COMMUTATIVE ORDERS REVISITED

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Abstract. This article studies commutative orders, that is, commutative semigroups having a semigroup of quotients. In a commutative order S, the square-cancellable elements $\mathcal{S}(S)$ constitute a well-behaved separable subsemigroup. Indeed, $\mathcal{S}(S)$ is also an order and has a maximum semigroup of quotients R, which is Clifford. We present a new characterisation of commutative orders in terms of semilattice decompositions of $\mathcal{S}(S)$ and families of ideals of S. We investigate the role of tensor products in constructing quotients, and show that all semigroups of quotients of S are homomorphic images of the tensor product $R \otimes_{\mathcal{S}(S)} S$. By introducing the notions of generalised order and semigroup of generalised quotients, we show that if S has a semigroup of generalised quotients, then it has a greatest one. For this we determine those semilattice congruences on $\mathcal{S}(S)$ that are restrictions of congruences on S.

1. Introduction

Commutative semigroups, in spite of possessing a well-developed theory, remain far from being fully understood. For a relatively recent general presentation, see [4]. Our aim here is to study a commutative semigroup S by dividing it into two parts. Namely, $S = \mathcal{S}(S) \cup T$ where $\mathcal{S}(S)$ is the subsemigroup of square-cancellable elements of S, and $T = S \setminus \mathcal{S}(S)$. Our tool is that of *quotients*: for the convenience of the reader we immediately recall the main relevant notions, beginning with that of square-cancellability.

An element $a \in S$ is *square-cancellable* if for all $x, y \in S^1$ we have that $xa^2 = ya^2$ implies $xa = ya$ and also $a^2x = a^2y$ implies $ax = ay$. It is clear that being square-cancellable is a necessary condition for an element to lie in a subgroup of an oversemigroup. Let S be a subsemigroup of a semigroup Q. Then S is a *left order* in Q and Q is a *semigroup of left quotients* of S if every $q \in Q$ can be written as $q = a^\sharp b$ where $a \in \mathcal{S}(S), b \in S$ and $a^\sharp$ is the inverse of a in a subgroup of Q and if, in addition, every square-cancellable element of S lies in a subgroup of Q. *Right orders* and *semigroups of right*

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quotients are defined dually. If S is both a left order and a right order in Q, then S is an *order* in Q and Q is a *semigroup of quotients* of S. We remark that if a commutative semigroup is a left order in Q, then Q is commutative [1, Theorem 3.1] so that S is an order in Q. A given commutative order S may have more than one semigroup of quotients. The semigroups of quotients of S are pre-ordered by the relation $Q \geq P$ if and only if there exists an onto homomorphism $\phi : Q \to P$ which restricts to the identity on S. Such a ϕ is referred to as an *S-homomorphism*; the classes of the associated equivalence relation are the S-isohomomorphism classes of orders, giving us a partially ordered set $\mathcal{Q}(S)$. In the best case, $\mathcal{Q}(S)$ contains maximum and minimum elements.

Our rationale is as follows. Let S be a commutative semigroup. The set $\mathcal{S}(S)$ is a subsemigroup of S and, if S is an order, then $\mathcal{S}(S)$ is also. In this case $\mathcal{S}(S)$ is a commutative separative semigroup and thus has a well-understood structure. Namely, $\mathcal{S}(S)$ is a semilattice of commutative cancellative semigroups and as such possesses a semigroup of quotients that is a semilattice of commutative groups, that is, a commutative Clifford semigroup [5]. Moreover, every semigroup of quotients of $\mathcal{S}(S)$ is a commutative Clifford semigroup and $\mathcal{Q}(\mathcal{S}(S))$ forms a complete lattice [1]. The subset T consists of what may be thought of as 'bad' elements, including any nilpotents. We aim to understand these elements in terms of their relation to elements of $\mathcal{S}(S)$, in the case S is an order.

Unfortunately, not all commutative semigroups are orders (not even all those in which every element is square-cancellable, see Example 2.5 below). Easdown and the second author [1] gave a description of those that are, in terms of compatible pre-orders, using a direct construction. They also give examples of commutative orders having, respectively, a maximum but no minimum and a minimum but no maximum semigroup of quotients. By using an entirely different approach we re-establish the description of orders given in [1] and give a deeper analysis of $\mathcal{Q}(S)$ for commutative orders S. We do so by using the decomposition $S = \mathcal{S}(S) \cup T$ mentioned above.

In Section 2 we give the necessary preliminaries, and recap our knowledge of commutative orders, summarising and clarifying existing results. We present the description of commutative orders S in terms of compatible pre-orders from [1], and then proceed in Section 3 to give a new characterisation via semilattice decompositions of $\mathcal{S}(S)$ and families of ideals of S.

Section 4 introduces a notion of a *generalised* quotient semigroup, which will be of use in our final results. We observe that a semigroup of generalised quotients of a commutative semigroup S is always commutative. If S is commutative and either S is a monoid or $S = \mathcal{S}(S)$, then the notions of quotient semigroup and generalised quotient semigroup coincide.

We proceed as follows in Section 5. Let S be a commutative order, so that $\mathcal{S}(S)$ is also an order. In particular, if S is an order in Q, then $\mathcal{S}(S)$

is an order in $\mathcal{H}(Q)$, the commutative Clifford semigroup of the group $\mathcal{H}$-classes of Q. Note that S may also be an order in another semigroup Q' (so that $\mathcal{S}(S)$ is an order in $\mathcal{H}(Q')$) such that $\mathcal{H}(Q) \cong \mathcal{H}(Q')$, without Q being isomorphic to Q' (see [1, Example 7.4]). Put $R = \mathcal{H}(Q)$. We construct the tensor product $R \otimes_{\mathcal{S}(S)} S$ and show that R embeds into $R \otimes_{\mathcal{S}(S)} S$ and that Q is a morphic image of $R \otimes_{\mathcal{S}(S)} S$. We thus obtain *all* quotient semigroups of S that induce the same quotient semigroup R of $\mathcal{S}(S)$ as morphic images of the fixed semigroup $R \otimes_{\mathcal{S}(S)} S$. Moreover, we can recover the characterisation of commutative orders given in [1]. A further consequence is that every semigroup of quotients of S is the image of $M \otimes_{\mathcal{S}(S)} S$, where M is the maximum semigroup of quotients of $\mathcal{S}(S)$.

Now let S be any commutative semigroup such that $\mathcal{S}(S)$ is an order in a semigroup R. Again in Section 5, we give a necessary and sufficient condition for R to embed into $R \otimes_{\mathcal{S}(S)} S$, namely, that if ρ is the semilattice congruence on $\mathcal{S}(S)$ induced by that of R, then $\overline{\overline{\rho}}|_{\mathcal{S}(S)} = \rho$[1], where $\overline{\overline{\rho}}$ is the congruence on S generated by ρ. This is easily seen to be a necessary condition for S to be an order in some Q such that $\mathcal{H}(Q) = R$. Our first aim in Section 6 is to find a straightforward condition in terms of elements of S for $\overline{\overline{\rho}}|_{\mathcal{S}(S)} = \rho$. If a congruence ρ on $\mathcal{S}(S)$ satisfies this property, then one further condition on ρ tells us when S has a semigroup of generalised quotients. Our final results show that if S has a generalised semigroup of quotients (for example, if S is an order), then it has a maximum one.

2. Preliminaries

We recall that a *pre-order* (or *quasi-order*) $\preceq$ on a set S is a reflexive, transitive relation. From a pre-order $\preceq$ we can define an equivalence relation $\equiv_{\preceq}$ by

$$a \equiv_{\preceq} b \text{ if and only if } a \preceq b \preceq a.$$

If S is a semigroup, then we say that a pre-order $\preceq$ is *compatible* if for any $a, b, c \in S$, we have that if $a \preceq b$, then $ac \preceq bc$ and $ca \preceq cb$. If $\preceq$ is a compatible pre-order, $a \preceq b$ and $c \preceq d$, then it is clear by transitivity that $ac \preceq bd$ and, in this case, the associated equivalence relation is a congruence.

Lemma 2.1. *Let κ be a relation on a semigroup S.*

(i) The smallest compatible pre-order $\overline{\kappa}$ containing κ is given by the rule that for any $a, b \in S$, $a \,\overline{\kappa}\, b$ if and only if $a = b$ or there exists a sequence

$$a = c_1u_1d_1,\ c_1v_1d_1 = c_2u_2v_2,\ \ldots,\ c_nv_nd_n = b,$$

where for $1 \le i \le n$ we have that $(u_i, v_i) \in \kappa$ and $c_i, d_i \in S^1$.

[1]A word on notation: for any relation μ on a set X, we denote by $\mu|_Y$ the restriction of μ to a subset Y of X.

(ii) The smallest congruence $\overline{\overline{\kappa}}$ *containing* κ *is given by the rule that for any* $a, b \in S$, $a \,\overline{\overline{\kappa}}\, b$ *if and only if* $a = b$ *or there exists a sequence*

$$a = c_1 u_1 d_1,\; c_1 v_1 d_1 = c_2 u_2 v_2,\; \ldots, c_n v_n d_n = b,$$

where for $1 \leq i \leq n$ *we have that* $(u_i, v_i) \in \kappa$ *or* $(v_i, u_i) \in \kappa$ *and* $c_i, d_i \in S^1$.

Where the notation $\overline{\overline{\kappa}}$ is not convenient, we may use the more standard $\langle \kappa \rangle$.

Let Q be a commutative semigroup. Clearly, Green's relations $\mathcal{H}, \mathcal{L}, \mathcal{R}$ and $\mathcal{J}$ all coincide on Q and we will denote this relation, which is a congruence, by $\mathcal{H}$. Moreover, $\mathcal{H}$ is the equivalence associated with the (compatible) pre-order $\leq_{\mathcal{H}}$, where for $a, b \in Q$ we have $a \leq_{\mathcal{H}} b$ if and only if $a = bq$ for some $q \in Q^1$. Where there is danger of ambiguity we will denote $\mathcal{H}$ and $\leq_{\mathcal{H}}$ on Q by $\mathcal{H}^Q$ and $\leq_{\mathcal{H}^Q}$, respectively, with corresponding notation for $\mathcal{H}$-classes.

The following result is folklore. Its straightforward proof runs as that of the corresponding statement for $\mathcal{H}$, which can be found in, for instance, [6, Proposition II.4.5].

Lemma 2.2. *Let* T *be a regular subsemigroup of a commutative semigroup* Q. *Then*

$$\leq_{\mathcal{H}^T} = \leq_{\mathcal{H}^Q}|_T.$$

We now explain the concept of square-cancellability. Let S be a semigroup. The relation $\leq_{\mathcal{R}^*}$ is defined on S by the rule that for $a, b \in S$ we have $a \leq_{\mathcal{R}^*} b$ in S if and only if $a \leq_{\mathcal{R}} b$ in some oversemigroup of S. It is well known, and easy to see from the right regular representation of S in $\mathcal{T}_{S^1}$, that $a \leq_{\mathcal{R}^*} b$ if and only if for all $x, y \in S^1$ we have that $xb = yb$ implies $xa = ya$. Clearly, $\leq_{\mathcal{R}^*}$ is a pre-order; we denote the associated equivalence relation by $\mathcal{R}^*$.

The relations $\leq_{\mathcal{L}^*}$ and $\mathcal{L}^*$ are defined dually and we let $\leq_{\mathcal{H}^*}$ and $\mathcal{H}^*$ be the intersections $\leq_{\mathcal{R}^*} \cap \leq_{\mathcal{L}^*}$ and $\mathcal{R}^* \cap \mathcal{L}^*$, respectively. It is clear from their second characterisations that if S is commutative then

$$\leq_{\mathcal{R}^*} = \leq_{\mathcal{L}^*} = \leq_{\mathcal{H}^*} \text{ and } \mathcal{R}^* = \mathcal{L}^* = \mathcal{H}^*$$

and moreover $\leq_{\mathcal{H}^*}$ is compatible, so that $\mathcal{H}^*$ is a congruence. From the definition given in the Introduction, $a \in S$ is square-cancellable if $a \,\mathcal{H}^*\, a^2$. We have already observed that being square-cancellable is a necessary condition for an element of S to lie in a subgroup of an oversemigroup.

We denote by $\mathcal{H}(S)$ the set of elements of S lying in group $\mathcal{H}$-classes, and by $\mathcal{S}(S)$ the set of square-cancellable elements of S. Recall from the definition that if S is an order in Q, then $\mathcal{S}(S) = \mathcal{H}(Q) \cap S$. The next lemma builds on the preceding remarks.

Lemma 2.3. [1] *Let* T *be a commutative semigroup. Then:*

(i) $\mathcal{H}^*$ *is a congruence on* T *and* $\mathcal{S}(T)$ *is empty or is a subsemigroup of* T;

(ii) $\mathcal{H}$ *is a congruence on* T *and* $\mathcal{H}(T)$ *is empty or is a subsemigroup and moreover a semilattice of the group* $\mathcal{H}$*-classes of* T;

(iii) for all $a, b \in \mathcal{H}(T)$, $(ab)^\sharp = a^\sharp b^\sharp = b^\sharp a^\sharp$.

Further, if S *is an order in a commutative semigroup* Q, *then* $\mathcal{S}(S)$ *is an order in* $\mathcal{H}(Q)$.

Recall that a subset X of a commutative semigroup S is *separative* if for all $x, y \in X$ with $x^2 = xy = y^2$, we have $x = y$. Since any Clifford semigroup is separative, and separability is clearly inherited by subsemigroups, the following lemma is clear.

Lemma 2.4. *Let S be a commutative subsemigroup of Q such that $\mathcal{S}(S) = \mathcal{H}(Q) \cap S$. Then $\mathcal{S}(S)$ is separative.*

The following is an example of a commutative semigroup $S = \mathcal{S}(S)$ which is not separative, hence, in view of Result 2.6, not an order.

Example 2.5. (Ruškuc) Let S be the semigroup defined by the presentation

$$S = \langle a, b \,|\, a^2 = ab = ba = b^2 \rangle.$$

It is readily seen that $S = \{a^i : i \in \mathbb{N}\} \cup \{b\}$ and that b and all powers of a are distinct. Hence S is commutative, but not separative.

Every element of S^1 has length $|a^i| = i$, $|b| = 1$, $|1| = 0$ and clearly $|uv| = |u| + |v|$ for all $u, v \in S^1$. Let $c \in S$ and $x, y \in S^1$ with $xc^2 = yc^2$. Then $|x| = |y|$ so that either $x = y = 1$ or $xc = a^{|xc|} = a^{|yc|} = yc$. Thus every element of S is square-cancellable.

On the positive side we have the following, which draws together relevant results from [1, 3, 5] and [4]. First, we recall that an order S in a commutative semigroup Q is said to be *straight* if every element of Q can be written in the form $q = a^\sharp b$ where $a \in \mathcal{S}(S)$, $b \in S$, and $a \,\mathcal{H}\, b$ in Q.

Result 2.6. *The following conditions are equivalent for a semigroup S:*

(i) S is commutative and separative;

(ii) S is a semilattice of commutative, cancellative semigroups;

(iii) S is an order in a commutative Clifford semigroup;

(iv) S is a subsemigroup of a commutative Clifford semigroup;

(v) S is a commutative order such that $S = \mathcal{S}(S)$;

(vi) S is a commutative straight order in some semigroup of quotients;

(vii) S is a commutative order which is straight in each of its semigroups of quotients;

(viii) S is commutative, $S = \mathcal{S}(S)$ and the $\mathcal{H}^*$-classes of S are cancellative.

If any (all) of the above conditions hold, then S has a semigroup of quotients Q such that $\leq_{\mathcal{H}^Q}|_S = \leq_{\mathcal{H}^*}$.

Proof. The equivalence of (i) to (iv) comes from [3, Corollary 6.1] (cf. [7, Theorem II.6.6] and that of $(v), (vi), (vii)$ was noted in the beginning of Section 7 in [1]. Corollary 4.4 of [1] shows that (v) and $(viii)$ are equivalent. Clearly, (iii) implies (v), so we need only show that (v) implies (iii). If $S = \mathcal{S}(S)$ is a commutative order in Q, then by [1, Theorem 3.1], Q is commutative. Let $q = a^\sharp b$ where $a, b \in S$. Then $q \,\mathcal{H}\, ab \in Q$ and ab lies in a subgroup of Q. Hence Q is a union of groups, and so Clifford. $\square$

Example 2.5 shows that in Condition (v) of Result 2.6 the requirement that S is an order cannot be omitted.

Our next result is taken from [3, Theorem 3.1] and [1, Corollary 4.4, Proposition 5.3]. Here $\mathcal{N}$ denotes the least semilattice congruence on a semigroup S.

Result 2.7. *Let $S = \mathcal{S}(S)$ be a commutative order. Then:*
(i) S is an order in a semilattice Y of groups $G_\alpha, \alpha \in Y$, if and only if S is a semilattice Y of cancellative semigroups $S_\alpha, \alpha \in Y$;
(ii) S is an order in Q where $\leq_{\mathcal{H}^Q}|_S = \leq_{\mathcal{H}^*}$, so that $\mathcal{H}^*$ is a semilattice congruence on S with cancellative classes;
(iii) $\mathcal{Q}(S)$ forms a complete lattice, isomorphic to the dual of the interval $[\mathcal{N}, \mathcal{H}^*]$ in the lattice of congruences of S. The isomorphism is given by

$$\rho \longleftrightarrow \mathcal{H}^Q|_S,$$

where Q is a representative of its equivalence class.

Part *(iii)* of the above is achieved from the following.

Result 2.8. [1, Theorem 5.1] *Let S be a commutative semigroup and an order in semigroups Q_1 and Q_2. The following conditions are equivalent:*
(i) $Q_2 \leq Q_1$;
(ii) for all $a, b \in S$,

$$a \leq_{\mathcal{H}} b \text{ in } Q_1 \text{ implies that } a \leq_{\mathcal{H}} b \text{ in } Q_2;$$

(iii) for all $a, b \in S$,

$$a\,\mathcal{H}\,b \text{ in } Q_1 \text{ implies that } a\,\mathcal{H}\,b \text{ in } Q_2.$$

We would like to say that every commutative order S has a maximum and a minimum semigroup of quotients. Unfortunately, this is not the case [1, Section 7]. One of our reasons in introducing 'generalised' semigroups of quotients is that in Section 6 we show that for an arbitrary commutative order S, we can find a semigroup Q that is a 'generalised' semigroup of quotients of S, and is such that every semigroup of quotients of S is an image of Q in a natural way.

The existence and behaviour of quotient semigroups of a commutative S is closely tied to that of pre-orders on S, as is already apparent from the last claim of Result 2.6. Let $\preceq$ be a compatible pre-order on S. We recall from [1] some conditions on $\preceq$ that are crucial in determining quotient semigroups of S. We supplement this list with a related condition that will be required later.
(A) For all $b, c \in S$, we have $bc \preceq b$.
(B) For all $b, c \in S$ and $a \in \mathcal{S}(S)$, if

$$b \preceq a, c \preceq a, \text{ and } ab = ac,$$

then

$$b = c \preceq ab.$$

Conditions (A) and (B) restricted to $\mathcal{S}(S)$ clearly imply that the semigroup $\mathcal{S}(S)$ is separative.
(C) For all $b \in S$ there exists $x \in \mathcal{S}(S)$ with $b \preceq x$.

(C$'$) For all $b, c \in S$, $b \preceq c$ implies that $bx = cy$ for some $x \in \mathcal{S}(S), y \in S$ with $b \preceq x$.

(C$''$) For all $b, c \in S$, $b \preceq c$ implies that $bx = cy$ for some $x, y \in S^1$ such that if $x \in S$, then $x \in \mathcal{S}(S)$ and $b \preceq x$.

The motivation for introducing conditions of the kind above is made clear by the next result.

Theorem 2.9. [1, Theorem 4.3] *Let S be a commutative semigroup and let $\preceq$ be a relation on S. Then S is an order in a semigroup Q such that $\leq_{\mathcal{H}^Q}|_S = \preceq$ if and only if $\preceq$ is a compatible pre-order on S satisfying Conditions (A), (B) and (C$'$).*

With the above result in mind we introduce some terminology. We say that a compatible pre-order $\preceq$ on a commutative semigroup S is a *quotient pre-order* or *q-pre-order* if it satisfies Conditions (A), (B) and (C$'$). We normally denote the associated congruence $\equiv_{\preceq}$ on S by $\mathcal{H}'$. The restriction of a q-pre-order and its associated congruence to $\mathcal{S}(S)$ will be normally denoted by $\leq$ and $\mathcal{H}''$ and we will refer to these as being *induced* by $\preceq$ and $\mathcal{H}'$. Before continuing we make some technical observations concerning Conditions (A) and (B).

Lemma 2.10. *Let $\preceq$ be a compatible pre-order on a commutative semigroup S satisfying (A) and (B), and let $\equiv_{\preceq}$ be the associated congruence. Let $a \in \mathcal{S}(S)$ and $b \in S$. Then the following conditions are equivalent:*

(i) $b \preceq a$;

(ii) $ba \equiv_{\preceq} b$;

(iii) $ca \equiv_{\preceq} b$ for some $c \in S$.

Proof. $(i) \Rightarrow (ii)$ We remark that by (A), $ba \preceq b$. If $b \preceq a$, then with $b = c$ in (B), we have $b \preceq ba$ so that $b \equiv_{\preceq} ba$ as required.

$(ii) \Rightarrow (iii)$ Clear.

$(iii) \Rightarrow (i)$ We have $b \equiv_{\preceq} ca \preceq a$ by (A), so that $b \preceq a$. □

Lemma 2.11. *Let $\preceq$ be a compatible pre-order on a commutative semigroup S satisfying (A) and (B) and let $\equiv_{\preceq}$ be the associated congruence. Let $u \in S$. If there exist $a_1, \dots, a_n \in \mathcal{S}(S)$ and $v_1, \dots, v_n \in S$ with $u \equiv_{\preceq} a_i v_i$, $1 \leq i \leq n$, then $u \preceq a_1 \dots a_n$.*

Proof. We proceed by induction. Clearly the result is true if $n = 1$, by Lemma 2.10.

Suppose now that $n > 1$ and the result is true for $n-1$. Then $u \preceq a_1 \dots a_{n-1}$. By Lemma 2.10,

$$u \equiv_{\preceq} v_n a_n \equiv_{\preceq} u a_n \preceq a_1 \dots a_{n-1} a_n,$$

so that the result follows by induction. □

Our aim in Section 4 is to show how Theorem 2.9 can be obtained with a rather different construction to that in [1]. In fact, we need only the characterisation of orders in commutative Clifford semigroups and a particular use of tensor products, which forms the major construction of this paper, to produce all semigroups of quotients of a given commutative order.

Let T and U be semigroups. We say that T is a *U-semigroup* if there is a homomorphism $\phi : U \to T$. The extension of ϕ to $U^1 \to T^1$ defines an action of U^1 on T given by $ut = (u\phi)t$. Throughout this article, U is a subsemigroup of T and ϕ is inclusion. If V is also a U-semigroup then we can form the *tensor product* $T \otimes_{U^1} V$, which for convenience we abbreviate as $T \otimes_U V$. Specifically, this is the set $T \times V$ factored by the equivalence relation $\mathcal{T}$ generated by

$$\{\big((tu, v), (t, uv)\big) : t \in T, u \in U^1, v \in V\}.$$

We write $t \otimes v$ for the $\mathcal{T}$-equivalence class of (t, v). Note that for elements $(p, s), (p', s') \in T \times V$ we have that $p \otimes s = p' \otimes s'$ if and only if there exists a system of equations

$$\begin{array}{rclrcl} & & & s & = & s_1 b_1 \\ p s_1 & = & a_2 t_1 & t_1 b_1 & = & s_2 b_2 \\ a_2 s_2 & = & a_3 t_2 & & & \\ & \vdots & & & \vdots & \\ a_{m-1} s_{m-1} & = & a_m t_{m-1} & t_{m-1} b_{m-1} & = & s_m b_m \\ a_m s_m & = & p' t_m & t_m b_m & = & s' \end{array} \tag{1}$$

for some $s_1, t_1, \ldots, s_m, t_m \in U^1, a_2, \ldots, a_m \in T$ and $b_1, \ldots, b_m \in V$.

The tensor product $T \otimes_U V$ comes with a tensor map $\tau : T \times V \to T \otimes_U V$ given by $(p, s)\tau = p \otimes s$. The map τ is *balanced*, that is, $(pu, s)\tau = (p, us)\tau$ for all $(p, s) \in T \times V$ and $u \in U^1$. Conversely, it is clear that every balanced mapping $\phi : T \times V \to X$ factors uniquely through τ, that is, there is a mapping $\psi : T \otimes_U V \to X$ that is unique with respect to $\tau\psi = \phi$.

If T and V are commutative semigroups, then so is the direct product $T \times V$ and $T \otimes_U V$, and in the above, τ, ϕ and ψ are homomorphisms.

Lemma 2.12. *Let T, V be commutative U-semigroups. Then $T \otimes_U V$ is a commutative semigroup under*

$$(p \otimes s)(q \otimes t) = pq \otimes st.$$

Clearly $\tau : T \times V \to T \otimes_U V$ is a homomorphism. If X is a commutative semigroup and $\phi : T \times V \to X$ is a balanced homomorphism, then the unique map $\psi : T \otimes_U V \to X$ such that $\tau\psi = \phi$ is a homomorphism.

Proof. It is clear that the set of generators of $\mathcal{T}$ is compatible, hence so is $\mathcal{T}$, giving that $\mathcal{T}$ is a congruence. Thus $T \otimes_U V$ is a commutative semigroup as in the statement, and τ is the natural homomorphism. Given that ϕ is a homomorphism, it follows from standard algebraic arguments that so also is ψ. □

Lemma 2.13. *Suppose that T_1, V_1, T_2, V_2 are commutative U-semigroups, and there are U-homomorphisms $\phi : T_1 \to T_2$ and $\psi : V_1 \to V_2$, that is, for all $u \in U, t_i \in T_i$, $(ut_1)\phi = u(t_1\phi)$ and $(ut_2)\psi = u(t_2\psi)$. Then $\phi \otimes \psi : T_1 \otimes_U V_1 \to$*

$T_2 \otimes_U V_2$ given by $(p \otimes s)(\phi \otimes \psi) = p\phi \otimes s\psi$ is a homomorphism. Further, if ϕ and ψ are onto, then so is $\phi \otimes \psi$.

Proof. The map $T_1 \times V_1 \to T_2 \otimes_U V_2$ given by $(p, s) \mapsto p\phi \otimes s\psi$ is a balanced homomorphism. Now call upon Lemma 2.12. $\square$

Example 2.14. We briefly consider the special case of a commutative cancellative semigroup S. Certainly $\mathcal{H}^* = S \times S$, $S = \mathcal{S}(S)$ and S is an order (in, for example, a group). From Result 2.7 we know that $\mathcal{Q}(S)$ is a lattice and is isomorphic to the dual of the interval $[\mathcal{N}, S \times S]$ in the lattice of congruences on S, and hence therefore to the dual of the lattice of semilattice congruences on S.

Put $Y = S/\mathcal{N}$, so that $\mathcal{Q}(S)$ is therefore isomorphic to the dual of the lattice of congruences on Y. Moreover, S has a greatest semigroup of quotients Q, where Q is a semilattice Y of groups $G_\alpha, \alpha \in Y$, and S is a semilattice Y of orders S_α in G_α. Let e_α denote the identity of G_α, $\alpha \in Y$. By definition of the ordering on $\mathcal{Q}(S)$, it is clear that $\mathcal{Q}(S)$ corresponds to the set of congruences on Q that restrict to the identity relation ι on S. We now show directly that these are exactly the congruences generated by sets of the form

$$C = \{(e_{\alpha_i}, e_{\beta_i}) : i \in I\}.$$

Proof. Let τ be the congruence on Q generated by a set C as above. We may assume that C is symmetric. If $u, v \in S$ and $u \,\tau\, v$, then $u = v$ or there is a sequence

$$u = e_{\alpha_{i_1}} q_1,\; e_{\beta_{i_1}} q_1 = e_{\alpha_{i_2}} q_2,\; \ldots,\; e_{\beta_{i_n}} q_n = v$$

where $n \in \mathbb{N}$, $q_1, \ldots, q_n \in Q$ and $i_1, \ldots, i_n \in I$. With $e = e_{\alpha_{i_1}} e_{\beta_{i_1}} \ldots e_{\alpha_{i_n}} e_{\beta_{i_n}}$ we have $eu = ev$. If $e \in G_\gamma$, then choosing $c \in S_\gamma$ we have $cu = cv$ so that as S is cancellative, $u = v$. Thus $\tau\,|_S = \iota$.

Conversely, let κ be a congruence on Q that restricts to the identity on S. Suppose that $a^\sharp b \,\kappa\, c^\sharp d$, where $a, b \in S_\alpha$ and $c, d \in S_\beta$. Then $cb \,\kappa\, ad$ so that $cb = ad$ by assumption. Moreover, $e_\alpha = (a^\sharp b)^\sharp (a^\sharp b) \,\kappa\, (a^\sharp b)^\sharp (c^\sharp d) = (bc)^\sharp ad = e_{\alpha\beta}$ and similarly, $e_\beta \,\kappa\, e_{\alpha\beta}$. Also from $cb = ad$ we have $e_{\alpha\beta} cb = e_{\alpha\beta} ad$ so that $e_{\alpha\beta} a^\sharp b = e_{\alpha\beta} c^\sharp d$. Put $\kappa_{\alpha,\beta} = \langle (e_\alpha, e_{\alpha\beta}), (e_\beta, e_{\alpha\beta}) \rangle$. Then

$$a^\sharp b = a^\sharp b e_\alpha \;\kappa_{\alpha,\beta}\; a^\sharp b e_{\alpha\beta} = c^\sharp d e_{\alpha\beta} \;\kappa_{\alpha,\beta}\; c^\sharp d e_\beta = c^\sharp d.$$

The result follows. $\square$

We end this section with an example which demonstrates the complexities that can arise for commutative orders, even when $S = \mathcal{S}(S)$.

Example 2.15. Consider the multiplicative semigroup of natural numbers $\mathbb{N}$ and denote by $\mathbb{P}$ the set of prime numbers. Since $\mathbb{N}$ is cancellative, we have $\mathbb{N} = \mathcal{S}(\mathbb{N})$. Let Θ be the smallest semilattice congruence on $\mathbb{N}$, then the Θ-classes are in a 1-1 correspondence with the finite subsets of $\mathbb{P}$ and $\mathbb{N}/\Theta$ is the free semilattice monoid (free semilattice with an identity adjoined) F_ω on countably many generators. Each Θ-class is uniquely determined by the smallest square-free number n_Θ in it, and thus by the set X of prime factors of

n_Θ; we may therefore write $n_\Theta = n_X$. For each non-empty finite subset X of $\mathbb{P}$, let $\mathbb{N}_X$ be the Θ-class containing n_X, and let G_X be the group of quotients of $\mathbb{N}_X$, with identity element e_X. Clearly, $\mathbb{N}_X$ is a free semigroup and G_X a free abelian group of rank $|X|$.

Let $\mathcal{P}_f(\mathbb{P})$ denote the set of finite subsets of $\mathbb{P}$ and let $Q = \bigcup_{X\in\mathcal{P}_f(\mathbb{P})} G_X$, where we take the union to be disjoint. The multiplication in Q works as follows. For $q_1, q_2 \in Q$ with $q_1 \in G_{X_1}, q_2 \in G_{X_2}$ we have $q_1 = a^\sharp b$, $q_2 = c^\sharp d$ for some $a, b \in \mathbb{N}_{X_1}$, $c, d \in \mathbb{N}_{X_2}$. Then $ac, bd \in \mathbb{N}_{X_1\cup X_2}$, and we put

$$q_1q_2 = (ac)^\sharp bd \in G_{X_1\cup X_2}.$$

Clearly, Q is the greatest element of $\mathcal{Q}(\mathbb{N})$.

Since $\mathbb{N}$ is cancellative, $\mathcal{H}^*$ is universal. It follows that the lattice $\mathcal{Q}(\mathbb{N})$ is isomorphic to the dual of the lattice of semilattice congruences on $\mathbb{N}$ and hence to the dual of the lattice of congruences on F_ω. The latter is a vast lattice known to satisfy no lattice identity [2].

3. Characterisation by ideals

The aim of this section is to give a new description of commutative orders, in terms of ideal decompositions.

Theorem 3.1. *Let S be a commutative semigroup. Then S is an order in a semigroup Q such that for each $e \in E = E(Q)$ we have*

$$C_e = \mathcal{S}(S) \cap H_e^Q \text{ and } I_e = S \cap eQ$$

if and only if S has a set $\{C_e, I_e : e \in E\}$ of subsets such that:

(1) $\mathcal{S}(S)$ is a semilattice E of subsemigroups $C_e, e \in E$ and $S = \bigcup_{e\in E} I_e$; and for any $e, f \in E$

(2) I_e is an ideal of S with $C_e \subseteq I_e$ and $I_e \cap I_f = I_{ef}$;

(3) if $C_e \cap I_f \neq \emptyset$, then $C_e \subseteq I_f$ and $e \leq f$ in E;

(4) if $x \in I_e, a \in C_e$ and $xa \in I_f$, then $x \in I_f$;

(5) if $a \in C_e$ and $x, y \in I_e$ with $ax = ay$, then $x = y$.

Proof. Suppose that S is an order in a semigroup Q and put $E = E(Q)$. For each $e \in E$ define

$$C_e = \mathcal{S}(S) \cap H_e^Q \text{ and } I_e = S \cap eQ.$$

It is straightforward to see that (1)–(5) hold, but to this end we note that for $e, f \in E$ we have

$$I_e \cap I_f = \{s \in S : es = s = fs\} = \{s \in S : efs = s\} = I_{ef}.$$

If $x \in I_e$ and $a \in C_e$, then $ex = x$ and $x \,\mathcal{H}\, ax$, so that if $ax \in I_f$, then $x \in I_f$. Moreover, if also $y \in I_e$ and $ax = ay$, then $x = ex = a^\sharp ax = a^\sharp ay = ey = y$.

Now suppose that S has a set of subsets $\{C_e, I_e : e \in E\}$ such that Conditions (1)–(5) hold.

Define a relation $\preceq$ on S by the rule that

$$x \preceq y \Leftrightarrow \exists\, e \in E, a \in C_e, z \in S \text{ with } x \in I_e \text{ and } ax = zy.$$

Note immediately that if $x \in I_e$ and $a \in C_e$ we have $x \preceq a$ and $x \preceq x$. Moreover, it is clear that (C′) holds.

Suppose now that $x, y, z \in S$ with $x \preceq y \preceq z$. Then $\exists\, e, f \in E$ with $x \in I_e, y \in I_f$ and $a \in C_e, b \in C_f$ with

$$ax = dy, \; by = hz$$

for some $d, h \in S$. It follows that

$$bax = bdy = dhz$$

and $ba \in C_{ef}$ by (1). From $ax = dy \in I_f$ and (4), we have $x \in I_f$ so that $x \in I_e \cap I_f = I_{ef}$. Hence $x \preceq z$ and $\preceq$ is a pre-order which is clearly compatible with multiplication.

Let $x, y \in S$, say $xy \in I_e$; choosing $a \in C_e$ the triviality $a(xy) = (ax)y$ gives that $xy \preceq y$ so that (A) holds. For (B), we note that if $x \preceq a$ where $x \in I_e$ and $a \in C_f$, then $bx = ay$ for some $b \in C_e$ and $y \in S$. Then $bx \in I_f$ so that by (4), $x \in I_f$. Now $a^2x = a(ax)$ tells us that $x \preceq ax$. The remaining part of (B) follows immediately from (5).

From Theorem 2.9 we have that S is an order in a commutative semigroup Q such that $\leq_{\mathcal{H}^Q}|_S = \preceq$.

Let $a, b \in \mathcal{S}(S)$ with $a \in C_e$ and $b \in C_f$. If $e \leq f$ then as $I_e = I_{ef} \subseteq I_f$ we have $a \preceq b$. On the other hand, if $a \preceq b$, then it follows as above that $a \in I_f$ so that $e \leq f$ by (3). We may therefore assume that E is the semilattice of idempotents of Q and for each $e \in E$ we have $C_e = S \cap H_e^Q$. Choose $a \in C_e$. Then for any $x \in S$ we have that

$$x \in eQ \Leftrightarrow x \leq_{\mathcal{H}^Q} e \Leftrightarrow x \leq_{\mathcal{H}^Q} a \Leftrightarrow x \preceq a.$$

Now if $x \preceq a$ then we have seen that $x \in I_e$. On the other hand, if $x \in I_e$ then we noted earlier that $x \preceq a$. It follows that $eQ \cap S = I_e$.

□

4. Generalised quotients

For later purposes we introduce and make some comments concerning a generalisation of the notion of order. For a subset X of a semigroup Q, we denote by $\langle X \rangle$ the subsemigroup of Q generated by X.

Definition 4.1. Let S be a subsemigroup of a semigroup Q. Then Q is a *generalised quotient semigroup* of S and S is a *generalised order* in Q if every square-cancellable element of S lies in a subgroup of Q and

$$Q = \langle S \cup \{a^\sharp : a \in \mathcal{S}(S)\} \rangle.$$

It is clear that semigroups of quotients are generalised quotient semigroups. We will see that the notions almost coincide when S is commutative. The methods in the lemma below are similar to those in [1, Theorem 3.1], but we give a proof for completeness.

Lemma 4.2. *If S is a commutative subsemigroup of Q and Q is generated by $S \cup \{a^\sharp : a \in S \cap \mathcal{H}(Q)\}$, then Q is commutative.*

Proof. Let $a, b \in S$ and suppose that $a^\sharp$ exists. Then

$$a^\sharp b = (a^\sharp)^2 ab = (a^\sharp)^2 ba = (a^\sharp)^2 ba^3 (a^\sharp)^2 = (a^\sharp)^2 a^3 b (a^\sharp)^2 = ab(a^\sharp)^2 = ba(a^\sharp)^2 = ba^\sharp.$$

If in addition we have that $c \in S$ and $c^\sharp$ exists, then a similar calculation, making use of the above, gives that

$$a^\sharp c^\sharp = (a^\sharp)^2 ac^\sharp = (a^\sharp)^2 c^\sharp a = (a^\sharp)^2 c^\sharp a^3 (a^\sharp)^2 =$$

$$(a^\sharp)^2 a^3 c^\sharp (a^\sharp)^2 = ac^\sharp (a^\sharp)^2 = c^\sharp a (a^\sharp)^2 = c^\sharp a^\sharp.$$

□

The following corollary now follows easily from Lemmas 2.3 and 4.2.

Corollary 4.3. *Let S be commutative and a subsemigroup of Q such that every square-cancellable element of S lies in a subgroup of Q. Then Q is a semigroup of generalised quotients of S if and only if for any $q \in Q$, either $q \in S$ or $q = a^\sharp b$ for some $a, b \in S$.*

If S is commutative, and is a monoid or $S = \mathcal{S}(S)$, then we get nothing new by moving to *generalised* quotients.

Lemma 4.4. *Let S be a commutative monoid. Then*
(i) S is a generalised order in Q if and only if S is an order in Q;
(ii) if S is an order in Q, then Q is a monoid.

Proof. (i) If S is a generalised order in Q and $s \in S$, then $s = 1^\sharp s$.
(ii) Let $a^\sharp b \in Q$, where $a, b \in S$. Then $(a^\sharp b)1 = a^\sharp b1 = a^\sharp b$ and $1(a^\sharp b) = 1a(a^\sharp)^2 b = a(a^\sharp)^2 b = a^\sharp b$.

□

Lemma 4.5. *Let S be commutative with $S = \mathcal{S}(S)$. Then S is a generalised order in Q if and only if S is an order in Q;*

Proof. Suppose that S is a generalised order in Q. If $s \in S$, then as $s \in \mathcal{S}(S) \subseteq \mathcal{H}(Q)$ we have $s = s^\sharp s^2$. □

In the commutative case, we can answer the question of whether a semigroup Q of (generalised) quotients of S is a semigroup of (generalised) quotients of itself.

Proposition 4.6. *Let S be a commutative (generalised) order in Q. Then Q is a (generalised) order in Q.*

Proof. Clearly we need only show that $\mathcal{S}(Q) \subseteq \mathcal{H}(Q)$. Let $q \in \mathcal{S}(Q)$. If $q \in S$ then clearly $q \in \mathcal{H}(Q)$. Otherwise, $q = a^\sharp b$ for some $a \in \mathcal{S}(S)$ and $b \in S$. Then $q = (a^2)^\sharp ab$, so that we can assume $b \leq_{\mathcal{H}} a$ and so $b \,\mathcal{H}\, a^\sharp b$ in Q. Consequently, $b^2 \,\mathcal{H}\, (a^\sharp b)^2$ in Q and so as $\mathcal{H} \subseteq \mathcal{H}^*$ we have that $b \,\mathcal{H}^*\, b^2$ in Q. Certainly then $b \,\mathcal{H}^*\, b^2$ in S, so that b lies in a subgroup of Q. Using Lemma 2.3 we conclude $q = a^\sharp b \in \mathcal{H}(Q)$. □

5. A construction

Let S be a commutative semigroup and let $\preceq$ be a q-pre-order, that is, a compatible pre-order satisfying Conditions (A), (B) and (C′). By Theorem 2.9, S is an order in a commutative semigroup Q such that $\leq_{\mathcal{H}^Q}|_S = \preceq$ and hence from Lemma 2.3, $\mathcal{S}(S)$ is an order in $\mathcal{H}(Q)$. It is this latter fact that we need, that can be shown independently of the main construction of [1].

Lemma 5.1. *Let S be a commutative semigroup and let $\preceq$ be a q-pre-order on S. Putting*

$$\leq = \preceq|_{\mathcal{S}(S)}$$

we have that $\mathcal{S}(S)$ is an order in a Clifford semigroup R such that

$$\leq_{\mathcal{H}^R}|_{\mathcal{S}(S)} = \leq .$$

Moreover, with $\mathcal{H}' = \equiv_{\preceq}$ and $\mathcal{H}'' = \equiv_{\leq}$ being the equivalence relations associated with $\preceq$ and $\leq$, respectively, we have that

$$\mathcal{H}'' = \mathcal{H}'|_{\mathcal{S}(S)} = \mathcal{H}^R|_{\mathcal{S}(S)}.$$

Proof. It is clear from the definitions that $\mathcal{H}'' = \mathcal{H}'|_{\mathcal{S}(S)}$. Suppose that $a \in \mathcal{S}(S)$; by Lemma 2.10 we have that $a\,\mathcal{H}''\,a^2$ so that $\mathcal{H}''$ is a semilattice congruence on $\mathcal{S}(S)$. Writing H''_u for the $\mathcal{H}''$-class of $u \in \mathcal{S}(S)$, and again using Lemma 2.10, we have that for any $a, b \in \mathcal{S}(S)$,

$$\begin{aligned} a \leq b \;&\Leftrightarrow\; a \preceq b \\ &\Leftrightarrow\; ab\,\mathcal{H}'\,a \\ &\Leftrightarrow\; ab\,\mathcal{H}''\,a \\ &\Leftrightarrow\; H''_a \leq H''_b \text{ in the semilattice } \mathcal{S}(S)/\mathcal{H}''. \end{aligned}$$

Consider an $\mathcal{H}''$-class H''. Clearly (B) gives that H'' is cancellative and it is right reversible, as it is commutative. By Result 2.7, we have that $\mathcal{S}(S)$ is an order in R, where R is a semilattice $\mathcal{S}(S)/\mathcal{H}''$ of commutative groups. It follows that $\mathcal{H}'' = \mathcal{H}^R|_{\mathcal{S}(S)}$. Moreover, for any $a, b \in \mathcal{S}(S)$ we have

$$a \leq_{\mathcal{H}^R} b \Leftrightarrow ab\,\mathcal{H}^R\,b \Leftrightarrow ab\,\mathcal{H}''\,a \Leftrightarrow a \leq b,$$

so that $\leq_{\mathcal{H}^R}|_{\mathcal{S}(S)} = \leq$. $\square$

The following lemma is clear, and certainly applies to the foregoing relations $\mathcal{H}''$ and $\mathcal{H}'$.

Lemma 5.2. *Let S be a commutative semigroup and let ρ be a congruence on $\mathcal{S}(S)$. Let $\overline{\overline{\rho}}$ be the congruence on S generated by ρ. If ρ is the restriction to $\mathcal{S}(S)$ of a congruence on S, then $\overline{\overline{\rho}}|_{\mathcal{S}(S)} = \rho$.*

Suppose now that S is a commutative semigroup and $\mathcal{S}(S)$ is an order in a (commutative Clifford) semigroup R. We are *not* assuming here that S is an order. From (*vii*) of Result 2.6 we have that $\mathcal{S}(S)$ is straight in R, that is, if $q \in R$ then $q = x^\sharp y$ where $x, y \in \mathcal{S}(S)$ and $x\,\mathcal{H}^R\,y$. We let

$$\leq = \leq_{\mathcal{H}^R}|_{\mathcal{S}(S)} \text{ and } \rho = \equiv_{\leq} = \mathcal{H}^R|_{\mathcal{S}(S)}.$$

From Lemma 2.12, $Q = R \otimes_{\mathcal{S}(S)} S$ is a commutative semigroup in which $(p \otimes s)(q \otimes t) = pq \otimes st$.

The next lemma is phrased in such a way that we can maximise its implications.

Lemma 5.3. *Let S, R and Q be as above and let $\overline{\overline{\rho}}$ be the congruence on S generated by ρ. Suppose that*

$$p^\sharp q \otimes s = x^\sharp y \otimes t$$

where $p, q, x, y \in \mathcal{S}(S)$, $s, t \in S$, $p \,\rho\, q$ and $x \,\rho\, y$. Then

(i) $qs \,\overline{\overline{\rho}}\, yt$;

(ii) if $\rho \subseteq \mathcal{H}''$, where $\mathcal{H}''$ is a congruence on $\mathcal{S}(S)$ induced by a q-pre-order on S, then

$$qs \,\mathcal{H}'\, yt \text{ and } xqs = pyt;$$

(iii) if $\rho = \overline{\overline{\rho}}|_{\mathcal{S}(S)}$, and $s, t \in \mathcal{S}(S)$, then

$$qs \,\rho\, yt \text{ and } xqs = pyt.$$

Proof. (i) We have a system of equalities

$$\begin{array}{rclrcl} & & & s &=& s_1 b_1 \\ p^\sharp q s_1 &=& a_2 t_1 & t_1 b_1 &=& s_2 b_2 \\ a_2 s_2 &=& a_3 t_2 & & & \\ &\vdots& & &\vdots& \\ a_{m-1} s_{m-1} &=& a_m t_{m-1} & t_{m-1} b_{m-1} &=& s_m b_m \\ a_m s_m &=& x^\sharp y t_m & t_m b_m &=& t \end{array} \tag{2}$$

for some $s_1, t_1, \dots, s_m, t_m \in \mathcal{S}(S)^1$, $a_2, \dots, a_m \in R$ and $b_1, \dots, b_m \in S$.

Since $\mathcal{S}(S)$ is a straight left order in R, we have that $a_i = {c_i}^\sharp d_i$ for some $c_i, d_i \in \mathcal{S}(S)$ with $c_i \,\rho\, d_i$, $2 \le i \le m$. Let $w = pc_2 \dots c_m x \in \mathcal{S}(S)$. Then, multiplying each of the equations in the left hand column of (2) by w, we have

$$\begin{array}{rcll} c_2 \dots c_m x q s_1 &=& pc_3 \dots c_m x d_2 t_1 & \text{as } p^\sharp pq = q \text{ and } {c_2}^\sharp c_2 d_2 = d_2 \\ pc_3 \dots c_m x d_2 s_2 &=& pc_2 c_4 \dots c_m x d_3 t_2 & \text{as } {c_2}^\sharp c_2 d_2 = d_2 \text{ and } {c_3}^\sharp c_3 d_3 = d_3 \\ &\vdots& & \\ pc_2 \dots c_{m-1} x d_m s_m &=& pc_2 \dots c_m y t_m & \text{as } c_m^\sharp c_m d_m = d_m \text{ and } x^\sharp x y = y. \end{array}$$

This gives us that

$$\begin{array}{rcl} c_2 \dots c_m x q s &=& c_2 \dots c_m x q s_1 b_1 \\ &=& pc_3 \dots c_m x d_2 t_1 b_1 \\ &=& pc_3 \dots c_m x d_2 s_2 b_2 \\ &=& pc_2 c_4 \dots c_m x d_3 t_2 b_2 \\ &\vdots& \\ &=& pc_2 \dots c_{m-1} x d_m t_{m-1} b_{m-1} \\ &=& pc_2 \dots c_{m-1} x d_m s_m b_m \\ &=& pc_2 \dots c_m y t_m b_m \\ &=& pc_2 \dots c_m yt. \end{array}$$

Again using our list of equalities (2), we have that

$$qs_1 \,\rho\, c_2t_1,\ c_2s_2 \,\rho\, c_3t_2,\ \dots, c_ms_m \,\rho\, yt_m,$$

so that

$$qs = qs_1b_1 \,\overline{\overline{\rho}}\, c_2t_1b_1 = c_2s_2b_2 \,\overline{\overline{\rho}}\, c_3t_2b_2 \,\overline{\overline{\rho}}\, \dots \overline{\overline{\rho}}\, c_mt_{m-1}b_{m-1} = c_ms_mb_m \,\overline{\overline{\rho}}\, yt_mb_m = yt.$$

(ii) Suppose now that $\rho \subseteq \mathcal{H}''$, where $\mathcal{H}''$ is a congruence on $\mathcal{S}(S)$ induced by a q-pre-order $\preceq$ on S. Then $\rho \subseteq \overline{\overline{\rho}} \subseteq \langle \mathcal{H}'' \rangle \subseteq \mathcal{H}'$.

From (i) we certainly have $qs \,\mathcal{H}'\, yt$. Hence $pqxs \,\mathcal{H}'\, pxyt$ and so $xqs \,\mathcal{H}'\, pyt$. From Lemma 2.11 we have that $qs \preceq c_2 \dots c_m \in \mathcal{S}(S)$. Now from $c_2 \dots c_mxqs = pc_2 \dots c_myt$, Conditions (A) and (B) give that $xqs = pyt$.

(iii) Suppose now that $\rho = \overline{\overline{\rho}}|_{\mathcal{S}(S)}$, and $s, t \in \mathcal{S}(S)$. We certainly have that $qs \,\rho\, yt$. We have observed that for any $i \in \{2, \dots, m\}$, we have that $qs \,\overline{\overline{\rho}}\, c_iw$ for some $w \in S$. Now $c_i \,\rho\, c_i^2$, so that

$$c_iqs \,\overline{\overline{\rho}}\, c_i^2w \,\overline{\overline{\rho}}\, c_iw \,\overline{\overline{\rho}}\, qs,$$

and $qs \,\rho\, c_2 \dots c_mqs$. With a familiar argument we see that $xqs \,\rho\, pyt$ and so from $c_2 \dots c_mxqs = pc_2 \dots c_myt$ and the fact that $\mathcal{S}(S)$ is an order in the Clifford semigroup R, we deduce that $xqs = pyt$. □

Lemma 5.4. *With notation as above, the map $\theta : R \to Q$ given by*

$$(a^\sharp b)\theta = a^\sharp \otimes b$$

where $a, b \in \mathcal{S}(S)$ and $a \,\rho\, b$, is a well-defined homomorphism.

Further, θ is an embedding if and only if $\overline{\overline{\rho}}|_{\mathcal{S}(S)} = \rho$.

Proof. Suppose that $a^\sharp b = c^\sharp d$ where $a, b, c, d \in \mathcal{S}(S)$, $a \,\rho\, b$ and $c \,\rho\, d$. Then a, b, c, d are all ρ-related and lie in the same subgroup of R. We then calculate that $cb = ad$ and

$$a^\sharp \otimes b = a^\sharp c^\sharp c \otimes b = a^\sharp c^\sharp \otimes cb = a^\sharp c^\sharp \otimes ad = a^\sharp c^\sharp a \otimes d = c^\sharp \otimes d,$$

so that θ is well defined.

To see that θ is a homomorphism, again let $a^\sharp b, c^\sharp d \in R$ where $a, b, c, d \in \mathcal{S}(S)$, $a \,\rho\, b$ and $c \,\rho\, d$, so that $ac \,\rho\, bd$. Using the fact that in R we have $(uv)^\sharp = u^\sharp v^\sharp$, we see that

$$\big((a^\sharp b)(c^\sharp d)\big)\theta = ((ac)^\sharp bd)\theta = (ac)^\sharp \otimes bd = a^\sharp c^\sharp \otimes bd = (a^\sharp \otimes b)(c^\sharp \otimes d) = (a^\sharp b)\theta(c^\sharp d)\theta,$$

so that θ is a homomorphism.

We remark that for *any* $u^\sharp v \in R$, where $u, v \in \mathcal{S}(S)$, we have that

$$\begin{aligned}(u^\sharp v)\theta = \big((u^2v)^\sharp uv^2\big)\theta = (u^2v)^\sharp \otimes uv^2 = (u^2)^\sharp v^\sharp v^2 \otimes u = \\ (u^2)^\sharp v \otimes u = (u^2)^\sharp \otimes vu = (u^2)^\sharp u \otimes v = u^\sharp \otimes v.\end{aligned}$$

Suppose now that $\overline{\overline{\rho}}|_{\mathcal{S}(S)} = \rho$. Again let $a^\sharp b, c^\sharp d \in R$ where $a, b, c, d \in \mathcal{S}(S)$, $a \,\rho\, b$ and $c \,\rho\, d$ and suppose that $(a^\sharp b)\theta = (c^\sharp d)\theta$. Then $a^\sharp \otimes b = c^\sharp \otimes d$ and, re-writing to fit in with the notation of Lemma 5.3, we have that $(a^2)^\sharp a \otimes b = (c^2)^\sharp c \otimes d$. From (iii) of Lemma 5.3, we have that $ab \,\rho\, cd$ and $c^2ab = a^2cd$.

Since a, b, c, d all lie in the same subgroup of R, we see that $a^\sharp b = c^\sharp d$ so that θ is an embedding.

Finally, let us assume that θ is an embedding and $u, v \in \mathcal{S}(S)$ are such that $u\,\overline{\overline{\rho}}\,v$. If $u = v$, then certainly $u\,\rho\,v$. Otherwise, there exists $n \in \mathbb{N}$ and elements $c_1, \dots, c_n \in S^1$ and $(x_1, y_1), \dots, (x_n, y_n) \in \rho$ such that

$$u = x_1c_1, y_1c_1 = x_2c_2, \dots, y_nc_n = v.$$

We have

$$\begin{aligned}
u\theta &= (u^\sharp u^2)\theta \\
&= u^\sharp \otimes u^2 \\
&= u^\sharp \otimes x_1c_1u \\
&= u^\sharp x_1 \otimes c_1u \\
&= u^\sharp x_1 y_1^\sharp y_1 \otimes c_1u \\
&= u^\sharp x_1 y_1^\sharp \otimes y_1c_1u \\
&= u^\sharp x_1 y_1^\sharp \otimes x_2c_2u \\
&\vdots \\
&= u^\sharp x_1 y_1^\sharp \dots x_n y_n^\sharp \otimes y_nc_nu \\
&= u^\sharp x_1 y_1^\sharp \dots x_n y_n^\sharp \otimes vu \\
&= (uy_1 \dots y_n)^\sharp \otimes x_1 \dots x_n vu \\
&= \big((uy_1 \dots y_n)^\sharp x_1 \dots x_n vu\big)\theta.
\end{aligned}$$

Since θ is an embedding, we deduce that

$$u = (uy_1 \dots y_n)^\sharp x_1 \dots x_n vu$$

and hence that $u \leq v$. Together with the dual we have that $u\,\rho\,v$ as required. □

We now apply Lemmas 5.2 and 5.4.

Corollary 5.5. *With notation as above, suppose that ρ is induced by a q-preorder on S. Then R embeds into Q.*

In view of Lemma 2.13 the following is clear.

Corollary 5.6. *Let S be a commutative semigroup and let R_1 and R_2 be semigroups of quotients of $\mathcal{S}(S)$, and suppose there is an S-homomorphism from R_1 to R_2. Then $p^\sharp q \otimes s \mapsto p^* q \otimes s$ is a homomorphism from $R_1 \otimes_{\mathcal{S}(S)} S$ onto $R_2 \otimes_{\mathcal{S}(S)} S$, where for clarity we write the inverse of $p \in \mathcal{S}(S)$ in R_2 as p^*.*

We will now suppose that our commutative semigroup S is an order, which is a stronger statement than saying that $\mathcal{S}(S)$ is an order. Again, our next result is phrased in such a way that we maximise its usage.

Theorem 5.7. *Let S be a commutative order in a semigroup W, such that W induces $\preceq$ and $\mathcal{H}'$ on S and $\leq$ and $\mathcal{H}''$ on $\mathcal{S}(S)$. Let ρ be any semilattice congruence on $\mathcal{S}(S)$ such that $\rho \subseteq \mathcal{H}''$, and let R be a semigroup of quotients of $\mathcal{S}(S)$ inducing ρ. Then $\psi : R \otimes_{\mathcal{S}(S)} S \to W$ given by $(a^\sharp b \otimes s)\psi = a^* bs$, where*

$a, b \in \mathcal{S}(S), a\,\rho\, b$ and a^* denotes the group inverse of a in W, is a well-defined onto homomorphism.

Proof. From Result 2.8, there is an S-homomorphism from R to $\mathcal{H}(W)$, which must be given by $a^\sharp b \mapsto a^* b$. Now from Corollary 5.6, we have that there is a homomorphism from $R \otimes_{\mathcal{S}(S)} S \to \mathcal{H}(W) \otimes_{\mathcal{S}(S)} S$ given by $a^\sharp b \otimes s \mapsto a^* b \otimes s$. Clearly the map from $\mathcal{H}(W) \otimes_{\mathcal{S}(S)} S$ to W given by $a^* b \otimes s \mapsto a^* bs$ is an onto homomorphism. $\square$

Corollary 5.8. *Let S be a commutative order, let ρ be the smallest semilattice congruence on $\mathcal{S}(S)$, and let R be a semigroup of quotients of $\mathcal{S}(S)$ inducing ρ. Then every semigroup of quotients of S is a morphic image of $Q = R \otimes_{\mathcal{S}(S)} S$ under $(a^\sharp b \otimes s)\psi = a^* bs$. Moreover, if $\theta : R \to Q$ is given by $(a^\sharp b)\theta = a^\sharp \otimes b$, then $(a^\sharp b)\theta\psi = a^* b$.*

Corollary 5.9. *Let S be a commutative order in W and let $\preceq$ be the q-pre-order on S induced by W. Let $R = \mathcal{H}(W)$ be a semigroup of quotients of $\mathcal{S}(S)$ corresponding to $\mathcal{H}''$. Then R embeds into $R \otimes_{\mathcal{S}(S)} S$ under $(a^\sharp b)\theta = a^\sharp \otimes b$, $\psi : R \otimes_{\mathcal{S}(S)} S \mapsto W$ given by $(a^\sharp b \otimes s)\psi = a^\sharp bs$ is an onto homomorphism, where $a, b \in \mathcal{S}(S)$ with $a\,\mathcal{H}''\, b$ and $s \in S$. Further, $(a^\sharp b)\theta\psi = a^\sharp b$ and $\theta\psi$ is the identity map on R.*

The diagram below represents the relation between the various semigroups constructed. Here, S is a commutative order and R, T are semigroups of quotients of $\mathcal{S}(S)$ with R being the greatest such. The semigroups Q_R and Q_T are any quotient semigroups of S such that $\mathcal{H}(Q_R)$ $(\mathcal{H}(Q_T))$ are isomorphic to R and T, respectively. In general we cannot deduce that Q_T is an image of Q_R, since this depends upon the pre-orders induced by Q_R and Q_T on the whole of S.

$$\begin{tikzcd}
R \arrow[r] \arrow[drr] \arrow[dd] & R \otimes_{\mathcal{S}(S)} S \arrow[dr] \arrow[dd] & \\
 & & Q_R \\
T \arrow[r] \arrow[drr] & T \otimes_{\mathcal{S}(S)} S \arrow[dr] & \\
 & & Q_T
\end{tikzcd}$$

We would like to say in Corollary 5.9 above that $W \cong R \otimes_{\mathcal{S}(S)} S$. However, this is not true, owing to the fact that $\mathcal{H}''$ on $\mathcal{H}(W)$ may be induced by *different* q-pre-orders on S and hence by *different* semigroups of quotients. We now show how to recover each such W by factoring $R \otimes_{\mathcal{S}(S)} S$. We hence recover the constructive part of Theorem 2.9.

To get the widest applications, we again proceed in the most general way. Let $\preceq$ be a q-pre-order on S, with associated equivalence relation $\mathcal{H}'$, and let $\leq$ and $\mathcal{H}''$ be the restrictions of $\preceq$ and $\mathcal{H}'$ to $\mathcal{S}(S)$. Let R be a semigroup of quotients of $\mathcal{S}(S)$, and let $\leq_\rho$ and ρ be the restrictions of $\leq_{\mathcal{H}}$ and $\mathcal{H}$ in R to $\mathcal{S}(S)$. Suppose that $\leq_\rho \subseteq \leq$, so that $\rho \subseteq \mathcal{H}''$.

Let $Q = R \otimes_{\mathcal{S}(S)} S$ and put

$$\overline{Q} = Q/\overline{\overline{K}}$$

where $\overline{\overline{K}}$ is the congruence generated by

$$K = \{(uu^\sharp \otimes s, vv^\sharp \otimes s) : u, v \in \mathcal{S}(S), s \preceq u, v\}.$$

By the standard construction of a semigroup congruence from a symmetric set of generators, $\overline{\overline{K}}$ is the reflexive transitive closure of

$$\overline{K} = \{((uu^\sharp \otimes s)\alpha, (vv^\sharp \otimes s)\alpha) : (uu^\sharp \otimes s, vv^\sharp \otimes s) \in K, \alpha \in Q^1\}.$$

Denoting the $\overline{\overline{K}}$-equivalence class of $p \otimes s \in Q$ by $[p \otimes s]$, let $\theta : S \to \overline{Q}$ be given by

$$s\theta = [uu^\sharp \otimes s] \text{ where } s \preceq u \in \mathcal{S}(S).$$

It is easy to see that Condition (C) may be deduced from (C$'$), so that $s\theta$ is defined for any $s \in S$. By definition of $\overline{\overline{K}}$, it is clear that θ is well defined. We proceed via a series of lemmas.

Lemma 5.10. *If $[u^\sharp x \otimes s] = [v^\sharp y \otimes t]$ where $u, x, v, y \in \mathcal{S}(S), u\,\rho\,x, v\,\rho\,y$ and $s, t \in S$, then $xs\,\mathcal{H}'\,yt$.*

Proof. If

$$u^\sharp x \otimes s = v^\sharp y \otimes t,$$

then using Lemma 5.3, we have $xs\,\rho\,yt$, so that $xs\,\mathcal{H}'\,yt$ as required.

Suppose now that

$$u^\sharp x \otimes s = \alpha(pp^\sharp \otimes r),\ \alpha(qq^\sharp \otimes r) = v^\sharp y \otimes t,$$

where $(pp^\sharp \otimes r, qq^\sharp \otimes s) \in K$ and $\alpha \in Q^1$. We have either $\alpha = 1$ or $\alpha = h^\sharp k \otimes z$ for some $h, k \in \mathcal{S}(S)$ with $h\,\rho\,k$ and $z \in S$. For $\alpha = 1$, let $h = k = z = 1$ in S^1. Then, by definition of multiplication in Q, we have in either case that

$$u^\sharp x \otimes s = h^\sharp kpp^\sharp \otimes zr,\ \ h^\sharp kqq^\sharp \otimes zr = v^\sharp y \otimes t.$$

Making use of Lemmas 2.10 and 5.3, we have

$$xs\,\mathcal{H}'\,kpzr\,\mathcal{H}'\,kzr\,\mathcal{H}'\,kqzr\,\mathcal{H}'\,yt.$$

The result now follows by transitivity. □

Lemma 5.11. *The function θ is an embedding of S in Q.*

Proof. Suppose first that $s\theta = t\theta$, that is,

$$[uu^\sharp \otimes s] = [vv^\sharp \otimes t]$$

for some $u, v \in \mathcal{S}(S)$ with $s \preceq u$ and $t \preceq v$.

By Lemmas 2.10 and 5.10, we have

$$s\,\mathcal{H}'\,us\,\mathcal{H}'\,vt\,\mathcal{H}'\,t.$$

If $u^\sharp u \otimes s = v^\sharp v \otimes t$, then from Lemma 5.3,

$$uvs = uvt$$

so that as $s, t \preceq uv$, Condition (B) gives that $s = t$.

Otherwise, since $\overline{\overline{K}}$ is the reflexive transitive closure of $\overline{K}$, there exist $n \in \mathbb{N}$ and for $1 \le i \le n$,

$$\alpha_i \in Q^1, {p_i}^\sharp p_i \otimes r_i, {q_i}^\sharp q_i \otimes r_i \in Q$$

where $p_i, q_i \in \mathcal{S}(S), r_i \in S$, with $r_i \preceq p_i, q_i$ such that

$$\begin{array}{rcl} u^\sharp x \otimes s &=& \alpha_1({p_1}^\sharp p_1 \otimes r_1) \\ \alpha_1({q_1}^\sharp q_1 \otimes r_1) &=& \alpha_2({p_2}^\sharp p_2 \otimes r_1) \\ &\vdots& \\ \alpha_{n-1}({q_{n-1}}^\sharp q_{n-1} \otimes r_{n-1}) &=& \alpha_n({p_n}^\sharp p_n \otimes r_n) \\ \alpha_n({q_n}^\sharp q_n \otimes r_n) &=& v^\sharp y \otimes t. \end{array}$$

For $1 \le i \le n$, either $\alpha_i = 1$ or $\alpha_i = {x_i}^\sharp y_i \otimes z_i$ for some $x_i, y_i \in \mathcal{S}(S)$ with $x_i \,\rho\, y_i$ and $z_i \in S$. For $\alpha_i = 1$, let $x_i = y_i = z_i = 1$ in S^1. Then by definition of multiplication in Q, we have

$$\begin{array}{rcl} u^\sharp x \otimes s &=& (p_1x_1)^\sharp p_1y_1 \otimes z_1r_1 \\ (q_1x_1)^\sharp q_1y_1 \otimes z_1r_1 &=& (p_2x_2)^\sharp p_2y_2 \otimes z_2r_2 \\ &\vdots& \\ (q_{n-1}x_{n-1})^\sharp q_{n-1}y_{n-1} \otimes z_{n-1}r_{n-1} &=& (p_nx_n)^\sharp p_ny_n \otimes z_nr_n \\ (q_nx_n)^\sharp q_ny_n \otimes z_nr_n &=& v^\sharp y \otimes t. \end{array} \tag{3}$$

Making use of Lemmas 2.10 and 5.3 (or Lemma 5.10), we have

$$xs\,\mathcal{H}'\,p_1y_1z_1r_1\,\mathcal{H}'\,y_1z_1r_1\,\mathcal{H}'\,q_1y_1z_1r_1\,\mathcal{H}'\,p_2y_2z_2r_2\,\mathcal{H}'\,\ldots\,\mathcal{H}'\,y_nz_nr_n\,\mathcal{H}'\,q_ny_nz_nr_n\,\mathcal{H}'\,yt.$$

From (3), we also have

$$\begin{array}{rcl} p_1x_1us &=& up_1y_1z_1r_1 \\ p_2x_2q_1y_1z_1r_1 &=& q_1x_1p_2y_2z_2r_2 \\ &\vdots& \\ p_nx_nq_{n-1}y_{n-1}z_{n-1}r_{n-1} &=& q_{n-1}x_{n-1}p_ny_nz_nr_n \\ vq_ny_nz_nr_n &=& q_nx_nvt. \end{array} \tag{4}$$

In view of Lemma 2.11, we can cancel $u, p_1, q_1, \ldots, p_n, q_n$ and v from the equalities (4) to obtain

$$\begin{array}{rcl} x_1 s & = & y_1 z_1 r_1 \\ x_2 y_1 z_1 r_1 & = & x_1 y_2 z_2 r_2 \\ & \vdots & \\ x_n y_{n-1} z_{n-1} r_{n-1} & = & x_{n-1} y_n z_n r_n \\ y_n z_n r_n & = & x_n t. \end{array}$$

We deduce that

$$x_1 \ldots x_n s = y_1 x_2 \ldots x_n z_1 r_1 = x_1 y_2 x_3 \ldots x_n z_2 r_2 = \ldots$$

$$= x_1 \ldots x_{n-1} y_n z_n r_n = x_1 \ldots x_n t.$$

If $x_1 \ldots x_n = 1$, clearly $s = t$. Otherwise, $x_1 \ldots x_n \in \mathcal{S}(S)$, and using Lemma 2.11, $s \,\mathcal{H}'\, t \preceq x_1 \ldots x_n$, yielding $s = t$. Thus θ is an injection.

To see that θ is an embedding, notice that if $a\theta = [c^\sharp c \otimes a]$ and $b\theta = [d^\sharp d \otimes b]$, where $a \preceq c$ and $b \preceq d$, then $ab \preceq cd$, so that

$$a\theta b\theta = [c^\sharp c \otimes a][d^\sharp d \otimes b] = [(c^\sharp c \otimes a)(d^\sharp d \otimes b)] = [(cd)^\sharp cd \otimes ab] = (ab)\theta.$$

□

Lemma 5.12. *Let*

$$[x^\sharp y \otimes s], [u^\sharp v \otimes t] \in \overline{Q},$$

where $x, y, u, v \in \mathcal{S}(S), x \,\rho\, y, u \,\rho\, v, s, t \in S$. *Then* $[x^\sharp y \otimes s] \leq_{\mathcal{H}} [u^\sharp v \otimes t]$ *if and only if* $ys \preceq vt$.

Proof. If $[x^\sharp y \otimes s] \leq_{\mathcal{H}} [u^\sharp v \otimes t]$, then either $[x^\sharp y \otimes s] = [u^\sharp v \otimes t]$ or there exists $[a^\sharp b \otimes c] \in \overline{Q}$, where $a, b \in \mathcal{S}(S), c \in S$ and $a \,\rho\, b$, such that

$$[x^\sharp y \otimes s] = [a^\sharp b \otimes c][u^\sharp v \otimes t] = [(ua)^\sharp bv \otimes ct].$$

In the first case, Lemma 5.10 gives directly that $ys \,\mathcal{H}'\, vt$ and in the second, we deduce that $ys \,\mathcal{H}'\, bvct \preceq vt$.

Conversely, suppose that $ys \preceq vt$. By (C′), there exist $a \in \mathcal{S}(S), b \in S$ with $ys \preceq a$, such that $ysa = vtb$. We then calculate that

$$[x^\sharp a^\sharp \otimes ub][u^\sharp v \otimes t] = [x^\sharp a^\sharp u^\sharp v \otimes ubt] = [x^\sharp a^\sharp u^\sharp uv \otimes bt] = [x^\sharp a^\sharp v \otimes bt] =$$

$$[x^\sharp a^\sharp \otimes vtb] = [x^\sharp a^\sharp \otimes ays] = [x^\sharp a^\sharp a \otimes ys] = [(x^\sharp)^2 aa^\sharp \otimes xys] = [(x^\sharp)^2 \otimes x][aa^\sharp \otimes ys] =$$

$$[(x^\sharp)^2 \otimes x][xx^\sharp \otimes ys] = [(x^\sharp)^3 x \otimes xys] = [x^\sharp y \otimes s],$$

since $(aa^\sharp \otimes ys, xx^\sharp \otimes ys) \in K$. We therefore deduce that $[x^\sharp y \otimes s] \leq_{\mathcal{H}} [u^\sharp v \otimes t]$ as required. □

The following corollary is now straightforward:

Corollary 5.13. *For any* $s, t \in S$,

$$s\theta \leq_{\mathcal{H}} t\theta \text{ in } \overline{Q} \text{ if and only if } s \preceq t.$$

Lemma 5.14. *The semigroup* $\overline{Q}$ *is a semigroup of quotients of* $S\theta$.

Proof. If $a \in \mathcal{S}(S)$, then an easy calculation gives that $a\theta = [aa^\sharp \otimes a]$ lies in a subgroup of $\overline{Q}$ with identity $[a^\sharp \otimes a]$, such that $[a^\sharp a \otimes a]^\sharp = [(a^\sharp)^2 \otimes a]$.

Suppose now that $[p^\sharp q \otimes s] \in \overline{Q}$, where $p, q \in \mathcal{S}(S), s \in S$ and $p \,\rho\, q$. Then
$(p\theta)^\sharp(qs)\theta = [(p^2)^\sharp \otimes p][q^\sharp q \otimes qs] = [(p^2)^\sharp q^\sharp q \otimes pqs] = [(p^2)^\sharp q^\sharp qpq \otimes s] = [p^\sharp q \otimes s]$. $\square$

Theorem 5.15. *Let $\preceq$ be a q-pre-order on a commutative semigroup S. Then S is an order in the semigroup $\overline{Q}$ inducing $\preceq$.*

Efffectively, what we have achieved in the preceding argument is to determine the kernel of ψ in Theorem 5.7. It is worth making specific one further consequence.

Corollary 5.16. *Let S be a commutative semigroup and let ρ be a congruence on $\mathcal{S}(S)$ induced by a q-pre-order on S. Let R be the corresponding semigroup of quotients of $\mathcal{S}(S)$. Then for* any *q-pre-order $\preceq$ inducing ρ we have that*

$$W \cong R \otimes_{\mathcal{S}(S)} S / \langle \{(u^\sharp u \otimes s, v^\sharp v \otimes s) : s \in S, u, v \in \mathcal{S}(S)\, s \preceq u, v\} \rangle.$$

6. Extension of semilattice congruences on $\mathcal{S}(S)$

Let S be a commutative semigroup. In view of Lemmas 5.2 and 5.4, we wish to determine under which conditions do we have that a semilattice congruence ρ on $\mathcal{S}(S)$ with associated preorder $\leq$ is such that $\overline{\overline{\rho}}|_{\mathcal{S}(S)} = \rho$. Further, if this holds, when is it the case that ρ and $\leq$ are induced by $\mathcal{H}'$ and $\preceq$, where $\preceq$ is a q-pre-order on S with associated congruence $\mathcal{H}'$? The first question we answer completely. As for the second, we get a full answer in the case where S is a monoid and show how to understand the result in the case that S is not.

Lemma 6.1. *Let S be a commutative semigroup and let ρ be a semilattice congruence on $\mathcal{S}(S)$ with associated compatible pre-order $\leq$. Then for any $c, d \in S$ with $c \overline{\leq} d$, where $\overline{\leq}$ is the smallest compatible pre-order on S containing $\leq$, we have*

(i) $c = d$ or

(ii) $c \,\overline{\overline{\rho}}\, xd$ and $yc = xd$ for some $x, y \in \mathcal{S}(S)$ with $x \leq y$.

Proof. Let $c \overline{\leq} d$. In view of Lemma 2.1, $c = d$ or there exist $n \in \mathbb{N}$, and for $1 \leq i \leq n$, elements $c_i \in S^1$ and $x_i, y_i \in \mathcal{S}(S)$ with $x_i \leq y_i$, such that

$$c = x_1c_1,\ y_1c_1 = x_2c_2,\ \dots,\ y_nc_n = d.$$

Suppose the latter holds. Notice that for $1 \leq i \leq n$ we have that $x_iy_i \,\rho\, x_i$. Then

$$c = x_1c_1 \,\overline{\overline{\rho}}\, x_1y_1c_1 = x_1x_2c_2 \,\overline{\overline{\rho}}\, \dots \,\overline{\overline{\rho}}\, x_1 \dots x_nc_n \,\overline{\overline{\rho}}\, x_1 \dots x_ny_nc_n = x_1 \dots x_nd,$$

so that $c \,\overline{\overline{\rho}}\, xd$ where $x = x_1 \dots x_n \in \mathcal{S}(S)$.

Let $y = y_1 \dots y_n$, so that $y \in \mathcal{S}(S)$ and $x \leq y$. We have

$$\begin{aligned} yc &= y_1 \dots y_nc = y_1 \dots y_nx_1c_1 = x_1x_2y_2 \dots y_nc_2 \\ &= \dots = x_1 \dots x_{n-1}y_{n-1}y_nc_{n-1} = x_1 \dots x_ny_nc_n = xd. \end{aligned}$$

□

We will return later to the $\leq$-sequence connecting c to d.

Lemma 6.2. *Let S be a commutative semigroup and let ρ be a semilattice congruence on S with associated compatible pre-order $\leq$. Then*
(i) $\overline{\overline{\rho}}$ is the congruence $\equiv_{\overline{\leq}}$ associated with $\overline{\leq}$;
(ii) $\overline{\overline{\rho}}|_{\mathcal{S}(S)} = \rho$ if and only if $\overline{\leq}|_{\mathcal{S}(S)} = \leq$.

Proof. (i) Suppose that $c\,\overline{\overline{\rho}}\,d$. Then either $c = d$ (so that clearly $c \equiv_{\overline{\leq}} d$) or there exists a ρ-sequence connecting c to d. As this sequence and its reverse are certainly $\leq$-sequences, we see that $c \equiv_{\overline{\leq}} d$.

Conversely, suppose that $c \equiv_{\overline{\leq}} d$. Then either $c = d$ (so that $c\,\overline{\overline{\rho}}\,d$), or by Lemma 6.1 we have that

$$c\,\overline{\overline{\rho}}\,ud \text{ and } d\,\overline{\overline{\rho}}\,vc$$

for some $u, v \in \mathcal{S}(S)$. Then $c\,\overline{\overline{\rho}}\,uvc$ so that

$$d\,\overline{\overline{\rho}}\,vc\,\overline{\overline{\rho}}\,uv^2c\,\overline{\overline{\rho}}\,uvc\,\overline{\overline{\rho}}\,c.$$

(ii) $(\Leftarrow)$ Let $a, b \in \mathcal{S}(S)$ and suppose that $a\,\overline{\overline{\rho}}\,b$. By (i), $a\,\overline{\leq}\,b\,\overline{\leq}\,a$ so that by assumption $a \leq b \leq a$ and so $a\,\rho\,b$.

$(\Rightarrow)$ Let $a, b \in \mathcal{S}(S)$ and suppose that $a\,\overline{\leq}\,b$. By Lemma 6.1 we have that either $a = b$ (so that $a \leq b$) or $a\,\overline{\overline{\rho}}\,ub$ for some $u \in \mathcal{S}(S)$. By assumption, $a\,\rho\,ub$, so that $a \leq b$ as required. □

We can now give the first result promised at the beginning of this section.

Proposition 6.3. *Let S be a commutative semigroup and let ρ be a semilattice congruence on $\mathcal{S}(S)$ with associated compatible pre-order $\leq$. Then $\overline{\overline{\rho}}|_{\mathcal{S}(S)} = \rho$ if and only if Condition (R) holds.*
(R) For all $a, b \in \mathcal{S}(S)$ and $c \in S$ with $a = bc$, we have that $a \leq b$.

Proof. Suppose that $\overline{\overline{\rho}}|_{\mathcal{S}(S)} = \rho$. Let $a, b \in \mathcal{S}(S)$ and $c \in S$ be such that $a = bc$. Then

$$a = bc\,\overline{\overline{\rho}}\,b^2c = ba,$$

so that $a\,\rho\,ab$ and consequently, $a \leq b$.

Conversely, suppose that (R) holds. Let $u, v \in \mathcal{S}(S)$ be such that $u\,\overline{\leq}\,v$. As in Lemma 6.1, either $u = v$ (and so $u \leq v$), or there exist $n \in \mathbb{N}$, and for $1 \leq i \leq n$, elements $c_i \in S^1$ and $x_i, y_i \in \mathcal{S}(S)$ with $x_i \leq y_i$, such that

$$u = x_1c_1,\ y_1c_1 = x_2c_2,\ \ldots\ , y_nc_n = v.$$

From $u = x_1c_1$ our given condition tells us that $u \leq x_1 \leq y_1$ and

$$u\,\rho\,y_1u = y_1x_1c_1.$$

Suppose that for some i with $1 \leq i < n$ we have that $u \leq x_j \leq y_j$ for all $1 \leq j \leq i$ and $y_1 \ldots y_iu = x_1 \ldots x_iy_ic_i$. Then

$$u\,\rho\,y_1 \ldots y_iu = x_1 \ldots x_ix_{i+1}c_{i+1}$$

and again using our given condition we find that $u \leq x_{i+1} \leq y_{i+1}$ and further, $y_1 \ldots y_{i+1}u = x_1 \ldots x_{i+1}y_{i+1}c_{i+1}$.

By finite induction we obtain that

$$u\,\rho\, y_1 \dots y_n u = x_1 \dots x_n y_n c_n = x_1 \dots x_n v.$$

Hence $u \leq v$ as required. The result now follows using Lemma 6.2 (ii).

□

We recall that a necessary condition for a semilattice congruence ρ on $\mathcal{S}(S)$ to be induced by $\mathcal{H}'$, where $\mathcal{H}'$ is $\equiv_{\preceq}$ for a q-pre-order on S, is that we have $\overline{\overline{\rho}}\,|_{\mathcal{S}(S)} = \rho$. We have now determined when the latter condition holds. If it does, what further conditions do we need in order that ρ be induced by $\mathcal{H}'$? Surprisingly, at least in the case where S is a monoid, only one. First, we examine how to find a compatible pre-order on S containing $\leq$ and satisfying Condition (A).

Lemma 6.4. *Let S be a commutative semigroup and let ρ be a semilattice congruence on $\mathcal{S}(S)$ with associated pre-order $\leq$. We define*

$$A = \{(bc, b) : b, c \in S\}$$

and let $\overline{\leq_A}$ be the compatible pre-order on S generated by $\leq \cup A$. Then
(i) for any $c, d \in S$, $c \overline{\leq_A} d$ if and only if $c \overline{\leq} wd$ for some $w \in S^1$;
(ii) $\overline{\leq}\,|_{\mathcal{S}(S)} = \leq$ if and only if $\overline{\leq_A}\,|_{\mathcal{S}(S)} = \leq$.

Proof. We remark that certainly $\overline{\leq} \subseteq \overline{\leq_A}$.
(i) Suppose that $c \overline{\leq} wd$ for some $w \in S^1$. Then using the definition of A,

$$c \;\overline{\leq_A}\; wd \;\overline{\leq_A}\; d.$$

Conversely, suppose that $c \overline{\leq_A} d$. Then either $c = d$ (so that $c \overline{\leq} d1$) or there exist $n \in \mathbb{N}$ and for $1 \leq i \leq n$, elements $c_i \in S^1$ and $(x_i, y_i) \in \leq \cup A$, such that

$$c = x_1 c_1,\; y_1 c_1 = x_2 c_2,\; \dots\;, y_n c_n = d.$$

If every $(x_i, y_i) \in \leq$, then $c \overline{\leq} d = d1$. Otherwise, let

$$i_1 < i_2 < \dots < i_m$$

be those integers in $\{1, \dots, n\}$ such that $(x_{i_j}, y_{i_j}) \in A$ for $1 \leq j \leq m$; write $(x_{i_j}, y_{i_j}) = (h_{i_j} k_{i_j}, k_{i_j})$.
We calculate:

$$\begin{aligned}
&c \overline{\leq} y_{i_1-1} c_{i_1-1} = x_{i_1} c_{i_1} = h_{i_1} k_{i_1} c_{i_1} = h_{i_1} y_{i_1} c_{i_1} = h_{i_1} x_{i_1+1} c_{i_1+1} \overline{\leq} \\
&h_{i_1} y_{i_2-1} c_{i_2-1} = h_{i_1} x_{i_2} c_{i_2} = h_{i_1} h_{i_2} k_{i_2} c_{i_2} = h_{i_1} h_{i_2} y_{i_2} c_{i_2} \overline{\leq} \dots \overline{\leq} \\
&h_{i_1} h_{i_2} \dots h_{i_m} y_{i_m} c_{i_m} \overline{\leq} h_{i_1} h_{i_2} \dots h_{i_m} y_n c_n = h_{i_1} h_{i_2} \dots h_{i_m} d,
\end{aligned}$$

so that $c \overline{\leq} dw$ for $w = h_{i_1} h_{i_2} \dots h_{i_m}$.
(ii) Suppose that $\overline{\leq}\,|_{\mathcal{S}(S)} = \leq$. Let $a, b \in \mathcal{S}(S)$ with $a \overline{\leq_A} b$. Then by (i), $a \overline{\leq} bw$ for some $w \in S^1$ and so by Lemma 6.1 we have that $a = bw$ or $a \overline{\overline{\rho}}\, bwx$ for some $x \in \mathcal{S}(S)$. In either case therefore we have that $a \overline{\overline{\rho}}\, yb$ for some $y \in S^1$. Then it is easy to see that $ab \overline{\overline{\rho}}\, a$ so that as $\overline{\overline{\rho}} = \equiv_{\overline{\leq}}$, our assumption gives that $ab\,\rho\, a$ and so $a \leq b$.
Conversely, if $\overline{\leq_A}\,|_{\mathcal{S}(S)} = \leq$ then as

$$\leq \subseteq \overline{\leq} \subseteq \overline{\leq_A}$$

is is clear that $\overline{\leq}\,|_{\mathcal{S}(S)} = \leq$. $\square$

Under the assumption that $\overline{\overline{\rho}}\,|_{\mathcal{S}(S)} = \rho$, our relation $\overline{\leq_A}$ automatically satisfies many of the conditions required to be a q-pre-order.

Lemma 6.5. *Suppose that $\overline{\overline{\rho}}\,|_{\mathcal{S}(S)} = \rho$. Then*
(i) $\equiv_{\overline{\leq_A}}|_{\mathcal{S}(S)} = \rho$;
(ii) $\overline{\leq_A}$ satisfies Conditions (A) and (C$''$);
(iii) for any $a \in \mathcal{S}(S)$ and $b \in S$,

$$b\,\overline{\leq_A}\,a \Leftrightarrow ab \equiv_{\overline{\leq_A}} b.$$

Proof. (i) By Lemmas 6.2 and 6.4 we have that $\overline{\leq_A}\,|_{\mathcal{S}(S)} = \leq$, so that as ρ is $\equiv_{\leq}$, the result is clear.

(ii) Clearly (A) holds by construction of $\overline{\leq_A}$. Suppose that $b, c \in S$ with $b\,\overline{\leq_A}\,c$. From Lemma 6.4 we have that $b\,\overline{\leq}\,cw$ for some $w \in S^1$. From Lemma 6.1, we have that either $b = cw$, or $b\,\overline{\overline{\rho}}\,xcw$ and $yb = xcw$ for some $x, y \in \mathcal{S}(S)$ with $x \leq y$. Since $\overline{\overline{\rho}} \subseteq \equiv_{\overline{\leq_A}}$, in the latter case we have that

$$b \equiv_{\overline{\leq_A}} xwc\,\overline{\leq_A}\,x\,\overline{\leq_A}\,y$$

so that $b\,\overline{\leq_A}\,y$.

(iii) Let $b \in S$ and $a \in \mathcal{S}(S)$. By definition we have that $ab\,\overline{\leq_A}\,b$.

If $ab \equiv_{\overline{\leq_A}} b$, then $b\,\overline{\leq_A}\,ab\,\overline{\leq_A}\,a$, by definition of $\overline{\leq_A}$.

On the other hand, if $b\,\overline{\leq_A}\,a$, then $b\,\overline{\leq}\,wa$ for some $w \in S^1$. From Lemma 6.1, we have that $b\,\overline{\overline{\rho}}\,va$ for some $v \in S^1$. We deduce that $ba\,\overline{\overline{\rho}}\,b$ so that certainly $b \equiv_{\overline{\leq_A}} ab$. $\square$

To get the smooothest final conclusions we make use of generalised orders. The motivation is as follows. In trying to construct a semigroup of quotients of S, this may be prevented by there being elements of S that are not less than any square-cancellable element in any suitable pre-order. If S is an order in Q, then for any $s \in S$ there must be an $a \in \mathcal{S}(S)$ such that $s \leq_{\mathcal{H}^Q} a$, simply because we must be able to write s as a quotient $a^{\sharp}b$, where $a, b \in S$. To artificially make a choice of pairs (s, a) to add to our relation $\overline{\leq_A}$ may destroy the nice properties of that relation.

We say that a compatible pre-order on a commutative semigroup S is a *generalised quotient pre-order* or *gq-pre-order* if it satisfies Conditions (A), (B) and (C$''$).

Lemma 6.6. *Let S be a commutative and such that S is monoid or $S = \mathcal{S}(S)$. Then a pre-order $\preceq$ on S is a gq-pre-order if and only if it is a q-pre-order.*

Proof. The result is clear if S is a monoid. If $S = \mathcal{S}(S)$, then just notice that (C$'$) holds for any pre-order. $\square$

For any pre-order on S we denote by $\preceq^1$ the relation $\preceq \cup\{(s,1) : s \in S^1\}$ on S^1. Notice that if S is a monoid and $\preceq$ a q-pre-order, then for any $s \in S = S^1$ we have that $s = s1 \preceq 1$, by Condition (A), so that $\preceq = \preceq^1$. We remark that from the definition of $\mathcal{H}^*$ it follows that $\mathcal{S}(S^1) = \mathcal{S}(S) \cup \{1\}$.

Proposition 6.7. *Let S be a commutative semigroup which is not a monoid. Then the following conditions are equivalent:*

(i) S is a generalised order in a semigroup Q such that $\leq_{\mathcal{H}^Q}|_S = \preceq$;

(ii) $\preceq$ is a gq-pre-order on S;

(iii) $\preceq^1$ is a q-pre-order on S^1;

(iv) S^1 is an order in a monoid P such that $P \setminus \{1\}$ is a semigroup Q and $\leq_{\mathcal{H}^P}|_{S^1} = \preceq^1$.

Proof. $(ii) \Rightarrow (iii)$ Suppose that (ii) holds. It is clear that $\preceq^1$ is a pre-order.

Let $u, v, w \in S^1$ with $u \preceq^1 v$. If $w = 1$, then clearly $uw \preceq^1 vw$; we suppose therefore that $w \neq 1$. If $u, v \in S$ then again it is clear that $uw \preceq^1 vw$. If $u = 1$, then from the definition of $\preceq^1$, we have that $v = 1$ so that certainly $uw \preceq^1 vw$. If $u \neq 1$ and $v = 1$, then $uw \preceq w = vw$, by Condition (A) for $\preceq$. Thus $\preceq^1$ is compatible.

For any $b, c \in S$ we know that $bc \preceq c$, so that $bc \preceq^1 c$. Clearly $1\,1 \preceq^1 1$. Also, $b1 = b \preceq^1 b$ and $b1 \preceq^1 1$, so that $\preceq^1$ satisfies Condition (A).

We remarked above that $\mathcal{S}(S^1) = \mathcal{S}(S) \cup \{1\}$. It is now easy to see that Condition (B) holds for $\preceq^1$.

Suppose now that $b, c \in S$ and $b \preceq^1 c$. Then $b \preceq c$ so by (C$''$), we have that $bu = vc$ for some $u, v \in S^1$, such that if $u \in S$, then $u \in \mathcal{S}(S)$ and $b \preceq u$. But if $u = 1$, then certainly $u \in \mathcal{S}(S^1)$ and we have $b \preceq^1 u$. We also have that for any $s \in S^1$, $s \preceq^1 1$, and $s1 = 1s$. Hence $\preceq^1$ satisfies Condition (C$'$).

$(iii) \Rightarrow (ii)$ This is clear.

$(iii) \Rightarrow (iv)$ From Theorem 2.9, S^1 is an order in a semigroup P such that $\leq_{\mathcal{H}^P}|_{(S^1} = \preceq^1$. From Lemma 4.4, we have that P is a monoid with identity 1; we claim that $P = Q^1$ for some proper subsemigroup Q of P. To see this, observe that if $1 = (a^\sharp b)(c^\sharp d)$ where $a, b, c, d \in S^1$, then we must have $1\,\mathcal{H}^P\, ac\,\mathcal{H}^P\, bd$ so that in S^1 we must have that $1 \preceq^1 ac$ and $1 \preceq^1 bd$. This tells us that $a = b = c = d = 1$. Consequently, $P = Q^1$ where $Q = P \setminus \{1\}$ is a semigroup containing S as a subsemigroup.

$(iv) \Rightarrow (iii)$ This is immediate from Theorem 2.9 or Theorem 5.15.

$(iv) \Rightarrow (i)$ If (iv) holds, then it is clear that S is a generalised order in Q, since if $q \in Q$, then $q = a^\sharp b$ where a, b cannot both be 1. If $b = 1$, then $q = (a^\sharp)^2 a$; otherwise, if $a = 1$, then $q = b \in S$. For $a, b \in S$ we have that

$$a \leq_{\mathcal{H}^Q} b \Leftrightarrow a = bq \text{ for some } q \in Q^1 \Leftrightarrow a \leq_{\mathcal{H}^{Q^1}} b \Leftrightarrow a \preceq^1 b \Leftrightarrow a \preceq b,$$

so that Q induces $\preceq$ on S.

$(i) \Rightarrow (iv)$ This is clear from the definitions.

□

Theorem 6.8. *Let S be a commutative semigroup and let ρ be a semilattice congruence on $\mathcal{S}(S)$ with associated pre-order $\leq$. Then S is a generalised order in a semigroup Q inducing $\leq$ if and only if Conditions (R) and (B$'$) hold.*

(B$'$) for all $b, c \in S$ and $a \in \mathcal{S}(S)$ with $b \overline{\leq} au, c \overline{\leq} av$ for some $u, v \in S^1$, if $ab = ac$, then $b = c$.

Proof. Let S be a generalised order in Q such that $\leq = \preceq|_{\mathcal{S}(S)}$ where $\preceq = \leq_{\mathcal{H}^Q}|_S$. By Lemma 5.2 we have that $\overline{\overline{\rho}}\,|_{\mathcal{S}(S)} = \rho$ and so by Proposition 6.3, we have that (R) holds.

Suppose now that $b, c \in S$ and $a \in \mathcal{S}(S)$ with $b\,\overline{\leq}\,au, c\,\overline{\leq}\,av$ for some $u, v \in S^1$ and $ab = ac$. Then as $\overline{\leq}\ \subseteq \preceq$, and $\preceq$ is a gq-pre-order, we have that $b \preceq au \preceq a$ and similarly, $c \preceq a$. As $\preceq$ satisfies (B), we deduce that $b = c$ so that (B$'$) holds.

Conversely, suppose that (R) and (B$'$) hold. By Proposition 6.3 we have that $\overline{\overline{\rho}}\,|_{\mathcal{S}(S)} = \rho$ and so Lemmas 6.2 and 6.4 give $\overline{\leq_A}\,|_{\mathcal{S}(S)} = \leq$. Moreover, by Lemma 6.5, $\overline{\leq_A}$ is a compatible pre-order satisfying Conditions (A) and (C$''$), and is such that if $b\,\overline{\leq_A}\,a \in \mathcal{S}(S)$, then $b \equiv_{\overline{\leq_A}} ab$. If $b \in S$ and $a \in \mathcal{S}(S)$ with $b\,\overline{\leq_A}\,a$, then again Lemma 6.4 $b\,\overline{\leq}\,au$ for some $u \in S^1$. Condition (B) for $\equiv_{\overline{\leq_A}}$ now follows from (B$'$). Thus $\overline{\leq_A}$ is a gq-pre-order, so by Proposition 6.7, S is a generalised order in Q inducing $\overline{\leq_A}$ on S and hence $\leq$ on $\mathcal{S}(S)$. □

The preceding theorem gives rise to the following:

Question Which commutative semigroups S have the property that every semilattice congruence on $\mathcal{S}(S)$ lifts to a congruence on S induced by a semigroup of quotients?

We are now able to present a series of corollaries that throw some light on the existence and structure of the set of quotients of a commutative order. First, we must extend Result 2.8 to generalised orders. If Q_1 and Q_2 are semigroups of generalised quotients of a commutative semigroup S, then as for quotient semigroups we write $Q_1 \geq Q_2$ if there is a homomorphism from Q_1 to Q_2 fixing the elements of S.

Proposition 6.9. *Let S be a commutative semigroup and a generalised order in semigroups Q_1 and Q_2. The following conditions are equivalent:*
(i) $Q_2 \leq Q_1$;
(ii) for all $a, b \in S$,

$$a \leq_{\mathcal{H}} b \text{ in } Q_1 \text{ implies that } a \leq_{\mathcal{H}} b \text{ in } Q_2;$$

(iii) for all $a, b \in S$,

$$a\,\mathcal{H}\,b \text{ in } Q_1 \text{ implies that } a\,\mathcal{H}\,b \text{ in } Q_2.$$

Proof. It is only necessary to prove (iii) implies (i). To do so, let us temporarily denote by T^* a semigroup T with an identity adjoined *whether or not T is a monoid*. Clearly S^* is an order in Q_i^* for $i = 1, 2$. If (iii) holds then certainly for all $a, b \in S^*$,

$$a\,\mathcal{H}\,b \text{ in } Q_1^* \text{ implies that } a\,\mathcal{H}\,b \text{ in } Q_2^*$$

so that by Result 2.8 we have that there exists a homomorphism $\theta : Q_1^* \to Q_2^*$ fixing the elements of S^*. Note that if $x \in Q_1$, then $x \in S$ or $x = u^\sharp v$ for some $u, v \in S$, so that in either case $x\theta \in Q_2$. Hence θ restricts to a homomorphism from Q_1 to Q_2 fixing the elements of S, as required. □

Corollary 6.10. *Let S be a commutative generalised order. Then S has a greatest generalised semigroup of quotients.*

Proof. Suppose that S is a generalised order: fix a semigroup Q of generalised quotients and let $\preceq$ be the induced pre-order on S. Let I index the set of semilattice congruences on $\mathcal{S}(S)$ induced by a semigroup of generalised quotients. For any $\rho_i, i \in I$, we have that (R) holds, so that if the associated pre-order on $\mathcal{S}(S)$ is denoted by $\leq_i$, then $a = bc$ $(a, b \in \mathcal{S}(S), c \in S)$ implies that $a \leq_i b$. Let $\rho = \bigcap_{i \in I} \rho_i$ and let $\leq$ be the associated pre-order. Clearly (R) holds for ρ. By Lemma 6.5, $\overline{\leq_A}$ satisfies (A) and (C$''$) and for $b \overline{\leq_A} a$ where $a \in \mathcal{S}(S)$, we have that $b \equiv_{\overline{\leq_A}} ab$. If $a, b \in \mathcal{S}(S)$ and $a \leq b$, then $ab \,\rho\, a$, so that in particular, $ab \,\tau\, a$ where τ is induced by $\preceq$; thus $a \preceq b$. Suppose now that $ab = ac$ where $b, c \in S, a \in \mathcal{S}(S)$ and $b, c \overline{\leq_A} a$. Since $\leq \subseteq \preceq$ and $\preceq$ satisfies (A), we must have that $\leq \cup A \subseteq \preceq$ and so $\overline{\leq_A} \subseteq \preceq$. It follows that $b = c$ by Condition (B) for $\preceq$. Thus $\overline{\leq_A}$ is a gq-pre-order and clearly is the smallest such. The result now follows from Proposition 6.9. $\square$

Corollary 6.11. *Let S be a commutative semigroup and let $\leq$ and ρ be a pre-order and its associated congruence on $\mathcal{S}(S)$ induced by a generalised semigroup of quotients. Then there is a greatest semigroup of generalised quotients Q^ρ inducing $\leq$ and ρ on $\mathcal{S}(S)$.*

Proof. Using Lemma 6.5 it is easy to check that $\overline{\leq_A}$ is the least gq-pre-order inducing $\leq$ and ρ on $\mathcal{S}(S)$. Denote the corresponding semigroup of generalised quotients by Q^ρ. $\square$

Our final result is now straightforward, using Propositions 6.7 and 6.9.

Corollary 6.12. *Let S be a commutative semigroup and let ρ_i for $i = 1, 2$ be semilattice congruences on $\mathcal{S}(S)$ induced by generalised semigroups of quotients Q_1 and Q_2. Then $\rho_1 \subseteq \rho_2$ if and only if $Q^{\rho_1} \geq Q^{\rho_2}$.*

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