_j(\vartheta_1) = 0,$$

therefore for both cases

(5.7) $$V = \sum_{j=b}^{b \cdot N} j \cdot P\big(k_1(\vartheta^1) = j\big).$$

Now we prove the following assertions.

Theorem 5. Under the conditions of Theorem 3

(5.8) $$V = \frac{1 - \beta^N}{1 - \beta} . \quad ⊠$$

Theorem 6. Under the conditions of Theorem 4

(5.9) $$V = b(1 + \beta). \quad ⊠$$

In the proofs of these theorems we shall use the following equalities which are provable for example by simple induction.

Let $\alpha > 0$, $\beta > 0$ be real numbers for which $\alpha + \beta = 1$ holds, and let $b \geq 1$ and $N \geq 2$ be integer numbers. Then

$$(5.10) \qquad N \cdot \beta^{N-1} + \sum_{i=1}^{N-1} i \cdot \alpha \cdot \beta^{i-1} = \sum_{i=0}^{N} \beta^i$$

and

$$(5.11) \qquad \sum_{i=1}^{b} i \binom{b}{i} \alpha^{b-i} \beta^i = b \cdot \beta \sum_{i=0}^{b} \binom{b}{i} \beta^i \alpha^{b-i}.$$

Proof of Theorem 5.

Using Theorem 3 and (5.7) we get

$$(5.12) \qquad V = N \cdot \beta^{N-1} + \sum_{i=1}^{N-1} i \cdot \alpha \cdot \beta^{i-1}.$$

From here due to (5.10)

$$(5.13) \qquad V = \sum_{i=0}^{N-1} \beta^i.$$

This sum forms a geometrical series, therefore we get (5.8) immediately. ☒

Proof of Theorem 6.

Using Theorem 4 and (5.7) we get

$$(5.14) \qquad V = \sum_{j=b}^{2b} j \binom{b}{b-j} \alpha^{2b-j} \beta^{j-b}$$

from where due to (5.11)

$$(5.15) \qquad V = b \cdot \sum_{i=0}^{b} \binom{b}{i} \alpha^{b-i} \beta^i + \sum_{i=1}^{b} i \binom{b}{i} \alpha^{b-i} \beta^i = b \sum\nolimits_1 + \sum\nolimits_2.$$

Here the members of $\sum_1$ are positive elements of the stationary distribution (with the proper multiple) in Theorem 4, therefore $\sum_1 = 1$. Then using (5.11) for $\sum_2$ we get our assertion. ☒

For a sequence

$$(5.16) \qquad r_1, \ldots, r_z \ (r_i \in \mathcal{R}, \ i = 1, 2, \ldots)$$

let τ_1 and τ_b denote the necessary processing points of time for the processing algorithms using $S^{(1)}$ and $S^{(b)}$ resp.

It is not hard to prove that for any sequence of form (5.16) $\tau_1 = b \cdot t$ implies $\tau_b \leq t$. From here under the conditions of Theorems 1 and 2 we have

(5.17) $V^{(b)} \geq b \cdot V^{(1)} ,$

where $V^{(b)}$ and $V^{(1)}$ denote the speeds corresponding to $S^{(b)}$ and $S^{(1)}$ resp.

Comparing (5.17) with (5.9) we get that the estimation (5.17) is not improvable.

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