Partitioning 2-edge-colored graphs by monochromatic paths and cycles

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Abstract

We present results on partitioning the vertices of 2-edge-colored graphs into monochromatic paths and cycles. We prove asymptotically the two-color case of a conjecture of Sárközy: the vertex set of every 2-edge-colored graph can be partitioned into at most $2\alpha(G)$ monochromatic cycles, where $\alpha(G)$ denotes the independence number of G. Another direction, emerged recently from a conjecture of Schelp, is to consider colorings of graphs with given minimum degree. We prove that apart from $o(|V(G)|)$ vertices, the vertex set of any 2-edge-colored graph G with minimum degree at least $\frac{(1+\varepsilon)3|V(G)|}{4}$ can be covered by the vertices of two vertex disjoint monochromatic cycles of distinct colors. Finally, under the assumption that $\overline{G}$ does not contain a fixed bipartite graph H, we show that in every 2-edge-coloring of G, $|V(G)| - c(H)$ vertices can be covered by two vertex disjoint paths of different colors, where $c(H)$ is a constant depending only on H. In particular, we prove that $c(C_4) = 1$, which is best possible.[1]

1 Background, summary of results.

In this paper, we consider some conjectures about partitioning vertices of edge-colored graphs into monochromatic cycles or paths. For simplicity, colored graphs means edge-colored graphs in this paper. In this context it is conventional to accept *empty graphs and one-vertex graphs* as a path or a cycle (of any color) and also *any edge* as a path or a cycle (in its color). With this convention one can define the *cycle (or path) partition number* of any colored graph G as the minimum number of vertex disjoint monochromatic cycles (or paths) needed to cover the vertex set of G. For complete graphs, [6] posed the following conjecture.

Conjecture 1.1. *The cycle partition number of any t-colored complete graph K_n is t.*

The $t = 2$ case of this conjecture was stated earlier by Lehel in a stronger form, requiring that the colors of the two cycles must be different. After some initial

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results [2, 8], Łuczak, Rödl and Szemerédi [22] proved Lehel's conjecture for large enough n, which can be considered as a birth of certain advanced applications of the Regularity Lemma. A more elementary proof, still for large enough n, was obtained by Allen [1]. Finally, Bessy and Thomassé [4] found a completely elementary inductive proof for every n.

The $t = 3$ case of Conjecture 1.1 was solved asymptotically in [15]. Pokrovskiy [24] showed recently (with a nice elementary proof) that the path partition number of any 3-colored K_n is at most three (for any $n \geq 1$). But then surprisingly Pokrovskiy [25] found a counterexample to Conjecture 1.1 for all $t \geq 3$. However, in the counterexample all but one vertex can be covered by t vertex disjoint monochromatic cycles.

For general t, the best bound for the cycle partition number is $O(t \log t)$, see [9]. Note that it is far from obvious that the cycle partition number of K_n can be bounded by *any* function of t.

We address the extension of the cycle and path partition numbers from complete graphs to arbitrary graphs G. If we want these numbers to be independent of $|V(G)|$, some other parameter of G must be included. We consider three of these parameters.

Let $\alpha(G)$ denote the independence number of G, the maximum number of pairwise non-adjacent vertices of G. The role of $\alpha(G)$ in results on colorings of non-complete graphs was observed in [10, 11, 16] and in Sárközy [27] who extended Conjecture 1.1 to the following.

Conjecture 1.2. *The cycle partition number of any t-colored graph G is $t\alpha(G)$.*

For $t = 1$, Conjecture 1.2 is a well-known result of Pósa [23] (and clearly best possible). For $t = 2$ it is also best possible, shown by vertex disjoint copies of triangles, each colored using two colors. To prove Conjecture 1.2 for $t = 2$ and arbitrary $\alpha(G)$ seems very difficult (considering the complexity of the proof for $\alpha(G) = 1$ in [4]). Then again the counterexample of Pokrovskiy [25] shows that the conjecture is not true in this form for any $t \geq 3$. Perhaps the following weakening of the conjecture is true.

Conjecture 1.3. *Let G be a t-colored graph with $\alpha(G) = \alpha$. Then there exists a constant $c = c(\alpha, t)$ such that $t\alpha$ vertex disjoint monochromatic cycles of G cover at least $n - c$ vertices.*

Pokrovskiy's example implies that $c \geq \alpha$ must be true. We cannot prove this conjecture even for $t = 2$, we can only prove the following weaker asymptotic result.

Theorem 1.4. *For every positive η and α, there exists an $n_0(\eta, \alpha)$ such that the following holds. If G is a 2-colored graph on n vertices, $n \geq n_0$, $\alpha(G) = \alpha$, then there are at most 2α vertex disjoint monochromatic cycles covering at least $(1 - \eta)n$ vertices of $V(G)$.*

Recently, Schelp [28] suggested in a posthumous paper to strengthen certain Ramsey problems from complete graphs to graphs of given minimum degree. In particular, he conjectured that with $m = R(P_n, P_n)$, minimum degree $\frac{3m}{4}$ is sufficient to find a monochromatic path P_n in any 2-colored graph of order m.[2] Influenced by this conjecture, here we pose the following conjecture.

Conjecture 1.5. *If G is an n-vertex graph with $\delta(G) > 3n/4$ then in any 2-edge-coloring of G, there are two vertex disjoint monochromatic cycles of different colors, which together cover $V(G)$.*

That is, the above mentioned Bessy-Thomassé result [4] would hold for graphs with minimum degree larger than $3n/4$. Note that the condition $\delta(G) \geq \frac{3|V(G)|}{4}$ is sharp. Indeed, consider the following n-vertex graph, where $n = 4m$. We partition the vertex set into four parts A_1, A_2, A_3, A_4 with $|A_i| = m$. There are no edges from A_1 to A_2 and from A_3 to A_4. Edges in $[A_1, A_3], [A_2, A_4]$ are red and edges in $[A_1, A_4], [A_2, A_3]$ are blue, inside the classes any coloring is allowed. In such an edge-colored graph, there are no two vertex disjoint monochromatic cycles of *different colors* covering G, while the minimum degree is $3m - 1 = \frac{3n}{4} - 1$.

We prove Conjecture 1.5 in the following asymptotic sense.

Theorem 1.6. *For every $\eta > 0$, there is an $n_0(\eta)$ such that the following holds. If G is an n-vertex graph with $n \geq n_0$ and $\delta(G) > (\frac{3}{4} + \eta)n$, then every 2-edge-coloring of G admits two vertex disjoint monochromatic cycles of different colors covering at least $(1 - \eta)n$ vertices of G.*

The proofs of Theorems 1.4 and 1.6 follow a method of Łuczak [21]. The crucial idea is that the words "cycles" or "paths" in a statement to be proved are replaced by the words "connected matchings". In a *connected matching*, the edges of the matching are in the same component of the graph.[3] We prove first this weaker result, then we apply to the cluster graph of a regular partition of the target graph. Through several technical details, the regularity of the partition is used to "lift back" the connected matching of the cluster graph to a path or cycle in the original graph. In our case, the relaxed versions of Theorems 1.4 and 1.6 for connected matchings are stated and proved in Section 2 (Theorem 2.4 and 2.5).

Another possibility to extend Conjecture 1.1 to more general graphs is to consider a graph G, whose complement does not contain a fixed bipartite graph H. This brings in a different flavor, since these graphs are very dense, they have $\binom{|V(G)|}{2} - o(|V(G)|^2)$ edges. In return, we prove sharper results in this case. We also state a more general conjecture.

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[2]Some progress towards this conjecture have been done in [17] and [3].

[3]When the edges are colored, a connected red matching is a matching in a red component.

Conjecture 1.7. *Let H be a graph with chromatic number $k+1$ and let G be an t-edge-colored graph on n vertices such that H is not a subgraph of $\overline{G}$. Then there exists a constant $c = c(H, k, t)$ such that kt vertex disjoint monochromatic paths of G cover at least $n - c$ vertices.*

In Section 4, we prove Conjecture 1.7 for $k = 1, t = 2$ (Theorem 4.6) and in particular, $c(C_4, 1, 2) = 1$ (Theorem 4.8). Note that this conjecture is related to Conjecture 1.3 by selecting H to be the complete graph of size $k+1$.

2 Partitioning into connected matchings.

In this section we prove Conjectures 1.2 and 1.5 in weakened forms, replacing cycles and paths with connected matchings (Theorems 2.4, 2.5). We notice first that the $t = 1$ case of Conjecture 1.2 is due to Pósa [23].[4]

Lemma 2.1. *The vertex set of any graph G can be partitioned into at most $\alpha(G)$ parts, where each part either contains a spanning cycle, or spans an edge or a vertex.*

For two colors, we need the following result, which is essentially equivalent to König's theorem. It was discovered in [11] and applied in [16].

Lemma 2.2. *Let the edge set of G be colored with two colors. Then $V(G)$ can be covered with the vertices of at most $\alpha(G)$ monochromatic connected subgraphs of G.*

Proof. For a graph G whose edges are colored with red and blue, let $\rho(G)$ denote the minimum number of monochromatic components covering the vertex set of G. Let $\alpha^*(G)$ be the maximum number of vertices in G so that no two of them is covered by a monochromatic component. Suppose that the red edges define connected components $C_1, \dots, C_p$ and the blue edges define connected components $D_1, \dots, D_q$. Define a bipartite multigraph B with vertex classes $C_1, \dots, C_p$ and $D_1, \dots, D_q$. For every vertex $v \in V(G), v \in A_i, v \in B_j$ we define the edge C_i, D_j in B. (In fact, B is the dual of the hypergraph formed by the monochromatic components on $V(G)$.)

Recall that $\nu(B)$ is the maximum number of pairwise disjoint edges in B and $\tau(B)$ is the minimum cover, i.e., the least number of vertices in B that meet all edges of B. From König's theorem and from easy observations follows that

$$\rho(G) = \tau(B) = \nu(B) = \alpha^*(G) \leq \alpha(G) \tag{1}$$

finishing the proof. □

Observe that (1) gives a stronger form of Lemma 2.2 (equivalent form of König's theorem).

[4] See also Exercise 3 on page 63 in [20].

Proposition 2.3. *For any 2-edge colored graph G, $\rho(G) = \alpha^*(G)$.*

Theorem 2.4. *If the edges of a graph G are colored red and blue, then $V(G)$ can be partitioned into at most $2\alpha(G)$ monochromatic parts, where each part is either an edge, or a single vertex, or contains a connected matching or a spanning cycle.*

It is worth noting that Theorem 2.4 is best possible, although it is weaker than Conjecture 1.2. Indeed, let G be formed by k vertex disjoint copies of K_s, where $s \geq 3$. We color $E(G)$ so that in each K_s the set of blue edges forms a K_{s-1}. Here $\alpha(G) = k$, and we need two parts to cover each K_s, one in each color.

Proof of Theorem 2.4. Set $V = V(G)$. By Lemma 2.2, we can cover V by the vertices of some p red and q blue monochromatic components, $C_1, \dots, C_p, D_1, \dots, D_q$, where $p + q \leq \alpha(G)$. We partition V into the doubly and singly covered sets. Let $A_{ij} = C_i \cap D_j$ and $S_i = C_i - \cup_j A_{ij}$, $T_j = D_j - \cup_i A_{ij}$, where $1 \leq i \leq p, 1 \leq j \leq q$.

Fix M_i, a largest red matching in C_i for every i, and then let N_j be a largest blue matching in $D_j - \cup_i V(M_i)$. These $p + q \leq \alpha(G)$ monochromatic matchings are connected. Delete the vertices of these matchings from V and for convenience keep the same notation for the truncated sets, so A_{ij}, S_i, T_j denote the sets remaining after all vertices of these matchings are deleted. Denote the remaining graph by G_1, and its vertex set by V_1. Partition V_1 into three sets, $A = \cup_{i=1}^{p} \cup_{j=1}^{q} A_{ij}, S = \cup_{i=1}^{p} S_i, T = \cup_{j=1}^{q} T_j$. Observe that there are no edges between S and T.

Edges of G_1 can only be inside S (colored blue) or inside T (colored red). Applying Lemma 2.1 for the blue and red graphs $G_1[S], G_1[T]$, we can cover $S \cup T$ by $\alpha(G_1[S]) + \alpha(G_1[T])$ parts, where each part contains a monochromatic spanning cycle or it is an edge or a vertex. Now A is a collection of isolated points in G_1; we just cover it with its vertices. Altogether, we partitioned V_1 into $|A| + \alpha(G_1[S]) + \alpha(G_1[T]) \leq \alpha(G_1) \leq \alpha(G)$ parts and together with the monochromatic connected matchings M_i, N_j, there are at most $2\alpha(G)$ parts as required. $\square$

Theorem 2.5. *Let $G = (V, E)$ be an n-vertex graph with $\delta(G) \geq 3n/4$, where n is even. If the edges of G are 2-colored with red and blue, then there exist a red connected matching and a vertex-disjoint blue connected matching, which together form a perfect matching of G.*

Proof. Let C_1 be a largest monochromatic component, say red. Theorem 1.4 in [17] yields $|C_1| \geq 3n/4$. Let $U = V \setminus V(C_1)$. Any vertex u in U can only have less than $n/4$ red neighbors. Therefore, the blue degree of u is at least $n/2$. This implies that the blue neighborhoods of any two vertices in U which are not connected with a blue edge intersect. Therefore, if $U \neq \emptyset$, then U is covered by a blue component of G, say C_2. If $U = \emptyset$, then define C_2 as a largest blue component in G. Set $p = |V(C_1) \setminus V(C_2)|, q = |V(C_2) \setminus V(C_1)|$, where $p \geq q$ by the choice of C_1. Let G_1

be the graph, which we get from G by deleting the blue edges inside $C_1 \setminus C_2$ and the red edges inside $C_2 \setminus C_1$. Note that in Cases 2 and 3 $C_2 \setminus C_1 = \emptyset$. We distinguish three cases.

Case 1: Suppose $|C_1| < n$. By the maximality of C_1 and C_2, there are no edges between $C_1 \setminus C_2$ and $C_2 \setminus C_1$. Therefore, $q < n/4$ and $p < n/4$. We claim that G_1 satisfies the Dirac-property[5], $\delta(G_1) \geq n/2$. Indeed, we deleted at most $n/4 - 1$ edges at any vertex, and thus the remaining degree is more than $n/2$ at each vertex. Therefore, there is a Hamiltonian cycle, that also contains a perfect matching. This perfect matching consists of a connected red matching and a connected blue matching covering G.

Case 2: Suppose $|C_1| = n$ and $p \leq n/2$. Now we claim that G_1 satisfies the Chvátal-property[6]: if the degree sequence in G_1 is $d_1 \leq d_2 \leq \ldots \leq d_n$, then $d_k + d_{n-k} \geq n$ for $k \leq n/2$. Indeed, the degrees of the p vertices in $C_1 \setminus C_2$ are at least $3n/4 - p + 1$, where $p \leq n/2$. The rest of the degrees are unchanged being at least $3n/4$. That yields $3n/4 - p + 1 + 3n/4 = 3n/2 - p + 1 > n$ in the Chvátal-condition. This implies the existence of a Hamiltonian cycle, which contains a perfect matching. This perfect matching contains a connected red matching and a connected blue matching, which together cover G.

Case 3: Suppose $|C_1| = n$ and $p > n/2$. That is, $|C_2| - p < n/2$. Again, we claim that there is a perfect matching in G_1. Assume to the contrary that the largest matching is imperfect. By Tutte's theorem, there exists a set X of vertices in G_1 such that the number of odd components in $G_1 \setminus X$ is larger than $|X|$, which implies that $|X| < n/2$. Let all the components (not just the odd ones) be $D_1, D_2, \ldots, D_\ell$ in increasing order of their size, $\ell \geq |X| + 1$. Note that $\ell \geq 2$ always holds, even for $X = \emptyset$, as n is even. Notice, that any potential edge in G between two components of $G_1 \setminus X$ is a blue edge inside $C_1 \setminus C_2$ that was deleted. Let H be the graph formed by the vertices in $G \setminus X$, and the blue edges in $C_1 \setminus C_2$. Since $|X| < n/2$, we have $|V(H)| > n/2$.

Suppose first that $|X| = x < n/4$. Let us consider the smallest component D_1 and put $|D_1| = d_1$. We claim that

$$d_1 + x \leq n - |D_1 \cup X|. \tag{2}$$

For $d_1 = 1$, using n being even, we also get (2) from $|D_1 \cup X| = 1 + x \leq n/2$. When $x = 0$ then (2) is true as $\ell \geq 2$, when $x = 1$ then (2) is true because n is even. For $x \geq 2$ and $d_1 \geq 2$ we have $|D_1 \cup X| = d_1 + x \leq d_1 x \leq n - |D_1 \cup X|$, implying (2).

From (2), the blue neighborhoods of any two vertices in D_1 intersect in H, and D_1 is covered by a blue component C_2'. Using $x < n/4$, we get $|C_2'| \geq 3n/4 - d_1 -$

[5] Exercise 21 on page 75 in [20].

[6] Exercise 21 on page 75 in [20].

$x+1+d_1-1 = 3n/4 - x > n/2$. That is a contradiction since C_2 was the largest blue component and $|C_2| < n/2$.

Now we may assume $n/4 \le |X| < n/2$. Since $|X| < n/2$ we have $V(H) > n/2$. If we prove that H is connected, then we get a contradiction again, since C_2 was the largest blue component, and $|C_2| < n/2$. Assume to the contrary that we can partition the vertices of H into A and B with no edges between them. We may assume $|A| \ge |B|$, and therefore $|A| > n/4$. We have two subcases.

Case 3.a: Suppose $A \cap D_i \ne \emptyset$ for $1 \le i \le \ell$. Let v be a vertex in B and assume $v \in D_j$. There is no edge of G from v to $A \cap D_i$, for each $i \ne j$, $1 \le i \le \ell$: An edge from G_1 is impossible, because $i \ne j$; a blue edge from $C_1 \setminus C_2$ is impossible, because (A, B) is a cut in H. Therefore, the degree of v in G is at most $n - 1 - \ell + 1 \le n - (|X| + 1) \le n - 1 - n/4 < 3n/4$, a contradiction.

Case 3.b: Suppose $A \cap D_j = \emptyset$ for a fixed j, $1 \le j \le \ell$. Let v be a vertex in D_j. There is no edge from v to any vertex u of A: An edge from G_1 is impossible, because $u \in D_i$, where $i \ne j$. A blue edge from $C_1 \setminus C_2$ is impossible, because (A, B) is a cut. Therefore, the degree of v in G is at most $n - 1 - |A| \le n - 1 - n/4 < 3n/4$, a contradiction. $\square$

3 Applying the Regularity lemma.

As in many applications of the Regularity Lemma, one has to handle irregular pairs, that translates to exceptional edges in the reduced graph. To prove such a variant of Theorem 2.4, first Lemma 2.2 is tuned up. A graph G on n vertices is *ε-perturbed* if at most $\varepsilon\binom{n}{2}$ of its edges are marked as exceptional (or perturbed). For a perturbed graph G, let G^- denote the graph obtained by removing all perturbed edges.

Lemma 3.1. *Suppose that G is a 2-edge-colored ε-perturbed graph on n vertices, $n \ge \varepsilon^{-1/2}$. Then all but at most $f(\alpha(G))\sqrt{\varepsilon}n$ vertices of G can be covered by the vertices of $\alpha(G)$ monochromatic connected subgraphs of G^-, where f is a suitable function.*

Proof. Set $\alpha = \alpha(G)$ and remove from $V(G)$ a set X of at most $\sqrt{\varepsilon}n$ vertices so that in the remaining graph H each vertex is incident to at most $\sqrt{\varepsilon}n$ perturbed edges.

Let $\mathcal{T}$ denote the (possibly edgeless) hypergraph whose edges are those sets $T \subset V(H)$ for which $|T| = \alpha+1$ and no monochromatic component of H^- covers more than one vertex of T. (Each $T \in \mathcal{T}$ is a witness showing $\alpha^*(H^-) \ge \alpha+1$.) We call pairwise disjoint hyperedges $T_1, T_2, \dots, T_k$ in $\mathcal{T}$ *independent*, if there are no perturbed edges in the k-partite graph defined by the T_i-s. Set $c = 3^{\alpha^2}$ and let $R = R(3, 3, \dots, 3, \alpha+1)$ be the c-color Ramsey number, the smallest m such that in every c-coloring of the edges of K_m either there is a triangle in one of the first $c-1$ colors or a $K_{\alpha+1}$ in color c.

Claim 3.2. *Select in $\mathcal{T}$ as many pairwise independent hyperedges as possible, say $T_1, T_2, \dots, T_k$. Then $k < R$.*

Proof. Fix an ordering within each of the sets T_i; if $x \in T_i$ is the j-th element in this order in T_i, we write $ind(x) = j$. Suppose for contradiction that $k \geq R$ and consider a coloring of the pairs among $T_1, T_2, \dots, T_k$ defined as follows. Color a pair T_i, T_j $(1 \leq i < j \leq k)$ by their "color pattern" on the pairs $x \in T_i, y \in T_j$ with $ind(x) \neq ind(y)$. There are α^2 such pairs (none of them is a perturbed edge) thus x, y is a red edge, a blue edge or not an edge in H. So we have a c-coloring on the pairs T_i, T_j, the color when all the α^2 pairs are not edges of H is called special. By the assumption $k \geq R$, we have either $\alpha + 1$ T_i-s with any pair of them colored with the special color or three T_i-s with all three pairs colored with the same non-special color. We show that both cases lead to contradiction.

In the latter case we have a triple, say T_1, T_2, T_3 and different indices i, j, such that $p \in T_1, q, r \in T_2, s \in T_3$, $ind(p) = ind(q) = i, ind(r) = ind(s) = j$ and pr, ps, qs are all edges of H colored with the same color. Thus r, p, s, q is a monochromatic path of H^-, intersecting T_2 in two vertices, contradicting to the definition of T_2.

In the former case we have say $T_1, T_2, \dots, T_{\alpha+1}$ pairwise colored with the special color. For $i = 1, 2, \dots, \alpha + 1$, select $v_i \in T_i$ such that $ind(v_i) = i$. Observe that $\{v_1, \dots, v_{\alpha+1}\}$ spans an independent set in G, contradicting the assumption that $\alpha(G) = \alpha$. □

Let Y denote the set of vertices in H sending at least one perturbed edge to $\cup_{i=1}^k T_i$. Observe that $|Y| \leq (\alpha+1)R\sqrt{\varepsilon}n$ and by the maximality of k, $Z = \cup_{i=1}^k T_i \cup Y$ meets all edges of $\mathcal{T}$, thus removing $X \cup Z$ from V(G) leaves a subgraph $F \subset G$ with $\alpha^*(F^-) \leq \alpha$. Therefore, applying Proposition 2.3 to F^-, $\rho(F^-) \leq \alpha$. The theorem follows, since (using the assumption $1 \leq \sqrt{\varepsilon}n$)

$$|X \cup Z| \leq \sqrt{\varepsilon}n + R(\alpha + 1) + (\alpha + 1)R\sqrt{\varepsilon}n \leq (1 + 2R(\alpha + 1)\sqrt{\varepsilon}n,$$

i.e. $f(\alpha) = (1 + 2R(\alpha + 1))$ is a suitable function. □

Now we are ready to prove a *perturbed* version of Theorem 2.4.

Theorem 3.3. *Let G be an ε-perturbed 2-edge-colored graph on n vertices, $n \geq \varepsilon^{-1/2}$. Then there exists a $Z \subset V(G)$ such that $|Z| \leq (f(\alpha(G)) + \alpha(G))\sqrt{\varepsilon}n$ and $V(G) \setminus Z$ can be partitioned into at most $2\alpha(G)$ classes, where each part in G^- either contains a connected monochromatic spanning matching or a monochromatic spanning cycle or it is an edge or a single vertex.*

Proof. Using Lemma 3.1, we can remove from $V(G)$ a set of at most $f(\alpha)\sqrt{\varepsilon}n$ vertices such that for the remaining graph H, the following holds. The vertices $V(H)$ can be covered by the vertices of at most $\alpha(G)$ monochromatic components of H^-, say with p red and q blue monochromatic components, $C_1, \dots, C_p, D_1, \dots, D_q$, where $p + q \le \alpha(G)$. We may suppose that each vertex of H is incident to at most $\sqrt{\varepsilon}n$ perturbed edges, as this is automatic from the proof of Lemma 3.1. The $p + q$ components yield a partition of $V(H)$ into doubly and singly covered sets. Let $A_{ij} = C_i \cap D_j$ and $S_i = C_i - \cup_j A_{ij}$, $T_j = D_j - \cup_i A_{ij}$, where $1 \le i \le p, 1 \le j \le q$. First let M_i be a largest red matching induced by H^- in C_i for every $1 \le i \le p$, and then N_j be a largest blue matching induced by H^- in $D_j - \cup_i V(M_i)$, for every $1 \le j \le q$. Observe that these matchings are connected in H^-. Delete all vertices of these matchings from $V(H)$ and for convenience keep the same notation for the truncated sets (so A_{ij}, S_i, T_j denotes the sets remaining after all vertices of these matchings are deleted). The remaining graph is denoted by F. Partition $V(F)$ into three sets, $A = \cup_{i=1}^p \cup_{j=1}^q A_{ij}, S = \cup_{i=1}^p S_i, T = \cup_{j=1}^q T_j$. Observe that edges of F^- can be only inside S (colored blue) or inside T (colored red). Now we follow the proof method of Lemma 2.1 (see Exercise 3 on page 63 in [20]) to partition most of the vertices in $V(F)$ into at most $\alpha(G)$ monochromatic cycles.

We apply the following procedure to subsets U of one of the sets A, S, T. Observe that $F^-[U]$ is an independent set if $U \subset A$, edges of $F^-[U]$ are all blue if $U \subset S$, edges of $F^-[U]$ are all red if $U \subset T$.

In any step of the procedure, consider a maximal path P of $F^-[U]$ and let x be one of its endpoints. If x is an isolated vertex in $F^-[U]$, define $C^* = \{x\}$. If x has degree one in F^-, let y be its neighbor on P and define $C^* = \{x, y\}$. If x has degree at least two in F^-, let z be the neighbor of x on P (in F^-), which is the furthest from x. Now C^* is defined as the cycle obtained by connecting the endpoints of the edge xz on the path P. Let Y be the set of perturbed neighbors of x in F^-. That is, the set of vertices in $V(F)$, which are adjacent to x by exceptional edges. The step ends with removing $C^* \cup Y$ from $V(F)$ and defining the new F, A, S, T as the truncated sets.

This procedure decreases $\alpha(F)$ at each step, because any independent set of the truncated set can be extended by x to an independent set of F. Therefore, at most $\alpha(G)$ steps can be executed. Now apart from the union of the sets Ys, at most $\alpha(G)$ monochromatic C^*-s partition $V(F)$. Together with the $p + q \le \alpha$ monochromatic connected matchings N_i, M_j we have the required covering. The number of uncovered vertices are at most $f(\alpha)\sqrt{\varepsilon}n$ (lost when the matchings were defined) plus $\alpha\sqrt{\varepsilon}n$ (when the cycles are defined). $\square$

3.1 Building cycles from connected matchings.

Next we show how to prove Theorem 1.4 from Theorem 3.3 and the Szemerédi Regularity Lemma [29]. The material of this section is fairly standard by now (see [9, 12, 13, 14, 15] so we omit some of the details. We need a 2-edge-colored version of the Szemerédi Regularity Lemma.[7]

Lemma 3.4. *For every integer m_0 and positive ε, there is an $M_0 = M_0(\varepsilon, m_0)$ such that for $n \geq M_0$ the following holds. For any n-vertex graph G, where $G = G_1 \cup G_2$ with $V(G_1) = V(G_2) = V$, there is a partition of V into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$ such that*

- *$m_0 \leq \ell \leq M_0$, $|V_1| = |V_2| = \ldots = |V_\ell|$, $|V_0| < \varepsilon n$,*
- *apart from at most $\varepsilon\binom{\ell}{2}$ exceptional pairs, all pairs $G_s|_{V_i \times V_j}$ are ε-regular, where $1 \leq i < j \leq \ell$ and $1 \leq s \leq 2$.*

Proof of Theorem 1.4. Given η and α, first we fix a positive ε sufficiently small so that the claimed bound $(f(\alpha) + \alpha)\sqrt{\varepsilon}$ in Theorem 3.3 is much smaller than η. Then we choose m_0 sufficiently large compared to $1/\sqrt{\varepsilon}$ (so Theorem 3.3 can be applied). Let G be a graph on n vertices with $\alpha(G) = \alpha$, where $n \geq M_0$ with M_0 coming from Lemma 3.4. Consider a 2-edge-coloring of G, that is $G = G_1 \cup G_2$. We apply Lemma 3.4 to G in order to obtain a partition of V, that is $V = \cup_{0 \leq i \leq \ell} V_i$. Define the following *reduced graph* G^R: The vertices of G^R are $p_1, \ldots, p_\ell$, and there is an edge between vertices p_i and p_j if the pair (V_i, V_j) is either exceptional[8], or if it is ε-regular in both G_1 and G_2 with density in G exceeding $1/2$. The edge $p_i p_j$ is colored with the color, which is used on the most edges from $G[V_i, V_j]$ (the bipartite subgraph of G with edges between V_i and V_j). The density of this majority color is still at least $1/4$ in $G[V_i, V_j]$. This defines a 2-edge-coloring $G^R = G_1^R \cup G_2^R$.

We claim that $\alpha(G^R) \leq \alpha(G) = \alpha$. Indeed, we apply the standard Key Lemma[9] in the complement of G^R and G. Note that a non-exceptional pair is 2ε-regular in $\overline{G}$ as well. If we had an independent set of size $\alpha + 1$ in G^R, then we would have an independent set of size $\alpha + 1$ in G, a contradiction.

We now apply Theorem 3.3 to the ε-perturbed 2-edge-colored G^R (note that the condition in Theorem 3.3 is satisfied since $\ell \gg 1/\sqrt{\varepsilon}$). We cover most of G^R by at most $2\alpha(G^R) \leq 2\alpha(G) = 2\alpha$ subgraphs of $(G^R)^-$, where each subgraph in $(G^R)^-$ is either a connected monochromatic matching or a monochromatic cycle or an edge or a single vertex. Finally, we lift the connected matchings back to cycles in the original

[7] For background, this variant and other variants of the Regularity Lemma see [18].

[8] That is, ε-irregular in G_1 or in G_2. Also, these edges are marked exceptional in G^R.

[9] Theorem 2.1 in [18].

graph using the following[10] lemma in our context, completing the proof. Indeed, the number of vertices left uncovered in G is at most

$$(f(\alpha)+\alpha)\sqrt{\varepsilon}n + 3\epsilon n + \epsilon n = (f(\alpha)+\alpha)\sqrt{\varepsilon}n + 4\epsilon n \leq \eta n,$$

using our choice of ϵ. Here the uncovered parts come from Theorem 3.3, from Lemma 3.5 and V_0. $\square$

Lemma 3.5. *Assume that there is a monochromatic connected matching M (say in $({G_1}^R)^-$) saturating at least $c|V(G^R)|$ vertices of G^R, for some positive constant c. Then in the original G there is a monochromatic cycle in G_1 covering at least $c(1-3\varepsilon)n$ vertices.*

Proof of Theorem 1.6. We combine the degree form and the 2-edge-colored version of the Regularity Lemma.

Lemma 3.6. *For every positive ε and integer m_0, there is an $M_0 = M_0(\varepsilon, m_0)$ such that for $n \geq M_0$ the following holds. For any n-vertex graph G, where $G = G_1 \cup G_2$ with $V(G_1) = V(G_2) = V$, and real number $\rho \in [0,1]$, there is a partition of V into $\ell+1$ clusters $V_0, V_1 \ldots, V_\ell$, and there are subgraphs $G' = G'_1 \cup G'_2$, $G'_1 \subset G_1$, $G'_2 \subset G_2$ with the following properties:*

- *$m_0 \leq \ell \leq M_0$, $|V_0| \leq \varepsilon|V|$, $|V_1| = \ldots = |V_\ell| = L$,*
- *$deg_{G'}(v) > deg_G(v) - (\rho+\varepsilon)|V|$ for all $v \in V$,*
- *the vertex sets V_i are independent in G',*
- *each pair $G'|_{V_i \times V_j}$ is ε-regular, $1 \leq i < j \leq \ell$, with density 0 or exceeding ρ,*
- *each pair $G'_s|_{V_i \times V_j}$ is ε-regular, $1 \leq i < j \leq \ell, 1 \leq s \leq 2$.*

Let $\varepsilon \ll \rho \ll \eta \ll 1$, m_0 sufficiently large compared to $1/\varepsilon$ and M_0 obtained from Lemma 3.6. Let G be a graph on $n > M_0$ vertices with $\delta(G) > (\frac{3}{4}+\eta)n$. Consider a 2-edge-coloring of G, that is $G = G_1 \cup G_2$. We apply Lemma 3.6 to G. We obtain a partition of V, that is $V = \cup_{0 \leq i \leq \ell} V_i$. We define the following *reduced graph* G^R: The vertices of G^R are $p_1, \ldots, p_\ell$, and there is an edge between vertices p_i and p_j if the pair (V_i, V_j) is ε-regular in G' with density exceeding ρ. Since $\delta(G') > (\frac{3}{4}+\eta-(\rho+\varepsilon))|V|$, calculation[11] shows that $\delta(G^R) \geq \left(\frac{3}{4}+\eta-2\rho\right)\ell > \frac{3}{4}\ell$. The edge p_ip_j is colored again with the majority color, and the density of this color is still at least $\rho/2$ in $K(V_i, V_j)$.

[10] As in [12, 13, 14, 15].

[11] See a similar computation in [26].

Applying Theorem 2.5 to G^R, we get a red connected matching and a vertex-disjoint blue connected matching, which together form a perfect matching of G^R. Finally we lift the connected matchings back to cycles in the original graph using Lemma 3.5. The number of vertices left uncovered in G is at most $\sqrt{\varepsilon}n \leq \eta n$. $\square$

4 Excluding bipartite graphs from the complement.

In what follows, we prove the $t = 2, k = 1$ case of Conjecture 1.7. As every bipartite graph is a subgraph of a complete bipartite graph, we may assume that the graph H forbidden in the complement of G is $K_{p,p}$. Note that the constant c we get could be greatly improved even using the same arguments with more involved calculations, however, it would be still far from being optimal. We use the following well-known theorems.

Theorem 4.1 (Erdős-Gallai [5]). [12] *If G is a graph on n vertices with $|E(G)| > \ell(n-1)/2$, then G contains a cycle of length at least $\ell + 1$.*

Theorem 4.2 (Kővári-T. Sós-Turán [19]). [13] *If G is a graph on n vertices such that $K_{p,p}$ is not a subgraph of G, then $|E(G)| \leq (p-1)^{1/p}n^{2-1/p} + (p-1)n \leq 2pn^{2-1/p}$.*

Lemma 4.3. *Let p and n be positive integers such that $n \geq (10p)^p$. Let G be an n-vertex graph such that $K_{p,p} \not\subset \overline{G}$. Then any 2-edge-coloring of G contains a monochromatic cycle of length at least $n/4$.*

Proof. By Theorem 4.2 and by the lower bound on n,

$$e(G) \geq \binom{n}{2} - 2pn^{2-1/p} = n^2/2 - n/2 - 2pn^{2-1/p} \geq n^2/2 - n/2 - n^2/5 \geq n^2/4,$$

so one of the colors, say red, is used at least $n^2/8$ times. Then using Theorem 4.1 in the red subgraph we get a red cycle of length at least $n/4$. $\square$

For a bipartite graph G with classes A, B, the *bipartite complement* $\overline{G}[A, B]$ of G is obtained via complementing the edges between A and B, and keeping A and B independent sets.

Lemma 4.4. *Let $0 < \epsilon < 1$ and $n \geq (50p)^p/\epsilon$. Let G be a bipartite graph with classes A and B, $|A| = |B| = n$ such that $K_{p,p} \not\subset \overline{G}[A, B]$. Then there is a path of length at least $(2-\epsilon)n$ in G.*

[12] See also Exercise 28 on page 76 in [20].
[13] See also Exercise 37 on page 77 in [20].

Proof. First we prove a weaker statement.

Claim 4.5. *Let G' be a bipartite graph with classes A' and B' with $|A'| = |B'| = m \geq (20p)^p$ such that $K_{p,p} \not\subset \overline{G}'[A', B']$. Then there is a path of length at least $m/2$ in G'.*

Proof. By Theorem 4.2, $e(G') \geq m^2 - 8pm^{2-1/p} > m^2/2 = (2m)^2/8$, so by Theorem 4.1 G' contains a path of length at least $m/2$. $\square$

Let P be a longest path in G. Using Claim 4.5 with $G = G'$, we have that $|P| \geq n/2$. Assume for a contradiction that P is shorter than $(2-\epsilon)n$. Because G is bipartite, we can choose $A' \subset (G-P) \cap A$ and $B' \subset (G-P) \cap B$ with $|A'| = |B'| > \epsilon n/3$. By Claim 4.5, $G[A', B']$ contains a path P' with at least $\epsilon n/6$ vertices.

Consider the last $2p$ vertices of P and the last $2p$ vertices of P'. There is an edge e between these set of vertices by the assumption. Adding e to $P \cup P'$, there is a path, which contains all but $2p$ vertices of P, and all but $2p$ vertices of P', hence it is longer than P, a contradiction. Here we used that $\epsilon n/6 > 4p$. $\square$

Theorem 4.6. *Let G be an n-vertex graph such that $K_{p,p} \not\subseteq \overline{G}$. Then any 2-edge-coloring of G contains two vertex disjoint monochromatic paths of distinct colors covering at least $n - 1000(50p)^p$ vertices.*

Proof. Consider the vertex disjoint blue path, red path pair (P_1, P_2), which cover the most vertices, and let $G' = G \setminus \{P_1 \cup P_2\}$. Suppose there are n_1 vertices in G', where $n_1 > 1000(50p)^p$. As $n > n_1 > 1000(50p)^p$, by Lemma 4.3 at least $n/4$ vertices are covered by $P_1 \cup P_2$. Let $t = 10(50p)^p < n_1/100$. We split the proof into two cases.

<u>Case 1:</u> One of the paths, P_2 say, is shorter than t. Using that $3t < n/4$ we have that the length of P_1 is at least $2t$ in this case. Now G' does not contain a red path of length t, but by Lemma 4.3 it contains a monochromatic cycle of length at least $n_1/4 > 4t$, which must be blue. Hence, G' contains a blue path, say P_3, of length at least $4t$.

Denote L_1, the set of last $2t$ vertices of P_1 and L_3, the set of last $2t$ vertices of P_3. There is an edge e between L_1 and L_3 as $2t > p$ and $K_{p,p} \not\subseteq \overline{G}$. If e was blue then we use e to connect the paths P_1, P_3, and we find a blue path longer than P_1 vertex disjoint from P_2, a contradiction.

Hence all edges between L_1 and L_3 are red, and we can apply Lemma 4.4 for the red bipartite graph between L_1 and L_3 with $\epsilon = 1/8$. (Note that $2t \geq 8(50p)^p$, so indeed the lemma is applicable.) It yields a red path P_4 of length $(2-1/8)2t$ in $L_1 \cup L_3$. Let P_1' be P_1 without the last $2t$ vertices. Now P_1' and P_4 are disjoint and cover more vertices than P_1 and P_2, which is a contradiction.

Case 2: Both P_1 and P_2 have length at least t. Without loss of generality, in G' Lemma 4.3 implies the existence of a blue cycle C of length at least $n_1/4 \geq 4t$. Denote R_1 the set of the last t vertices of P_1, R_2 the set of the last $t/2$ vertices of P_2, and C_1 any set of consecutive t vertices of C. There are no blue edges between R_1 and C_1, otherwise P_1 could be replaced with a longer blue path. Now by Lemma 4.4, with $\epsilon = 1/8$, there is a red path P_3 in $G(R_1, C)$ of length $15t/8$. Let B be the set of the first and last $t/4$ vertices of P_3. For each vertex v in B, there is a red path P_v of length $13t/8$ starting at v, which is a subpath of P_3. If there is a red edge $e = (u, v)$ between R_2 and B, then $P_2 \cup e \cup P_v$ contains a red path with at least $|P_2| + 13t/8 - t/2$ vertices which together with the disjoint $P_1 - R_1$ cover more vertices than the pair (P_1, P_2), a contradiction.

Therefore, there are only blue edges between B and R_2. Since $|B \cap P_1| \geq p$, there are at least $t/2 - p + 1$ vertices of R_2 having neighbors in $B \cap P_1$. Let R_2' be the set of those vertices. If there is a blue edge f between R_2' and C, then $P_1 \cup f \cup C$ contains a blue path which together with the disjoint $P_2 - R_2$ cover more vertices than the pair (P_1, P_2), a contradiction.

Therefore, all the edges between R_2' and C are red. We already know that there are no red edges from R_2' and $B \cap C$. But we have that $|R_2'| \geq p$ and $|B \cap C| \geq p$, which is a contradiction. $\square$

The following proposition, which is a 1-colored version of one of our main results, Theorem 4.8, is also a special case of $R(P_m, C_n)$, determined in [7].

Proposition 4.7. *If G is a graph on n vertices and $C_4 \not\subseteq \overline{G}$, then G contains a path, which covers $n-1$ vertices.*

Proof. Denote by P a longest path of G. Let a and b be the first and last vertex of P. If P contains less than $n-1$ vertices, then there are two vertices x and y not in P. Let us consider the pairs ax, xb, by, ya. If none of them spans an edge in G, then they span a C_4 in $\overline{G}$, which is a contradiction. If any of them spans an edge in G, then it extends P, which is again a contradiction. $\square$

The following result, the two-color version of Proposition 4.7, shows that Conjecture 1.7 is true for $H = C_4$ with $c(C_4)$=1.

Theorem 4.8. *Let G be a graph such that $|V(G)| \geq 7$ and $C_4 \not\subseteq \overline{G}$. If the edges of G are colored red and blue, then there exist two vertex-disjoint monochromatic paths of different colors covering $n-1$ vertices.*

For simplicity, we refer to edges of $\overline{G}$ as black edges, and think of G as K_n with a 3-edge-coloring, but monochromatic paths should be blue or red, and sometimes when we write "edge of G" we mean "red or blue edge of G". We trust that this will not confuse the reader.

Remark 2. The value $n-1$ in Theorem 4.8 is best possible, as shown by the following example. Let v_1 and v_2 be two different vertices in K_n. If v_1x is black for all x, and v_2y is red for all y, $y \in V(K_n) \setminus v_1$, and all other edges are blue, then any two monochromatic paths can only cover at most $n-1$ vertices.

The condition $|V(G)| \geq 7$ is somewhat unexpected, since the statement is true if $|V(G)| \leq 4$. On five vertices, let $G_5 = K_1 \cup C_4$ and color the edges of C_4 alternately red and blue. On six vertices, let G_6 be the complement of C_6 and color the long diagonals red and the short diagonals blue. One can easily check that pairs of vertex disjoint red and blue paths must leave two vertices uncovered in these graphs.

Proof Theorem 4.8. Fix a blue path $P_1 = a_1 \dots a_i$ and a red path $P_2 = b_1 \dots b_j$ such that $i+j$ is as large as possible, and under this condition $|i-j|$ is as small as possible. Let G' be $G \setminus (P_1 \cup P_2)$. If G' contains only one vertex, then we are done. Therefore, we may choose a $U \subseteq V(G')$ such that $U = \{x, y\}$ for some $x \neq y$. Since $i+j$ is maximal, there are no blue edges between $\{a_1, a_i\}$ and G' and there are no red edges between $\{b_1, b_j\}$ and G'. We consider two cases, according whether $\min i, j = 1$ (say then $i = 1$).

Case 1: $i = 1$. If there is a blue edge between b_1 and G', then that one edge and $b_2 \dots b_j$ would be a better pair of paths (with smaller difference of the sizes), which is a contradiction, unless $j = 2$. In this case, $X = V(G) \setminus \{b_1, b_2\}$ has at least five vertices and (using that no C_4 in $\overline{G}$) one can easily see that X has either a blue edge or a red P_3 and both contradicts the choice of P_1, P_2.

Case 2: $i, j \geq 2$. Since there is no black C_4, there is an non-black edge of G between some of the endpoints of P_1 and some of the endpoints of P_2. We call such an edge a *cross-edge.*

Claim 4.9. *If both endpoints of a cross-edge are connected to G' by a non-black edges of G, then we can increase the number of vertices covered by the two monochromatic paths.*

We may assume that a_1b_1 is a cross-edge and it is blue. There is a blue edge between b_1 and G', say b_1z. Now $zb_1a_1 \dots a_i$ and $b_2 \dots b_j$ are two monochromatic paths, which cover more vertices than P_1 and P_2. $\square$

In what follows, we may assume that a_1b_1 is a blue cross-edge, and b_1z is black for any vertex z of G'. Let $v \in V(P_1) \cup V(P_2) \setminus b_1$. If vz_1 and vz_2 were two black edges for some $z_1, z_2 \in G'$, then $vz_1b_1z_2$ would be a black 4-cycle, a contradiction. Therefore, v is adjacent to all but one vertex in G'. In particular, there are red edge from both a_1 and a_i to G' and a blue edge from b_j to G'. Therefore, the edges a_1b_j and a_ib_j are both black by Claim 4.9.

Case 2.1: $j = 2$. If there were two (red) edges between a_i and G', say a_iz_1 and a_iz_2, then $b_1a_1 \dots a_{i-1}$ and $z_1a_iz_2$ would cover more vertices than $P_1 \cup P_2$, a contradiction. Therefore, $|V(G')| = 2$, that is $U = G'$. We may assume a_ix is red and a_iy is black. It follows that a_1y is red and a_1x is black, otherwise $a_1ya_ib_j$ would be a black C_4. contradiction. Since $|V(G)| \geq 7$, we now get $i > 2$. Therefore, $a_{i-1} \neq a_1$.

Case 2.1.1: a_1x is black. Consider the edges $a_{i-1}x$ and $a_{i-1}b_2$. If both of them were black, then $a_1xa_{i-1}b_2$ would be a black C_4. If both of them were red, then $b_1b_2a_{i-1}xa_i$ and $a_1 \dots a_{i-2}$ would cover more vertices than $P_1 \cup P_2$. If b_2a_{i-1} is blue, then $b_1a_1 \dots a_{i-1}b_2$ and a_ix cover more vertices than $P_1 \cup P_2$.

If $a_{i-1}x$ is blue, then consider the existing blue edge between b_2 and U. If b_2x were blue, then $b_1a_1 \dots a_{i-1}xb_2$ and a_i would cover more vertices than $P_1 \cup P_2$. Therefore, b_2y is a blue edge. Consider now the edge b_1a_i. If b_1a_i were red, then $b_2b_1a_ix$ and $a_1 \dots a_{i-1}$ would cover more vertices than $P_1 \cup P_2$. If b_1a_i were blue, then $xa_{i-1}a_ib_1a_1 \dots a_{i-2}$ and b_2 would cover more vertices than $P_1 \cup P_2$. Therefore, $b_1a_i \in \overline{G}$. Now we consider the edge xy. If xy is blue, then $a_1 \dots a_{i-1}xy$ and b_1b_2 cover more vertices than $P_1 \cup P_2$. If xy is red, then $b_1a_1 \dots a_{i-1}$ and a_ixy cover more vertices than $P_1 \cup P_2$. Finally, if $xy \in \overline{G}$, then xya_ib_1 is a black 4-cycle. This shows that $a_{i-1}x$ is not blue.

Now one of $a_{i-1}x$ and $a_{i-1}b_2$ is red and the other one is black. If $a_{i-1}x$ is red, then consider $a_{i-1}y$. If $a_{i-1}y$ is red, then $a_ixa_{i-1}y$ and $b_1a_1 \dots a_{i-2}$ cover more vertices than $P_1 \cup P_2$. If $a_{i-1}y$ is blue, then $b_1a_1 \dots a_{i-1}y$ and a_ix cover more vertices than $P_1 \cup P_2$. If $a_{i-1}y \in \overline{G}$, then $b_2a_{i-1}ya_i$ is a black 4-cycle.
If $a_{i-1}b_2$ is red and $a_{i-1}x$ is black, then look at $a_{i-1}y$. If $a_{i-1}y$ is black, then $xa_{i-1}yb_1$ is a black C_4. If $a_{i-1}y$ is blue, then $b_1a_1 \dots a_{i-1}y$ and a_ix cover more vertices than $P_1 \cup P_2$. If $a_{i-1}y$ is red, then $b_1a_1 \dots a_{i-2}$ and $b_2a_{i-1}y$ cover the same number of vertices as $P_1 \cup P_2$. At the same time, if $i \geq 4$, $|i-j|$ is smaller, giving a contradiction. On the other hand, if $i = 3$, then a_i and $b_1b_2a_{i-1}ya_1$ and a_i cover more vertices than $P_1 \cup P_2$.

Case 2.1.2: a_1x is red. If $i \geq 4$, then $a_2 \dots a_i$ and xa_1y cover the same number of vertices as $P_1 \cup P_2$ with a smaller $|i-j|$, a contradiction. Therefore, $i = 3$ that is $|V(G)| = 7$. If b_2y is blue, then look at a_2y. If a_2y is blue, then $b_1a_1a_2yb_2$ and a_3x cover more vertices than $P_1 \cup P_2$. If a_2y is red, then b_1 and $a_3xa_1ya_2$ cover more vertices than $P_1 \cup P_2$. Therefore, a_2y is black. Now if a_2b_2 is black, then $b_2a_3ya_2$ is a black C_4. If a_2b_2 is blue, then $b_1a_1a_2b_2y$ and a_3x cover more vertices than $P_1 \cup P_2$. Therefore, a_2b_2 is red. Now a_2x must be blue and b_2x black. Consider now b_1a_3. If b_1a_3 is blue, then $a_3b_1a_1a_2x$ and b_2 cover more vertices than $P_1 \cup P_2$. If b_1a_3 is red, then $a_2b_2b_1a_3xa_1y$ cover $V(G)$. Finally if b_1a_3 is black, then $b_1a_3b_2x$ is a black C_4.

Therefore, b_2y is black and b_2x is blue. Consider b_1a_3. If b_1a_3 is black, then $b_1a_3b_2y$ is a black C_4. If b_1a_3 is red, then $b_2b_1a_3xa_1y$ and a_2 cover more vertices than $P_1 \cup P_2$.

If b_1a_3 is blue, then $b_1a_3a_2$ and xa_1y cover more vertices than $P_1 \cup P_2$.

Case 2.2: $j > 2$. Consider the edge b_1b_j. If b_1b_j is blue, then $a_i \dots a_1b_1b_j$ plus a blue edge from b_j to G' and $b_2 \dots b_{j-1}$ cover more vertices than $P_1 \cup P_2$, a contradiction. If b_1b_j is red, then consider $b_2b_1b_j \dots b_3$, a red path of length j. By Claim 4.9, there is a cross-edge adjacent to two of a_1, a_i, b_2, b_3, and one of these vertices, say c (different from b_1) is non-adjacent to G'. That is, b_1xcy is a C_4 in $\overline{G}$, a contradiction. We conclude $b_1b_j \in \overline{G}$. Now $a_ib_jb_1z$ is a path on 4 vertices in $\overline{G}$, for any $z \in G'$. Therefore, any edge a_iz, where $z \in G'$, is a red edge. If there is a red edge b_2z, where $z \in G'$, then $b_1a_1 \dots a_{i-1}$ and $xa_izb_2 \dots b_j$ cover more vertices than $P_1 \cup P_2$, a contradiction. Thus there is a blue edge e from b_2 to G'. Now consider the edge b_2a_i. If it were blue, then $b_1a_1 \dots a_ib_2$ extended with e and $b_3 \dots b_j$ would cover more vertices than $P_1 \cup P_2$, a contradiction. If b_2a_i was red, then $b_1a_1, \dots a_{i-1}$ and $xa_ib_2, \dots b_j$ would cover more vertices than $P_1 \cup P_2$, a contradiction. We conclude that $b_2a_i \in \overline{G}$.

Next look at the pair a_1, b_2. It must be an edge G, otherwise $a_1b_2a_ib_j$ is a C_4 in $\overline{G}$, a contradiction. If a_1b_2 is red, then let f be a red edge from a_1 to U, say $f = a_1x$. Now $a_2 \dots a_{i-1}$ and $ya_ixa_1b_2 \dots b_j$ cover more vertices than $P_1 \cup P_2$, a contradiction. We conclude that a_1b_2 is blue.

Consider the edge a_ib_1. If it is red, then $a_1 \dots a_{i-1}$ and $xa_ib_1 \dots b_j$ form a better pair. If a_ib_1 is blue, then $b_1a_i \dots a_1b_2e$ and $b_3 \dots b_j$ form a better pair. We conclude $a_ib_1 \in \overline{G}$. Now the $b_ja_ib_1z$ is a path on 4 vertices in $\overline{G}$, for any $z \in G'$. Therefore any b_jz in $\overline{G}$ would form a C_4. That is, all b_jz are blue edges.

Let z be the endvertex of e in G'. Now $a_i \dots a_1b_2zb_jx$ and $b_3 \dots b_{j-1}$ cover more vertices than $P_1 \cup P_2$, giving a final contradiction. $\square$

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