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Density of safe matrices

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Abstract. A binary matrix A of size $m \times n$ is called *r-good* if it contains in each column at most r 1's; the matrix is called *r-schedulable* if, by deleting some zeros, the matrix becomes *r-good*; A is called *r-safe* if the first k $(1 \le k \le n)$ columns of the matrix contain at most kr 1's.

Let $Z = [z_{ij}]_{m \times n}$ be a matrix of independent random variables, having the common distribution $P(z_{ij} = 1) = p$ and $P(z_{ij} = 0) = 1 - p$, where $0 \le p \le 1$. For $m \ge 1$, lower and upper bounds are presented for the asymptotic probability of the event that a concrete realization of Z is 1-schedulable: the lower bound is connected with good, and the upper bound with safe matrices. Further exact formula is given for the critical probabilities $s_{crit}(m)$ defined as the supremum of probabilities, guaranteeing that the matrix Z is 1-safe with positive probability for arbitrary value of n and m.

1 Introduction

Percolation is a very popular research area of combinatorics [2, 3, 5, 6, 9, 10, 11, 18, 19, 20, 22, 23, 24, 25, 26, 27, 29, 52] and physics [15, 28, 36, 37, 38, 39, 42, 46].

In this paper we use and extend a mathematical model proposed by Peter Winkler [53] and studied later among others in [13, 14, 19, 20, 33, 35, 46]. This model is also useful for the investigation of some scheduling problems of parallel processes [40, 51] using resources requiring mutual exclusion [1, 7, 16, 21, 34, 45, 50].

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According to the Winkler model, two processes share one unit of a resource. We extend this model for $m \geq 2$ processes and $r > 0$ units of the resource requiring mutual exclusion. The rise of the number of processes results a model describing the percolation in three or more dimensions.
Estimations of the probability of schedulability of processes are derived using different methods, first of all by investigating of asymmetric random walks across the x axis.

2 Formulation of the problem

Let m and n be positive integers, let r $(0 \leq r \leq m)$ and p $(0 \leq p \leq 1)$ be real numbers and let

$$Z = \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1n} \\ z_{21} & z_{22} & \dots & z_{2n} \\ \dots & & & \\ z_{m1} & z_{m2} & \dots & z_{mn} \end{pmatrix}$$

be a matrix of independent random variables with the common distribution

$$P(z_{ij} = k) = \begin{cases} p, & \text{if } k = 1 \text{ and } 1 \leq i \leq m,\ 1 \leq j \leq n, \\ q = 1 - p, & \text{if } k = 0 \text{ and } 1 \leq i \leq m,\ 1 \leq j \leq n. \end{cases}$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

be a concrete realization of Z.
The good, safe and schedulable matrices are defined as follows.
Matrix A is called ***r*-good** if the number of the 1's is at most r in each column. The number of different r-good matrices of size $m \times n$ is denoted by $G_r(m, n)$, and the probability that Z is good is denoted by $g_r(m, n, p)$.
Matrix A is called ***r*-safe** if

$$\sum_{i=1}^{m} \sum_{j=1}^{k} a_{ij} \leq kr \quad (k = 1, 2, \dots, n).$$

The number of different r-safe matrices of size $m \times n$ is denoted by $S_r(m, n)$ and the probability that Z is safe, is denoted by $s_r(m, n, p)$.

If $a_{ij} = 0$, then it can be deleted from A. Deletion of a_{ij} means that we decrease the second indices of $a_{i,j+1}, \ldots, a_{im}$ and add $a_{im} = 0$ to the i^{th} row of A.

Matrix A is called **Winkler r-schedulable** (shortly **r-schedulable** or r-compatible) if it can be transformed into an r-good matrix B using deletions. The number of different r-schedulable matrices of size $m \times n$ is denoted by $W_r(m, n)$, and the probability that Z is r-schedulable is denoted by $w_r(m, n, p)$. The function $w_r(m, n, p)$ is called **r-schedulability function.** The functions $g_r(m, n, r)$, $w_r(m, n, r)$ and $s_r(m, n, r)$ are called the density of the corresponding matrices. The **asymptotic density** of the good, safe and schedulable matrices are defined as:

$$g_r(m, p) = \lim_{n \to \infty} g_r(m, n, p),$$

$$s_r(m, p) = \lim_{n \to \infty} s_r(m, n, p),$$

$$w_r(m, p) = \lim_{n \to \infty} w_r(m, n, p).$$

The critical probabilities defined as

$$w_{crit,r}(m) = \sup\{p \mid w_r(m, p) > 0\},$$

$$g_{crit,r}(m) = \sup\{p \mid g_r(m, p) > 0\},$$

and

$$s_{crit,r}(m) = \sup\{p \mid s_r(m, p) > 0\}$$

represent special interest for some applications.

The aim of this paper is to characterise the density, asymptotic density and critical probability of good, schedulable and safe matrices.

The starting point of our research is due to Péter Gács [20], proving that $w_1(2, p)$ is positive for p small enough. His proof implies that $w_{crit,1}(2) \geq 10^{-400}$.

2.1 Interpretation of the problem

Although the Winkler model was proposed to study the percolation, we describe a possible interpretation as a model of parallel processes. Let m processes use r units of some resource R. The requirements of the process P_i are modelled by the sequence $a_{i1}, a_{i2}, \ldots, a_{im}$. $a_{ij} = 1$ means that the process P_i needs a unit of the given resource in the j^{th} time unit. $a_{ij} = 0$ means that the process P_i executes some background work in the j^{th} time unit which can be

delayed and executed after the last usage of R.
The special case $m = 1$ and $r = 1$ is the well-known ticket problem [52] or ballot problem [17], while the special case $m = 2$ and $r = 1$ is the Winkler model of percolation [20, 53].
The good matrices are schedulable without deletion of zeros. But some not good matrices are schedulable, since they can be transformed into good matrices using the permitted deletion operation. Safeness is a necessary condition of schedulability. Therefore, the number of good matrices gives a lower bound and the number of safe matrices results an upper bound for the number of schedulable matrices.
Since we handle the model as a model of informatics, in the sequel we follow the terminology used by Feller [17] in queueing theory.

3 Analysis

In this section first of all we investigate – using different methods – the function of the asymptotic density of 1's as the function of the probability p of the appearance of 1's and of the number of sequences m.

Some basic properties of the investigated functions ($g_r(m, n, p)$, $w_r(m, n, p)$ and $s_r(m, n, p)$) are the following:

- $n \in \mathbb{N}^+$, $r \in \mathbb{R}$ and $r \in [0, m]$, $p \in \mathbb{R}$ and $p \in [0, 1]$;
- as the functions of n they are monotonically decreasing;
- as the functions of p they are monotonically decreasing;
- as the functions of m they are monotonically decreasing;
- as the functions of r they are monotonically increasing;

In the following we suppose that $r = 1$, that is in the column of the good matrices at most one 1, and in the first k columns of the safe matrices at most k 1's are permitted. Since r everywhere equals 1, it is omitted as an index.

3.1 Preliminary results

In the further sections we need the following assertions.
Let C_n ($n \in \mathbb{N}^+$) denote the number of binary sequences $a_1, a_2, \ldots, a_{2n}$, containing n ones and n zeros in such a manner that each prefix $a_1, a_2, \ldots, a_k$ ($1 \leq k \leq 2n$) contains at most so many ones as zeros.

Lemma 1 *If* $n \geq 0$, *then*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

It is worth remark that C_n is the n^{th} Catalan number, whose explicit form appears in numerous books and papers [8, 30, 31, 32, 48, 52].

Lemma 2 *If* $0 \leq x \leq 1$, *then*

$$f(x) = x\sum_{k=0}^{\infty}\frac{1}{k+1}\binom{2k}{k}(x(1-x))^k = \begin{cases} \dfrac{x}{1-x}, & \text{if } 0 \leq x < \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Proof. See [47]. ■

If $m \geq 2$, then the columns containing only 0's are called **white** (W), the columns containing only 1's are called **black** (B) and the remaining columns are called **gray** (G).

If $m \geq 2$, then each column of the matrix A is white or gray with probability $q^m + mpq^{m-1}$, therefore $g(m,n,p) = \left(q^m + mpq^{m-1}\right)^n$. If $p > 0$, then

$$g(m,p) = \lim_{n\to\infty}\left(q^m + pq^{m-1}m\right)^n = 0,$$

so the density of the good matrices tends to zero, when the number of the columns tends to infinity.

If in the case $m = 2$ we delete the white columns from a good matrix, then only gray columns remain in the matrix, that is, each row of the matrix is *the complementer* of the other row.

The following simple assertion plays an important role in the following.

Lemma 3 *If* $m \geq 2$, *then the good matrices are schedulable, and the schedulable matrices are safe.*

Proof. If in every column of matrix A is at most one 1, then the first k columns contain at most k 1's.

If there is a k ($1 \leq k \leq n$), that the first k columns of matrix A contains more 1's than k, then – according to the pigeonhole principle – there is at least one column containing two 1's. If we delete a zero from A, then the number of the 1's in the first k columns does not decrease, therefore A is not schedulable. ■

A useful consequence of this assertion is the following corollary.

Corollary 1 *If* $m \geq 2$, *then*

$$g(m,n,p) \leq w(m,n,p) \leq s(m,n,p),$$

$$g(m,p) \leq w(m,p) \leq s(m,p),$$

$$g_{crit}(m) \leq w_{crit}(m) \leq s_{crit}(m).$$

3.2 Matrices with two rows

For the simplicity of the notations we analyse the function $u(2,n,p) = 1 - s(2,n,p)$ instead of $s(2,n,p)$. At first we derive a closed formula for $u(2,n,0.5)$.

Lemma 4 *If* $n \geq 1$, *then*

$$u(2,n,0.5) = \sum_{i=1}^{n} \sum_{j=0}^{\lfloor (i-1)/2 \rfloor} 2^{i-1-2j} C_j \binom{i-1}{2j} 4^{n-i}. \tag{1}$$

Proof. Let's classify the possible matrices of size $2 \times n$ according to their first such column, in which the cumulated number of 1's became greater than the number of 0's. This column is called **the deciding column** of the matrix.
The index of the deciding column is $1, 2, \ldots, n-1$ or n. The matrices of the received classes can be further classified according to the number of black columns before the deciding column: the possible values of this number are $0,\ 1, \ldots,\ \lfloor (n-1)/2 \rfloor$.
The outer summing takes into account the deciding columns, while the inner summing does the black columns before the deciding column. The binomial coefficient mirrors the number of possibilities for the placement of the $2j$ black and white columns in the $i-1$ columns preceding the deciding column. The j^{th} Catalan number C_j gives the number of corresponding sequence of the black and white columns. The power of base 2 gives the number of possible arrangements of the gray columns. Finally the power of base 4 takes into account the fact, that the columns after the deciding one can be filled in arbitrary manner – the matrix will be unsafe in any case. ■

It seems that it would be hard to handle the formula (1) for $u(2,n,0.5)$. Therefore, we present a combinatorial method and three further ones based on random walks to get the explicit form of $s(2,p)$.

Lemma 5 *If* $0 \leq p \leq 1$, *then*

$$1 - s(2,p) = u(2,p) = \begin{cases} \dfrac{p^2}{q^2}, & \text{if } 0 \leq p < \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} \leq p \leq 1. \end{cases}$$

Proof. Some part of the unsafe matrices is unsafe due to the first black column. The general form of such matrices is G^aBA^b, where $a+b+1=n$, further G means a gray, B means a black and A means an arbitrary column. The asymptotic fraction of such columns is

$$\sum_{a=0}^{\infty} C_0(2pq)^a p^2 = \frac{p^2}{1-2pq}C_0.$$

The general form of the following group of the unsafe matrices is G^aBG^bW G^cBA^d, where $a+b+c+d+3=n$. The fraction of such matrices asymptotically equals to

$$\sum_{a=0}^{\infty}(2pq)^a p^2 \sum_{b=0}^{\infty}(2pq)^b q^2 \sum_{c=0}^{\infty}(2pq)^c p^2 = \frac{p^2}{1-2pq}C_1\frac{p^2}{1-2pq}\frac{q^2}{1-2pq}\ .$$

Generally, if the $(i+1)^{th}$ black column is deciding, then the asymptotic contribution of such matrices to the probability of the unsafe matrices equals to

$$\frac{p^2}{1-2pq}C_i\left(\frac{p^2}{1-2pq}\frac{q^2}{1-2pq}\right)^2,$$

and so

$$u(2,p) = \sum_{i=0}^{\infty}\frac{p^2}{1-2pq}C_i\left(\frac{p^2}{1-2pq}\frac{q^2}{1-2pq}\right)^i.$$

Lemma 2, gives the required formula with the substitutions $p^2/(p^2+q^2)=x$ and $q^2/(p^2+q^2)=1-x$. ■

We get a useful method for the investigation of our matrices assigning to each matrix a random walk [17, 43] on the real axis containing a sink at the point -1.

Another proof of Lemma 5 is as follows. In the following proofs of Lemma 5 we consider only the case $0 \le p \le 1/2$, since if $1/2 \le p \le 1$, then the following famous result of György Pólya [41, 43] implies our assertion.

Lemma 6 *The probability that the moving point performing a random walk over the real axis returns infinitely often to its initial position is equal to one.*

Second proof of Lemma 5. Let's assign a random walk to matrix A so that a black column implies a step to left, a white column implies a step to right

and a gray column results that the moving point preserves its position.
Let $b_k(A)$ denote the number of 1's in the first k columns of matrix A. Then

$$b_k = \sum_{i=1}^{k} (a_{1i+2i}).$$

If $b_i \leq k$ for $i = 1, 2, \ldots, k$, then after k time units the moving point is in the point $(k - b_k, 0)$ of the real axis, otherwise the point is absorbed by the sink at -1.
We wish to determine the probability of the absorption of the moving point.
The probability of a step to left is p^2, the probability of a step to right is q^2 and $2pq$ is the probability of the event that the point does not change its position.
Using the notation $u(2, p) = x$ we have

$$x = p^2 + 2pqx + q^2x^2.$$

The roots of this equation are

$$x_{1,2} = \frac{1 - 2pq \pm \sqrt{(1 - 2pq)^2 - 4p^2q^2}}{2q^2} = \frac{p^2 + q^2 \pm \sqrt{(p^2 - q^2)^2}}{2q^2},$$

from where we get

$$x_1 = \frac{p^2}{q^2} \text{ and } x_2 = 1. \tag{2}$$

This formula and $s(2, p) = 1 - u(2, p)$ result the required formula. ■
Since first of all we are interested in the probability of the absorption, we can assign a random walk to matrix Z neglecting the gray columns, as the gray columns have no influence on the limit probability of the absorption (they only make the convergence slower).
Another proof of Lemma 5 is the following.
Third proof of Lemma 5. Dividing the probability of the gray columns among the black and white columns in the corresponding ratio we get for the probability a of the step to left and for the probability of the step to right that

$$a = \frac{p^2}{p^2 + q^2} \quad \text{és} \quad b = \frac{q^2}{p^2 + q^2}. \tag{3}$$

Using these probabilities, we get the equation

$$x = a + bx^3.$$

Substituting a and b into the roots of this equation, according to (3), we also get here the roots corresponding to (2). ■

Finally we present such a method, which later can be extended to arbitrary $m \geq 2$ sequences.

Fourth proof of Lemma 5. Let x_k $(k = -1, 0, 1, 2, \ldots)$ denote the probability of the event that the point starting at point k will be absorbed by the sink at $x = -1$. Let's assign again a step to left to the columns containing two 1's, a step to right to the columns containing two 0's and preserve of the position to the mixed columns.

Then we can write the following system of equations.

$$\begin{array}{lclclcl} x_0 & = & q^2x_1 & + & 2qpx_0 & + & p^2, \\ x_1 & = & q^2x_2 & + & 2qpx_1 & + & p^2x_0, \\ x_2 & = & q^2x_3 & + & 2qpx_2 & + & p^2x_1, \\ x_3 & = & q^2x_4 & + & 2qpx_3 & + & p^2x_2, \\ \ldots & & & & & & \end{array} \tag{4}$$

Let

$$G(z) = \sum_{i=1}^{\infty} x_i z^i$$

be the generator function of sequence $x_0, x_1, x_2, \ldots$. Multiplying the equation beginning with x_i I $= 1, 2, \ldots$ of the system of equations (4) by z^i and summing up the new equations, we get the equation:

$$G(z) = q\frac{G(z) - x_0}{z} + 2pqG(z) + p^2(1 + zG(z)).$$

From this equation $G(z)$ can be expressed in the form

$$G(z) = \frac{P(z)}{Q(z)},$$

where

$$P(z) = q^2x_0 - p^2z$$

and

$$Q(z) = p^2z^2 + 2pqz + q^2 - z.$$

In the zero places z_o with $|z_0| \leq 1$ of the polynomial $Q(z)$, according to Cauchy-Hadamard theorem [44, page 69] it must hold $P(z) = 0$. Writing the equation $Q(z) = 0$ in the form

$$(pz + q)^2 = z$$

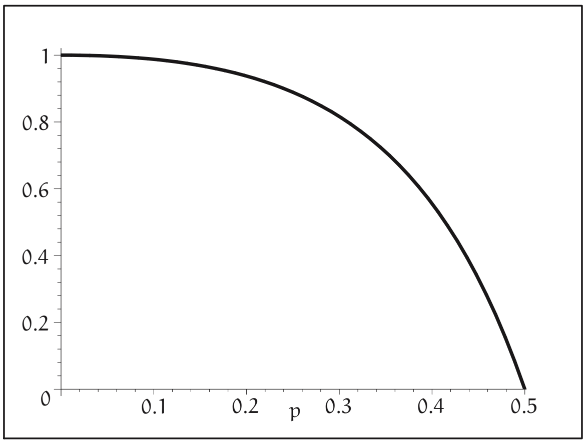

Figure 1: The curve of the schedulability function $s(2, p, 1)$ in the interval $p \in [0, 0.5]$.

we directly get that $z = 1$ is a root of the polynomial $Q(z)$. From the equation $P(1) = 0$ we get the root

$$x_0 = \frac{p^2}{q^2},$$

implying

$$s(2, p) = 1 - \frac{p^2}{q^2}, \qquad \blacksquare$$

Figure 1 shows the part belonging to the interval $p \in [0, 0.5]$ of the curve of the function $s(2, p, 1)$ defined in the interval $[0, 1]$.
According to the properties of the functions $g(2, p)$ and $s(2, p)$, the critical probabilities satisfy the following inequalities:

$$0 = g_{crit}(2) \leq w_{crit}(2) \leq s_{crit}(2) = \frac{1}{2}.$$

Let's remind that Gács proved $w_{crit}(2) \geq 10^{-400}$ [20].
Let $T(m, n)$ denote the number of binary matrices of size $m \times n$. Then $T(m, n) = 2^{mn}$.
Figure 2 contains the number and fraction of the good, schedulable and safe matrices for the case $m = 2$, $p = 0.5$, and $n = 1, 2, \ldots, 15$. In this case

n	$G(2,n)$	$\frac{G(2,n)}{T(2,n)}$	$W(2,n)$	$\frac{W(2,n)}{T(2,n)}$	$S(2,n)$	$\frac{S(2,n)}{T(2,n)}$	$\frac{W(2,n)}{S(2,n)}$
1	3	0.750	3	0.750	3	0.750	1
2	9	0.562	10	0.625	10	0.625	1
3	27	0.452	35	0.547	35	0.547	1
4	81	0.316	124	0.484	126	0.492	0.984
5	243	0.237	444	0.434	462	0.451	0.961
6	729	0.178	1592	0.389	1716	0.419	0.927
7	2187	0.133	5731	0.350	6435	0.393	0.890
8	6561	0.100	20671	0.315	24310	0.371	0.850
9	19683	0.075	74722	0.285	92378	0.352	0.808
10	59049	0.056	270521	0.258	352716	0.336	0.767
11	177147	0.042	980751	0.234	1352078	0.322	0.725
12	531441	0.032	3559538	0.212	5200300	0.310	0.684
13	1594323	0.022	12931155	0.193	20058300	0.299	0.646
14	4782969	0.018	47013033	0.175	77558760	0.289	0.606
15	14348907	0.013	171036244	0.159	300540195	0.280	0.568

Figure 2: Rounded data belonging to the parameters $m = 2$ and $p = 0.5$.

the fractions equal to the probability of the corresponding matrices. According to Lemma 5 in this case $G(m,n)/T(m,n)$, $W(m,n)/T(m,n)$, and $S(m,n)/T(m,n)$ tend to zero when n tends to infinity.

Figure 3 contains the fractions of the good, schedulable and safe matrices for the case $m = 2$, $p = 0.4$, and $n = 1, 2, \ldots, 16$. In column $s(2,n,0.4)$ of Table 3 the computed limit is $5/9 \sim 0.555$.

Figure 4 contains the fractions of the good, schedulable and safe matrices for the case $m = 2$, $p = 0.35$, and $n = 1,\ 2,\ \ldots,\ 17$. For the column $s(2,n,0.35)$ of Table 4 the computed limit is $120/169 \sim 0.710$.

3.3 Matrices with three rows

If $m = 3$, then the possible ratios of the 1's and 0's are 3:0, 2:1, 1:2 or 0:3. We assign such random walk to the investigated matrix, in which the walking point jumps by two to left with the probability p^3 of the column containing three 1's; the point makes a step to left with the probability $3p^2q$; the position is preserved with the probability q^3 of the column containing only zeros.

Using the notation x_k introduced in the fourth proof of Lemma 5, we get the

n	$T(2,n)$	$g(2,n,0.4)$	$w(2,n,0.4)$	$s(2,n,0.4)$	$\frac{w(2,n,0.4)}{s(2,n,0.4)}$
1	4	0.8400	0.8400	0.8400	1
2	16	0.7056	0.7632	0.7632	1
3	64	0.5927	0.7171	0.7171	1
4	256	0.4979	0.6795	0.6862	0.9902
5	1024	0.4182	0.6487	0.6639	0.9771
6	4096	0.3513	0.6206	0.6470	0.9592
7	16384	0.2951	0.5957	0.6339	0.9397
8	65536	0.2479	0.5731	0.6234	0.9193
9	262144	0.2082	0.5524	0.6149	0.8984
10	1048576	0.1749	0.5332	0.6078	0.8773
11	4194304	0.1469	0.5155	0.6019	0.8565
12	16777216	0.1234	0.4988	0.5967	0.8359
13	67108864	0.1037	0.4832	0.5924	0.8157
14	268435456	0.0871	0.4685	0.5886	0.7960
15	1073741824	0.0731	0.4545	0.5854	0.7764
16	4294967296	0.0644	0.4412	0.5825	0.7574
17	169779869184	0.0516	0.4286	0.5800	0.7390

Figure 3: Rounded data belonging to the parameters $m = 2$ and $p = 0.4$.

following equations:

$$\begin{array}{lllllllll} x_0 & = & q^3x_1 & + & 3q^2px_0 & + & 3qp^2 & + & p^3, \\ x_1 & = & q^3x_2 & + & 3q^2px_1 & + & 3qp^2x_0 & + & p^3, \\ x_2 & = & q^3x_3 & + & 3q^2px_2 & + & 3qp^2x_1 & + & p^3x_0, \\ x_3 & = & q^3x_4 & + & 3q^2px_3 & + & 3qp^2x_2 & + & p^3x_1, \\ \ldots \end{array} \tag{5}$$

Let

$$G(z) = \sum_{i=0}^{\infty} x_n z^n$$

be the generator function of the sequence $x_0, x_1, x_2, \ldots$. Then multiplying the equations of the system (5) with the corresponding powers of z and summing up the received equations, we get:

$$G(z) = q^3 \frac{G(z) - x_0}{z} + 3q^2pG(z) + 3qp^2(1 + zG(z)) + p^3 \left(1 + z + z^2G(z)\right),$$

n	$T(2,n)$	$g(2,0.35)$	$w(2,0.35)$	$s(2,n,0.35)$	$\frac{w(2,0.35)}{s(2,0.35)}$
1	4	0.8775	0.8775	0.8775	1
2	16	0.7700	0.8218	0.8218	1
3	64	0.6757	0.7901	0.7901	1
4	256	0.5929	0.7645	0.7699	0.9930
5	1024	0.5203	0.7441	0.7561	0.9841
6	4096	0.4565	0.7255	0.7462	0.9723
7	16384	0.4006	0.7094	0.7389	0.9601
8	65536	0.3515	0.6949	0.7334	0.9475
9	262144	0.3085	0.6817	0.7291	0.9350
10	1048576	0.2707	0.6696	0.7258	0.9226
11	4194304	0.2375	0.6585	0.7231	0.9107
12	16777216	0.2084	0.6481	0.7210	0.8989
13	67108864	0.1839	0.6383	0.7192	0.8875
14	268435456	0.1605	0.6291	0.7178	0.8764
15	1073741824	0.1401	0.6204	0.7166	0.8658
16	4294967296	0.1236	0.6122	0.7156	0.8555

Figure 4: Rounded data belonging to the parameters $m = 2$ and $p = 0.35$.

from where $G(z)$ can be expressed as the fraction of two polynomials:

$$G(z) = \frac{P(z)}{Q(z)},$$

where

$$P(z) = q^3 x_0 - 3qp^2 z - p^3(z + z^2)$$

and

$$Q(z) = p^3 z^3 + 3p^2 q z^2 + 3pq^2 z + q^3 - z.$$

The equation $Q(z) = 0$ can be transformed into the form

$$(q + pz)^3 = z,$$

from where the root $z_1 = 1$ follows immediately. Expressing x_0 from the equation $P(1) = 0$, we get:

$$x_0 = \frac{3p^2}{q^2} + \frac{2p^3}{q^3}, \tag{6}$$

n	$T(3,n)$	$g(3,n,0.5)$	$w(3,n,0.5)$	$s(3,n,0.5)$	$\frac{w(3,n,0.5)}{s(3,n,0.5)}$
1	8	0.5000	0.5000	0.5000	1.0000
2	64	0.2500	0.2969	0.2969	1.0000
3	512	0.1250	0.1914	0.1914	1.0000
4	4 096	0.0625	0.1282	0.1296	0.9892
5	32 768	0.0312	0.0880	0.0907	0.9702
6	262 144	0.0156	0.0612	0.0651	0.9401
7	2 097 152	0.0078	0.0429	0.0475	0.9032
8	16 777 216	0.0039	0.0303	0.0352	0.8594

Figure 5: Rounded data belonging to the parameters $m = 3$ and $p = 0.5$.

implying

$$s(3,p) = 1 - \frac{3p^2}{q^2} - \frac{2p^3}{q^3}, \tag{7}$$

The value of the function $1 - x_0 = x_0(p/q)$ is 1 at $p/q = 0$ and it is decreasing if $0 \le p/q \le 1/2$. With the multiplication by $q = (1-p)^3$ we get the equation

$$\frac{3p^2}{(1-p)^2} + \frac{2p^3}{(1-p)^3} = 1,$$

which – by algebraic manipulations – results the value $p = 1/3$, that is $s_{crit}(3) = 1/3$.

Figure 5 contains the fraction of the good, schedulable and safe matrices for the case $m = 3$, $p = 0.5$, and $n = 1, 2, \ldots, 8$.

In this table $g(3,n,0.5)$, $w(3,n,0.5)$, and $s(3,n,0.5)$ all have to tend to zero when n tends to infinity.

Figure 6 contains fraction of the good, schedulable and safe matrices for the case $m = 3$, $p = 0.25$, and $n = 1, 2, \ldots, 5$.

In this table $g(3,n,0.25)$ has to tend to zero, if n tends to infinity, but according to formula (7) $16/23 \sim 0.593$ is the computed limit for $s(3,n,0.25)$ when n tends to infinity.

We remark that the master thesis of Rudolf Szendrei [49] contains further simulation results.

n	$T(3,m)$	$g(3,n,0.25)$	$w(3,n,0.25)$	$s(3,n,0.25)$	$\frac{w(3,n,0.25)}{s(3,n,0.25)}$
1	8	0.8437	0.8437	0.8437	1.0000
2	64	0.7119	0.7712	0.7712	1.0000
3	512	0.6007	0.7286	0.7286	1.0000
4	4 096	0.5068	0.6981	0.7004	0.9967
5	32 768	0.4276	0.6748	0.6804	0.9917

Figure 6: Rounded data belonging to the parameters $m = 3$ and $p = 0.25$.

4 Main result

The analysis of the safe matrices of size $m \times n$ in the case of $m \geq 4$ is similar. If a column of matrix A contains at least $b \geq 3$ 1's, then the walking point jumps $(b-2)$ positions to left; if the column contains two 1's then the point makes a step to left; in the case of one 1 the point preserves its position and if the column contains only 0's, then the point makes a step to right. The corresponding probabilities are $\binom{m}{b}p^{b-2}q^{n-b+2}$, $\binom{m}{2}p^{m-2}q^2$, $\binom{m}{1}pq^{m-1}$ and $\binom{m}{0}q^m$,. So we get the following equations:

$$\begin{array}{rcccccc} x_0 & = & \binom{m}{0}q^m x_1 & + & \binom{m}{1}pq^{m-1}x_0 & + & \binom{m}{2}p^2q^{m-2} \\ & + & \binom{m}{3}p^3q^{m-3} & + & \dots & + & \binom{m}{m}p^m, \\ x_1 & = & \binom{m}{0}q^m x_2 & + & \binom{m}{1}pq^{m-1}x_1 & + & \binom{m}{2}p^2q^{m-2}x_0 \\ & + & \binom{m}{3}p^3q^{m-3} & + & \dots & + & \binom{m}{m}p^m, \\ x_2 & = & \binom{m}{0}q^m x_3 x_3 & + & \binom{m}{1}pq^{m-1}x_2 & + & \binom{m}{2}p^2q^{m-2}x_1 \\ & + & \binom{m}{3}p^3q^{m-3}x_0 & + & \dots & + & \binom{m}{m}p^m, \\ \dots & & & & & & \end{array} \tag{8}$$

Let

$$G(z) = \sum_{i=0}^{\infty} x_n z^n$$

be the generator function of the sequence $x_0, x_1, x_2, \dots$. Then multiplying the equations in (8) with the corresponding powers of z and summing up them, we get:

$$G(z) = \binom{m}{0}q^m \frac{G(z) - x_0}{z} + \binom{m}{1}pq^{m-1}G(z) + \binom{m}{2}p^2q^{m-2}(1 + zG(z))$$

$$+\binom{m}{3}p^3q^{m-3}(1+z+z^2G(z))+\dots+\binom{m}{m}p^m\left(1 + z + \dots + z^{m-2} + z^{m-1}G(z)\right),$$

from where one can express $G(z)$ as the fraction of two polynomials:

$$G(z) = \frac{P(z)}{Q(z)},$$

where

$$P(z) = \binom{m}{0} q^m x_0 - \sum_{i=2}^{m} \left(\binom{m}{i} p^i q^{m-i} \sum_{j=0}^{i-2} z^j \right).$$

If the denominator has a root x with $|x| \leq 1$, then the value of the nominator at x must be zero.
Reordering the equation $Q(z) = 0$ to the form

$$(q + pz)^m = 1$$

we get the root $z_1 = 1$. Division of the equation $P(1) = 0$ by q^m results the equation

$$x_0 = \sum_{i=2}^{m} \binom{m}{i} \left(\frac{p}{1-p} \right)^i (i-1).$$

The value of the function $x_0 = x_0(p)$ is zero at $p = 0$, and the function is increasing, if p is positive. From the equation $x_0 = 1$ we get $p = 1/m$.
Taking into account the results received above for cases $m = 2$ and $m = 3$, we received the following result.

Theorem 1 *If* $m \geq 2$ *and* $0 \leq p \leq m$, *then*

$$s_{crit}(m) = \frac{1}{m}. \tag{9}$$

and

$$s(m,p) = \begin{cases} 1 - \sum_{i=2}^{m} \binom{m}{i} \left(\frac{p}{1-p} \right)^i (i-1), & \text{if } 0 \leq p < \frac{1}{m}, \\ 0, & \text{if } \frac{1}{m} \leq p \leq 1. \end{cases} \tag{10}$$

Proof. a) The special case $m = 2$ is equivalent with Lemma 5.
b) The special case $m = 3$ is equivalent with formula (7).
c) For the case $m \geq 4$, see the proof before the theorem. ■

5 Summary

We determined the explicit form of the asymptotic density $s(m, p)$ for every number of the rows $m \geq 2$ and probability of 1's p. Furthermore we gave the exact values of the critical probabilities $s_{crit}(m)$ for $m \geq 2$. The value of $s_{crit}(2)$ is 0.5, which is characteristic to several other two dimensional critical probabilities. The further critical probabilities are decresse when m grows.
According to the simulation experiments the critical probabilities are near to the received upper bounds: Table 2 shows the data belonging to $m = 2$ and $p = 0.5$, Table 3 the data belonging to $m = 2$ and $p = 0.4$, Table 4 the data for $m = 2$ and $p = 0.35$, Table 5 the data belonging to $m = 3$ and $p = 0.5$, and Table 6 presents the data belonging to $m = 3$ and $p = 0.25$.
On the base of the data of the figures we suppose that the bound $p \geq 10^{-400}$ in [20] can be improved, but the analysis of the behaviour of fraction $w(m, p)/s(m, p)$ requires further work.
We are able to give a bit better lower and upper bounds of the investigated $w_r(m, n, p)$ probabilities, but the more precise characterisation of the critical probabilities requires more useful matrices than the good and safe ones.

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