Kesten's theorem for Invariant Random Subgroups

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January 18, 2012

Abstract

An invariant random subgroup of the countable group Γ is a random subgroup of Γ whose distribution is invariant under conjugation by all elements of Γ.

We prove that for a nonamenable invariant random subgroup H, the spectral radius of every finitely supported random walk on Γ is strictly less than the spectral radius of the corresponding random walk on Γ/H. This generalizes a result of Kesten who proved this for normal subgroups.

As a byproduct, we show that for a Cayley graph G of a linear group with no amenable normal subgroups, any sequence of finite quotients of G that spectrally approximates G converges to G in Benjamini-Schramm convergence. In particular, this implies that infinite sequences of finite d-regular Ramanujan Schreier graphs have essentially large girth.

1 Introduction

For a d-regular, countable, connected undirected graph G, we define the *spectral radius* of G, denoted $\rho(G)$, to be the norm of the Markov averaging operator on $\ell^2(G)$. This norm can also be expressed as

$$\rho(G) = \lim_{n\to\infty} \left(p_{x,x,2n}\right)^{1/2n} \qquad \text{(Rho)}$$

where $p_{x,y,k}$ denotes the probability that a simple random walk of length k starting at x ends at y.

Let Γ be a group generated by the symmetric set S and let $\mathrm{Cay}(\Gamma, S)$ denote the Cayley graph of Γ with respect to S. Let H be a subgroup of Γ and let $\mathrm{Sch}(\Gamma/H, S)$ be the Schreier graph of Γ defined on the coset space Γ/H. Since the map $g \mapsto Hg$ extends to a covering map from $\mathrm{Cay}(\Gamma, S)$ to $\mathrm{Sch}(\Gamma/H, S)$, using (Rho) we get

$$\rho(\mathrm{Sch}(\Gamma/H, S)) \geq \rho(\mathrm{Cay}(\Gamma, S)).$$

In his seminal papers [11] and [12], Kesten proved the following result.

Theorem 1 (Kesten) *Let* Γ *be a group generated by a finite symmetric set* S *and let* N *be a normal subgroup of* Γ*. Then the following are equivalent:*
1) $\rho(\mathrm{Cay}(\Gamma/N, S)) = \rho(\mathrm{Cay}(\Gamma, S))$*;*
2) N *is amenable.*

Two special cases of this theorem are particularly well known in the literature. First, applying the result to $N = \Gamma$ we get that Γ is amenable if and only if it has spectral radius 1. Second, applying it to the free group and noting that free groups have no nontrivial amenable normal subgroups, gives that if a d-regular infinite Cayley graph has the same spectral radius as the d-regular tree, then it is isomorphic to it. This has been later generalized to vertex transitive graphs as well (see e.g. [17]).

It is natural to ask whether Theorem 1 holds without the normality assumption, replacing $\mathrm{Cay}(\Gamma/N, S)$ with the Schreier graph $\mathrm{Sch}(\Gamma/H, S)$. It can be shown that if H is amenable, then $\rho(\mathrm{Sch}(\Gamma/H, S)) = \rho(\mathrm{Cay}(\Gamma, S))$. Already Kesten's original proof does not use normality at this point. However, the reverse implication (and hence Theorem 1) fails miserably for subgroups in general. For instance, when Γ is a free group on a large enough finite set S and H is the subgroup generated by the first 2 generators of Γ, then $\mathrm{Sch}(\Gamma/H, S)$ has the same spectral radius as $\mathrm{Cay}(\Gamma, S)$, although H is obviously not amenable.

Nevertheless, Theorem 1 does hold for a natural stochastic generalization of normal subgroups.

Definition 2 *Let* Γ *be a countable group. An* invariant random subgroup *(IRS) of* Γ *is a random subgroup of* Γ *whose distribution is invariant under conjugation by* Γ*.*

The main result of this paper is the following.

Theorem 3 *Let* Γ *be a group generated by a finite symmetric set* S *and let* H *be an invariant random subgroup of* Γ*. Then the following are equivalent:*
1) $\rho(\mathrm{Sch}(\Gamma/H, S)) = \rho(\mathrm{Cay}(\Gamma, S))$ *a.s.*
2) H *is amenable a.s.*

Applying Theorem 3 to the Dirac measure on a fixed normal subgroup of Γ, we get back Theorem 1.

For a finite, regular graph G we have $\rho(G) = 1$. Let $\rho_0(G)$ denote the norm of the Markov averaging operator acting on $\ell_0^2(G)$, the space of vectors with zero sum. Let Γ be a group generated by a finite symmetric set S and let $H \leq \Gamma$ be a subgroup of finite index. A priori, it may happen that $\rho_0(\mathrm{Sch}(\Gamma/H, S)) < \rho(\mathrm{Cay}(\Gamma, S))$, but the following analogue of the Alon-Boppana theorem [16] shows that asymptotically, $\mathrm{Sch}(\Gamma/H, S)$ can not beat $\mathrm{Cay}(\Gamma, S)$ spectrally.

Proposition 4 *Let* Γ *be an infinite group generated by the finite symmetric set* S *and let* $H_n \leq \Gamma$ *be a sequence of subgroups of finite index with* $|\Gamma : H_n| \to \infty$*. Then*

$$\liminf \rho_0(\mathrm{Sch}(\Gamma/H_n, S)) \geq \rho(\mathrm{Cay}(\Gamma, S)).$$

We say that the sequence H_n of subgroups of finite index *locally approximates* Γ, if $\mathrm{Sch}(\Gamma/H_n, S)$ converges to $\mathrm{Cay}(\Gamma, S)$ in Benjamini-Schramm convergence, that is, for every $R > 0$ we have

$$\lim_{n\to\infty} \frac{|\{v \in \mathrm{Sch}(\Gamma/H_n, S) \mid B_R(v) \cong B_R\}|}{|\Gamma : H_n|}$$

where $B_R(v)$ denotes the ball of radius R centered at v and B_R denotes the ball of radius R in $\mathrm{Cay}(\Gamma, S)$. This is equivalent to the following condition: for every $1 \neq g \in \Gamma$, we have

$$\lim_{n\to\infty} \frac{|\mathrm{Fix}(\Gamma/H_n, g)|}{|\Gamma : H_n|} = 0$$

where

$$\mathrm{Fix}(\Gamma/H, g) = \{Hx \mid Hxg = Hx\}$$

is the set of H-cosets fixed by g. This in particular shows that the notion of local approximation is independent of S.

It is easy to see that every normal chain with trivial intersection locally approximates its ambient group. Another family of examples is when Γ is an arithmetic group in zero characteristic and H_n is an arbitrary sequence of congruence subgroups in Γ. See [3] for a proof using Strong Approximation and [1] for explicit estimates on the essential injectivity radius.

Using [3], Theorem 3 now implies that for a finitely generated linear group Γ with no nontrivial amenable normal subgroups, every sequence that spectrally approximates Γ must locally approximate it.

Theorem 5 *Let Γ be a finitely generated linear group with no nontrivial amenable normal subgroups and let S be a finite symmetric generating set for Γ. Let $H_1, H_2, \ldots$ be a sequence of subgroups of finite index with $|\Gamma : H_n| \to \infty$ such that*

$$\lim_{n\to\infty} \sup \rho_0(\mathrm{Sch}(\Gamma/H_n, S)) \leq \rho(\mathrm{Cay}(\Gamma, S)).$$

Then H_n locally approximates Γ.

Note that the assumption of having no nontrivial amenable normal subgroups is not very restrictive, as for any finitely generated linear group Γ, the group $\Gamma/A(\Gamma)$ is also linear, where $A(\Gamma)$ denotes the maximal amenable normal subgroup of Γ.

We can apply Theorem 5 to the special case when Γ is a nonamenable free group. We call the finite d-regular graph G *Ramanujan*, if $\rho_0(G) \leq \rho(T_d)$ where T_d denotes the d-regular tree. Lubotzky, Philips and Sarnak [13], Margulis [14] and Morgenstein [15] have constructed sequences of d-regular Ramanujan graphs for $d = p^\alpha + 1$. Also, Friedman [8] showed that random d-regular graphs are close to being Ramanujan.

All the known Ramanujan graphs have *essentially large girth*, that is, for all L, we have

$$\lim_{n\to\infty} \frac{c_L(G_n)}{|G_n|} = 0$$

where $c_L(G_n)$ denotes the number of cycles of length L in G_n. Theorem 5 shows that this is not a coincidence.

Theorem 6 *Let Γ be a group generated by a finite symmetric set S and let $H_n \leq \Gamma$ be a sequence of subgroups of finite index with $|\Gamma : H_n| \to \infty$ such that the Schreier graphs $\mathrm{Sch}(\Gamma/H_n, S)$ are Ramanujan ($n \geq 1$). Then Γ is a free group freely generated by S and $\mathrm{Sch}(\Gamma/H_n, S)$ has essentially large girth.*

The result follows from the original Kesten theorem in the case when the H_n are normal in Γ, but for arbitrary subgroups, one needs Theorem 3.

Note that in the forthcoming paper [2], which focuses on graph theory and is not assuming the graphs to be Schreier graphs, we prove more: we give explicit estimates between the spectral radius and the density of short cycles. In particular, we show that the essential girth of a Ramanujan graph G is at least $\log\log|G|$.

Invariant Random Subgroups: history and background. Although we are coining the name IRS in this paper, they have been around in various forms.

A random sample of an IRS H is not just an arbitrary subgroup of Γ. This can be visualized by looking at the Schreier graph $\mathrm{Sch}(\Gamma/H, S)$. This graph exhibits a 'statistical homogeneity' in the sense that every event that occurs in it locally will appear in it again with some measurable frequency. This phenomenon is pointed out by Aldous and Lyons in their starting paper on unimodular random networks [4]. It turns out that for a random subgroup H of Γ, $\mathrm{Sch}(\Gamma/H, S)$ forms a unimodular random network if and only if H is an IRS. See Section 3 for details.

Another source of invariant random subgroups is probability measure preserving actions: the stabilizer of a random point of the underlying space always forms an IRS, and vice versa, every IRS arises this way (see Proposition 14). A result of Stück and Zimmer [18] says that in a higher rank simple real Lie group G, every ergodic IRS equals a Haar-random conjugate of a lattice in G. This generalizes the Margulis normal subgroup theorem [14] and is the deepest result on invariant random subgroups so far. Although the result itself is not about countable groups, it implies that in a *lattice* Γ in a Lie group as above, like $SL_3(\mathbb{Z})$, every ergodic IRS is either trivial or has finite index in Γ. Since these lattices satisfy the congruence subgroup property, this means that we have a complete control on their invariant random subgroups.

The stabilizer of a random point of a probability measure preserving action has also been studied by Bergeron and Gaboriau [6], who pointed out that these groups tend to behave similarly to normal subgroups. They present this in [6, Theorem 5.4] where they prove that if Γ has positive first L^2 Betti number, then any IRS in Γ which is nontrivial and of infinite index has infinite first L^2 Betti number. Another recent result, using the language of measure preserving actions is due to Vershik [19], who classified invariant random subgroups of the finitary symmetric group of countable rank.

Exploiting the Stück-Zimmer result, it is proved in [1] that for a higher rank simple real Lie group with symmetric space X, every sequence of symmetric

X-manifolds with volume tending to infinity must converge to X in Benjamini-Schramm convergence. With some additional assumptions, this implies convergence of normalized Betti numbers and multiplicities of certain unitary representations. Also, in [3] it is shown that in a linear group Γ, every amenable IRS of Γ lies in the amenable radical, the maximal amenable normal subgroup of Γ. Note that we do not know any counterexamples to this phenomenon, even when we do not assume linearity. This result is used in Theorem 5 but not in Theorem 6, as it is easy to see that non-Abelian free groups do not possess nontrivial amenable invariant random subgroups.

2 Preliminaries

In this section we define the notions and state some basic results used in the paper.

A subset S of the group Γ is called *symmetric*, if for all $s \in S$ we have $s^{-1} \in S$. Let Γ be a group generated by the finite symmetric subset S and let H be a subgroup of Γ. We define the *Schreier graph* $\mathrm{Sch}(\Gamma/H, S)$ as follows: the vertex set is the right coset space $\{Hg \mid g \in \Gamma\}$ and for each $s \in S$ and vertex Hg, there is an s-labeled edge going from Hg to Hgs. This defines a directed graph where the s-labeled edges are inverses of the s^{-1}-labeled edges. The *root* of $\mathrm{Sch}(\Gamma/H, S)$ is defined as the trivial coset H.

When H is the trivial subgroup of Γ, the Schreier graph is particularly nice, and we call it the *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ of Γ with respect to S. Cayley graphs are vertex transitive and carry geometric information on the group Γ. The map $g \longmapsto Hg$ extends to a covering map from $\mathrm{Cay}(\Gamma, S)$ to $\mathrm{Sch}(\Gamma/H, S)$ and so $\mathrm{Cay}(\Gamma, S)$ can also be thought of as the 'universal cover relative to Γ' of $\mathrm{Sch}(\Gamma/H, S)$ as H varies over all subgroups of Γ.

Let $G = \mathrm{Sch}(\Gamma/H, S)$. Let $\ell^2(G)$ be the Hilbert space of all square summable functions on the vertex set of G. Let us define the Markov operator $M : \ell^2 \to \ell^2$ as follows:

$$(Mf)(x) = \frac{1}{d}\sum_{s \in S} f(xs)$$

We define the *spectral radius* of G, denoted $\rho(G)$, to be the norm of M.

For a graph G and $x, y \in V(G)$ let $P_{x,n}$ denote the set of walks of length n starting at x and let $P_{x,y,n}$ denote the elements in $P_{x,n}$ that end at y. A *random walk of length n starting at x* is a uniform random element of $P_{x,n}$. Let the *probability of return*

$$p_{x,n} = \frac{|P_{x,x,n}|}{|P_{x,n}|}$$

denote the probability that a random walk of length n starting at x ends at x. Now we prove two general lemmas on vertex transitive graphs that will be used in the proof of Theorem 3.

Lemma 7 *Let G be a d-regular vertex transitive graph and let $n \geq 2$ be a positive even integer. Then*

$$|P_{x,y,n}| \leq |P_{x,x,n}| \leq d^2 \, |P_{x,x,n-2}|$$

Proof. Let $A = dM$ where M is the Markov operator on $l^2(G)$ and let δ_x be the characteristic function of x. Then using the Cauchy-Schwarz inequality we get

$$|P_{x,y,n}| = \left\langle \delta_x A^{n/2}, \delta_y A^{n/2} \right\rangle \leq \left\langle \delta_x A^{n/2}, \delta_x A^{n/2} \right\rangle = |P_{x,x,n}|$$

since $\delta_x A^n$ is a translate of $\delta_y A^n$. For the second inequality, we have

$$P_{x,x,n} = \bigcup_{y,z \text{ are neighbours of } x} P_{y,z,n-2}$$

which, using the first inequality and vertex transitivity yields the second one. $\square$

Lemma 8 *Let G be a d-regular vertex transitive graph, let $x \in V(G)$ and let a be a walk of length l starting at x. Let w be a uniform random element of $P_{x,x,n}$. Then the probability*

$$\mathbf{P}\left((w_1, \ldots, w_l) = a\right) \geq \frac{1}{d^{2l}}.$$

Proof. The number of walks $w \in P_{x,x,n}$ of length n such that $(w_1, \ldots, w_l) = a$ and also $(w_{l+1}, \ldots, w_{2l}) = a^{-1}$ is exactly $|P_{x,x,n-2l}|$, which, using Lemma 7, is at least

$$|P_{x,x,n-2l}| \geq \frac{1}{d^{2l}} \, |P_{x,x,n}|$$

which proves the lemma. $\square$

Let Γ be a group and let g be a random variable with values in Γ. Then g defines the Markov operator $M(g)$ on $l^2(\Gamma)$ as follows. For $f \in l^2(\Gamma)$ let

$$fM(g) = \sum_{h \in \Gamma} \boldsymbol{P}(g = h) xh$$

where fh is the right translate of f by h. If g is symmetric, that is, for all $h \in \Gamma$ we have $P(g = h) = P(g = h^{-1})$, then $M(g)$ is self-adjoint.

Lemma 9 *Let $g_1, g_2, \ldots, g_n$ be independent random elements of the group Γ and let $g = g_1 g_2 \cdots g_n$. Let e denote the identity element of Γ. Then*

$$\boldsymbol{P}(g = e) \leq \prod_{i=1}^{n} \|M(g_i)\| .$$

Proof. If g_1, g_2 are independent, then $M(g_1g_2) = M(g_1)M(g_2)$. This implies that

$$M(g) = \prod_{i=1}^{n} M(g_i).$$

Let x_e be the characteristic function of $l^2(\Gamma)$. Then, using the submultiplicativity of norm, we get

$$\boldsymbol{P}(g = e) = \langle x_e M(g), x_e \rangle \leq \|x_e M(g)\| \, \|x_e\| \leq \|M(g)\| \leq \prod_{i=1}^{n} \|M(g_i)\|$$

as claimed. □

Lemma 10 *Let Γ be a group, let $S \subseteq \Gamma$ be a finite symmetric subset and let $K = \langle S \rangle$ be the subgroup of Γ generated by S. Let g be a uniform random element of S and let M be the corresponding Markov operator on $l^2(\Gamma)$. Then the norm*

$$\|M(g)\| = \rho(\mathrm{Cay}(K, S)).$$

Proof. The space $l^2(\Gamma)$ is the orthogonal sum of countably many isomorphic copies of $l^2(K)$ (corresponding to the cosets of K in Γ) and M acts diagonally on this space. Hence, the norm equals the norm of M acting on $l^2(K)$, which by definition is the spectral radius of the corresponding Cayley graph. □

A countable group Γ is *amenable*, if there is a Γ-invariant finitely additive probability measure (called a mean) on all subsets of Γ. By Kesten's original theorem (Theorem 1), a group Γ is amenable, if and only if for all symmetric probability distributions on Γ, the corresponding Markov operator has norm 1. So a finitely generated group Γ is amenable if and only if $\rho(\mathrm{Cay}(\Gamma, S)) = 1$ for some (and thus, for every) finite symmetric generating set S of Γ.

Amenability is preserved under various group operations, like taking an ascending union, subgroups or quotient groups or extensions [10]. Finite and Abelian (or in general, solvable) groups are amenable and hence the amenability of Γ is preserved under taking a subgroup of finite index.

3 Invariant Random Subgroups and Probability Measure Preserving Actions

In this section we introduce invariant random subgroups. For a countable group Γ let

$$\mathrm{Sub}_\Gamma = \{H \subseteq \Gamma \mid H \text{ is a subgroup of } \Gamma\}$$

be the set of subgroups of Γ, endowed with the product topology inherited from the space of subsets of Γ. That is, a sequence of subgroups H_n converges, if for all $g \in \Gamma$, the event $g \in H_n$ stabilizes as n tends to infinity. This is called the

Chabauty topology [7]. Since Sub_Γ is closed in the space of subsets, Sub_Γ is also compact. The group Γ acts continuously on Sub_Γ by conjugation.

Assume now that Γ is generated by the finite symmetric set S. Let

$$\mathrm{SC}_\Gamma(S) = \{\mathrm{Sch}(\Gamma/H, S) \mid H \leq \Gamma\}$$

be the set of all connected Schreier graphs of Γ with respect to S. For $G_1, G_2 \in \mathrm{SC}_\Gamma(S)$ let $d(G_1, G_2) = 1/k$ where k is the maximal natural number such that the k-balls around the root of G_1 and G_2 are isomorphic as rooted, S-labeled graphs. Then d is a metric on $\mathrm{SC}_\Gamma(S)$ and the topology defined by d is compact. The group Γ acts on $\mathrm{SC}_\Gamma(S)$ as follows: for $s \in S$, $\mathrm{Sch}(\Gamma/H, S)s$ is isomorphic to $\mathrm{Sch}(\Gamma/H, S)$ but we move the root along the s-labeled edge. This extends to an action of Γ on $\mathrm{SC}_\Gamma(S)$ by continuous maps. This can also be expressed as

$$\mathrm{Sch}(\Gamma/H, S)g = \mathrm{Sch}(\Gamma/H^g, S)$$

for $g \in \Gamma$. One can check that the map

$$H \longmapsto \mathrm{Sch}(\Gamma/H, S) \tag{S}$$

is a Γ-equivariant homeomorphism between Sub_Γ and $\mathrm{SC}_\Gamma(S)$ that commutes with the Γ-action. So, the spaces $\mathrm{SC}_\Gamma(S)$ and Sub_Γ are isomorphic as Γ-spaces. Note that the inverse of the map (S) can be described as follows: for $G \in \mathrm{SC}_\Gamma(S)$ let H be the set of S-evaluations of returning walks starting at the root in G.

Definition 11 *An* invariant random subgroup *(IRS) of* Γ *is a random subgroup of* Γ *whose distribution is a* Γ*-invariant Borel probability measure on* Sub_Γ*.*

For finitely generated groups, using the above bijection between Sub_Γ and $\mathrm{SC}_\Gamma(S)$, every IRS gives rise to a unimodular random Schreier graph of Γ, that is, a Borel probability distribution on $\mathrm{SC}_\Gamma(S)$ that is invariant under moving the root.

A natural way to obtain an invariant random subgroup is to take the stabilizer of a random point in a measure preserving action of Γ on a Borel probability space.

Proposition 12 *Let* Γ *act on the Borel probability space* (X, μ) *by measure preserving maps. Let* H *be the stabilizer of a* μ*-random point in* X*. Then* H *is an IRS.*

Proof. Let $g \in \Gamma$ be fixed. Then

$$H^g = Stab_\Gamma(xg)$$

where x is μ-random in X. But g preserves μ, so xg is also uniform μ-random in X. So the distribution of H and H^g are equal. □

Our next proposition says that every IRS actually arises this way.

Proposition 13 *Let H be an invariant random subgroup of the finitely generated group Γ. Then there exists a measure preserving action of Γ on a Borel probability space (X, μ) such that H is the stabilizer of a μ-random point of X in Γ.*

Proof. Fix S to be some finite generating set for Γ. Let

$$X = \{(G, f) \mid G \in \mathrm{SC}_\Gamma(S), f : V(G) \to [0, 1]\}$$

the set of Schreier graphs of Γ vertex labeled by elements of the unit interval. We can endow X with the product topology. The group Γ naturally acts on X by moving the root.

Now take our random H and let $G = \mathrm{Sch}(\Gamma/H, S)$ be the Schreier graph of our random H. Label the vertices of this random G using an i.i.d. with uniform random values in the unit interval $[0, 1]$ (according to the Lebesque measure). This will give a probability measure μ on X that will be invariant under the Γ-action.

Let B be a μ-random element of X. Then almost surely, all the labels of the vertices of B are different. An element $g \in \Gamma$ stabilizes B if and only if when moving the root along a word in S representing g gives an isomorphic rooted, edge and vertex labeled graph. This implies that g fixes the root, that is, $g \in H$. On the other hand, H trivially stabilizes G. □

Note that for non-finitely generated groups, the same proof works: we can take the random $[0, 1]$-labeling simply on the cosets of H. We chose to write out the Schreier graph proof because it is more visual.

Summarizing the above gives us the following.

Proposition 14 *Let H be a random subgroup of the group Γ, generated by the finite symmetric set S. Then the following are equivalent:*
1) H is an invariant random subgroup;
2) $\mathrm{Sch}(\Gamma/H, S)$ is a unimodular random network;
3) H is the stabilizer of a random point for some measure preserving action of Γ.

The natural convergence notion for the space of random subgroups is weak convergence of measures. It turns out that this corresponds to Benjamini-Schramm convergence [5] on the level of Schreier graphs. For a finite graph G let $\widetilde{G}$ denote the random rooted graph that we get by assigning the root of G uniformly randomly.

Definition 15 *Let $G_n \in \mathrm{SC}_\Gamma(S)$ be a sequence of Schreier graphs and let G be a random graph in $\mathrm{SC}_\Gamma(S)$. We say that G_n* Benjamini-Schramm converges to *G, if $\widetilde{G_n}$ weakly converges to G.*

For a random rooted graph G, a finite rooted graph α and $R > 0$ let $P(G, R, \alpha)$ denote the probability that the R-ball around the root of G is isomorphic to α. Since the topology on $\mathrm{SC}_\Gamma(S)$ is generated by the closed-open sets

$$U(R,\alpha) = \{G \in \mathrm{SC}_\Gamma(S) \mid \text{the } R\text{-ball of } G \text{ equals } a\}$$

we get that G_n Benjamini-Schramm converges to G, if and only if

$$\lim_{n\to\infty} P(G_n, R, \alpha) = P(G, R, \alpha)$$

for all R and α.

For a group Γ, a subgroup $H \leq \Gamma$ and $g \in \Gamma$ let

$$\mathrm{Fix}(\Gamma/H, g) = \{Hx \mid Hxg = Hx\}$$

denote the set of H-cosets fixed by g.

Lemma 16 *Let Γ be a group generated by a finite symmetric set S and let $H_n \leq \Gamma$ be a sequence of subgroups of finite index with $|\Gamma : H_n| \to \infty$. Then the following are equivalent:*
1) $\mathrm{Sch}(\Gamma/H_n, S)$ *converges to* $\mathrm{Cay}(\Gamma, S)$ *in Benjamini-Schramm convergence;*
2) H_n locally approximates Γ;
3) for every $1 \neq g \in \Gamma$, we have

$$\lim_{n\to\infty} \frac{|\mathrm{Fix}(\Gamma/H_n, g)|}{|\Gamma : H_n|} = 0.$$

Proof. Let $G_n = \mathrm{Sch}(\Gamma/H_n, S)$ $(n \geq 1)$ and let α_R be the R-ball in $\mathrm{Cay}(\Gamma, S)$. Then

$$P(G_n, R, \alpha_R) = \frac{|\{v \in \mathrm{Sch}(\Gamma/H_n, S) \mid B_R(v) \cong \alpha_R\}|}{|\Gamma : H_n|}$$

where $B_R(v)$ denotes the ball of radius R centered at v. This gives 1) $\iff$ 2).

Let $g \in \Gamma$ with $g \neq 1$ and let $g = w(S)$ be a word of length R representing g in S. Then for every $v \in \mathrm{Fix}(\Gamma/H_n, g)$, the path starting at v along the word w is returning, but w does not return in $\mathrm{Cay}(\Gamma, S)$, so $B_R(v)$ is not isomorphic to α_R. Hence

$$\frac{|\mathrm{Fix}(\Gamma/H_n, g)|}{|\Gamma : H_n|} \leq 1 - P(G_n, R, \alpha_R).$$

This yields 1) $\Longrightarrow$ 3).

Vice versa, for a given R and $v \in \mathrm{Sch}(\Gamma/H_n, S)$, if $B_R(v)$ is not isomorphic to α_R, then there exists a word w of length at most $2R$ such that w starting at v is returning but $w(S) \neq 1$. This gives

$$P(G_n, R, \alpha_R) \geq 1 - \sum_{|w| \leq 2R,\ w(S) \neq 1} \frac{|\mathrm{Fix}(\Gamma/H_n, w(S))|}{|\Gamma : H_n|}$$

which yields 3) $\Longrightarrow$ 1). $\square$

4 Amenable stabilizers and spectral radius

In this section we prove Theorem 3. Note that in the proof, we are using Kesten's original theorem saying that nonamenable groups have spectral radius less than 1, but not Theorem 1. Hence we provide an alternate proof for Theorem 1.

Now we start working towards Theorem 3. The next lemma proves the easy implication. It is essentially the same as Corollary 2 in Kesten's original paper [11]; since it is only stated there for normal subgroups, we include a quick proof.

Lemma 17 *Let Γ be a group generated by a finite symmetric set S and let H be an amenable subgroup of Γ. Then*

$$\rho(\mathrm{Cay}(\Gamma, S)) = \rho(\mathrm{Sch}(\Gamma/H, S)).$$

Proof. Let R_n denote the endpoint of the standard random S-walk on Γ starting at the identity e. Then $\mathbf{P}\left(R_n \in H\right)$ equals the probability of return for the standard random S-walk on Γ/H starting at the coset H. Thus, for $\varepsilon > 0$ there exists an integer n_0 such that for all $n \geq n_0$ we have

$$q_{2n} = \mathbf{P}\left(R_{2n} \in H\right) \geq ((1-\varepsilon)\rho)^{2n}$$

where $\rho = \rho(\mathrm{Sch}(\Gamma/H, S))$. Now let $n > n_0$ to be chosen later and for $h \in H$ let

$$Q(h) = \frac{\mathbf{P}\left(R_{2n} = h\right)}{q_{2n}}$$

Then Q is a finite symmetric probability distribution on H and since H is amenable, using the original Kesten's theorem, the random walk R' on H with respect to Q has spectral radius 1. In particular, there exists an integer m such that

$$\mathbf{P}\left(R'_{2m} = e\right) \geq (1-\varepsilon)^{2m}.$$

Now let us consider the probability, that the walk R of length $4nm$ returns to H in every $2n$-th step and the product of the segments is e. This gives

$$\mathbf{P}\left(R_{4nm} = e\right) \geq q_{2n}^{2m} \mathbf{P}\left(R'_{2m} = e\right) \geq \rho^{4nm}(1-\varepsilon)^{2m(2n+1)}$$

Taking $4nm$-t roots, and using that n is arbitrarily large, we get

$$\rho(\mathrm{Cay}(\Gamma, S)) \geq (1-\varepsilon)\rho$$

Hence, the lemma holds. □

Let Γ be a group and let $S \subseteq \Gamma$ be a finite symmetric multiset. Let H be a subgroup of Γ. Let S^n denote the set of n-tuples in S and for $a \in S^n$ let $[a] = a_1 a_2 \cdots a_n \in \Gamma$ be the product of elements in the tuple. Let

$$A_H(S, n) = \{a \in S^n \mid [a] \in H\}.$$

Let $A(S, n) = A_1(S, n)$. The set $A(S, n)$ can be identified with returning walks of length n in $\mathrm{Cay}(\Gamma, S)$.

Lemma 18 *Let $(a_0, \ldots, a_{n-1})$ be a uniform random element of $A(S,n)$. Then for any $1 \leq k$ and $0 \leq t \leq n-1$, the distribution of the segment $(a_t, \ldots, a_{t+k-1}) \in S^k$, where the indices are understood modulo n, is independent of l.*

Proof. The set $A(S,n)$ is invariant under cyclic rotations. □

Lemma 19 *Let $k > 0$ be an integer and let $(a_0, \ldots, a_{n-1})$ be a uniform random element of $A(S,n)$ with $n > 2k$. Then for any $b \in S^k$, the probability*

$$\mathbf{P}\left((a_0, a_1, \ldots, a_{k-1}) = b\right) \geq |S|^{-2k}.$$

Proof. Let $b = (b_0, b_1, \ldots, b_{k-1})$. Then for any $(c_0, \ldots, c_{n-2k-1}) \in A(S, n - 2k)$, we have

$$(b_0, \ldots, b_{k-1}, b_{k-1}^{-1}, \ldots, b_0^{-1}, c_0, c_1, \ldots, c_{n-2k-1}) \in A(S,n)$$

which gives

$$\mathbf{P}\left((a_0, a_1, \ldots, a_{k-1}) = b\right) \geq \frac{|A(S, n-2k)|}{|A(S,n)|} = \frac{p_{n-2k}}{p_n} |S|^{-2k} \geq |S|^{-2k}$$

where p_n denotes the probability of return on $\mathrm{Cay}(\Gamma, S)$ in n steps. In the last inequality we are using Lemma 7. □

For $a \in S^n$ we shall look at the event

$$Ha_0 \cdots a_{t-1} = Ha_0 \cdots a_t$$

as right cosets of H. Equivalently, the walk corresponding to a in $\mathrm{Sch}(\Gamma/H, S)$ is at the same coset at time $t-1$ and t. Yet another way of writing this is that $a_t \in H^{a_0 \cdots a_{t-1}}$.

For $a \in S^n$ let the index set

$$I(S,H,a) = \left\{ t \in \{0, \ldots, n-1\} \mid a_t \in H^{a_0 \cdots a_{t-1}} \right\}.$$

Definition 20 *We call $a, b \in S^n$ H-*equivalent *if*

$$Ha_0 \cdots a_t = Hb_0 \cdots b_t \ (0 \leq t \leq n-1) \text{ and } a_t = b_t \ (t \notin I(S,H,a))$$

Let $C(a)$ denote the H-equivalence class of a.

Note that H-equivalent sequences have the same index set.

Lemma 21 *Let Γ be a group and let $S \subseteq \Gamma$ be a finite symmetric multiset. Let H be a subgroup of Γ and let $n > 0$ be an integer. Then*

$$|A_H(S,n)| \geq \sum_{a \in A(S,n)} [p(S,H,a)]^{-1}$$

where

$$p(S,H,a) = \prod_{t \in I(S,H,a)} \rho(\mathrm{Cay}(H^{a_0 \cdots a_{t-1}}, S \cap H^{a_0 \cdots a_{t-1}})).$$

Proof. For $0 \le t \le n-1$ let

$$X_t = S \cap H^{a_0 \cdots a_{t-1}}.$$

Then X_t is a symmetric multiset. The definition of H-equivalence implies that $C(a) \subseteq A_H(S,n)$ for all $a \in A_H(S,n)$ and thus, $A_H(S,n)$ is the disjoint union of its H-equivalence classes. A uniform random element x of $C(a)$ is of the form $(x_1, x_2, \ldots, x_n)$ where $x_t = a_t$ is fixed for $t \notin I(S,H,a)$ and x_t is a uniform random element of X_t for $t \in I(S,H,a)$. Hence, using Lemma 9 and Lemma 10, for a uniform random element x of $C(a)$ we have

$$\boldsymbol{P}\left(x \in A(S,n)\right) \le \prod_{t \in I(S,H,a)} \rho(\mathrm{Cay}(H^{a_0 \cdots a_{t-1}}, S \cap H^{a_0 \cdots a_{t-1}})) = p(S,H,a).$$

We can estimate the size of $A_H(S,n)$ by the sum of the sizes of equivalence classes of tuples in $A(S,n)$. We count $C(a)$ exactly $|C(a)|\, \boldsymbol{P}\left(x \in A(S,n)\right)$ times, which gives

$$|A_H(S,n)| \ge \sum_{a \in A(S,n)} \left[\boldsymbol{P}(x \in A(S,n))\right]^{-1} \ge \sum_{a \in A(S,n)} \left[p(S,H,a)\right]^{-1}.$$

The lemma holds. □

Let us denote

$$[S^k] = \left\{a_0 \cdots a_{k-1} \mid a \in S^k\right\}$$

where we look at $[S^k]$ as a multiset in Γ. Let $\{S^k\}$ denote the set of elements in $[S^k]$.

Lemma 22 *Let Γ be a group generated by the finite symmetric set S and let H be an invariant random subgroup of Γ that is nonamenable with positive probability. Then there exists an integer $k > 0$ and $p, r > 0$ such that*

$$\rho(\mathrm{Cay}(H, [S^k] \cap H)) < 1 - r$$

with probability at least p.

Proof. By adding ss^{-1} to words, we see that for $k \ge 0$ we have $\{S^k\} \subseteq \{S^{k+2}\}$. Since S generates Γ, the subgroup

$$N = \bigcup_{k=0}^{\infty} \{S^{2k}\}$$

has index at most 2 in Γ. Let H_{2k} be the subgroup generated by $\{S^{2k}\} \cap H$. The group H_{2k} is a well-defined subgroup of the random subgroup H of Γ, and the union $\bigcup H_{2k}$ has index at most 2 in H. Hence, there exists k such that H_{2k} is non-amenable with positive probability, using that the ascending union of amenable groups is amenable and that amenability does not change when passing to a finite index subgroup. By Kesten's theorem on amenability [11], H_{2k} is nonamenable if and only if $\rho(\mathrm{Cay}(H_{2k}, [S^{2k}] \cap H)) < 1$. Since $\rho(\mathrm{Cay}(H, [S^{2k}] \cap H))$ is a measurable function of H, there exists $p, r > 0$ such that $\rho(\mathrm{Cay}(H_{2k}, [S^{2k}] \cap H)) < 1 - r$ with probability at least p. □

We are ready to prove the main theorem of this section after a trivial lemma.

Lemma 23 *Let X be a random variable with $0 \leq X \leq R$. Then*

$$\boldsymbol{P}\left(X \geq \frac{\boldsymbol{E}[X]}{2}\right) \geq \frac{\boldsymbol{E}[X]/R}{2 - \boldsymbol{E}[X]/R}.$$

Proof of Theorem 3. If H is amenable a.s., then by Lemma 17, we have

$$\rho(\mathrm{Sch}(\Gamma/H, S)) = \rho(\mathrm{Cay}(\Gamma, S))$$

a.s. This part of the theorem does not require invariance or random subgroups.

Assume H is nonamenable with positive probability. Then by Lemma 22 there exists an integer $k > 0$ and $p, r > 0$ such that

$$\rho(\mathrm{Cay}(H, [S^k] \cap H)) < 1 - r$$

with probability at least p.

Let $T = [S^k]$. Fix a positive integer n. Let

$$J(T, H, a) = \left\{t \in I(T, H, a) \mid \rho(\mathrm{Cay}(H^{a_0 \cdots a_{t-1}}, T \cap H^{a_0 \cdots a_{t-1}})) < 1 - r\right\}.$$

Then we have

$$p(T, H, a) = \prod_{t \in I(a)} \rho(\mathrm{Cay}(H^{a_0 \cdots a_{t-1}}, T \cap H^{a_0 \cdots a_{t-1}})) < (1 - r)^{|J(T,H,a)|}$$

and hence, by Lemma 21, for any $H \leq \Gamma$ we have

$$|A_H(T, n)| \geq \sum_{a \in A(T,n)} p^{-1}(T, H, a) > \sum_{a \in A(T,n)} (1 - r)^{-|J(T,H,a)|}. \tag{A}$$

For any fixed element $a \in A(T, n)$ and $0 \leq t \leq n - 1$, over the random subgroup H, the probability

$$\begin{aligned}\boldsymbol{P}\left(t \in J(T, H, a)\right) = &\\ &= \boldsymbol{P}\left(a_t \in H^{a_0 \cdots a_{t-1}},\ \rho(\mathrm{Cay}(H^{a_0 \cdots a_{t-1}}, T \cap H^{a_0 \cdots a_{t-1}})) < 1 - r\right) = \\ &= \boldsymbol{P}\left(a_t \in H \text{ and } \rho(\mathrm{Cay}(H, T \cap H)) < 1 - r\right)\end{aligned}$$

by the invariance of H. Thus, the expected value of $|J(T, H, a)|$ over the random subgroup H equals

$$\boldsymbol{E}\left[|J(T, H, a)|\right] = \sum_{t=0}^{n-1} \boldsymbol{P}\left(a_t \in H \text{ and } \rho(\mathrm{Cay}(H, T \cap H)) < 1 - r\right). \tag{B}$$

For any fixed subgroup H such that $\rho(\mathrm{Cay}(H, T \cap H)) < 1 - r$ we have $T \cap H \neq \varnothing$. In particular, for these H, using Lemma 18 and Lemma 19, we have

$$\frac{1}{|A(T,n)|} \sum_{a \in A(T,n)} \mathbf{1}(a_t \in H) \geq |T|^{-2}$$

for all $0 \leq t \leq n-1$. Since $\boldsymbol{P}(\rho(\mathrm{Cay}(H, T \cap H)) < 1 - r) > p$, we get

$$\frac{1}{|A(T,n)|} \sum_{a \in A(T,n)} \boldsymbol{P}(a_t \in H \text{ and } \rho(\mathrm{Cay}(H, T \cap H)) < 1 - r) \geq p\,|T|^{-2}$$

and so by summing with respect to t and using (B) we yield

$$\boldsymbol{E}\left[\frac{1}{|A(T,n)|} \sum_{a \in A(T,n)} |J(T,H,a)|\right] \geq p\,|T|^{-2}\,n.$$

Using

$$\frac{1}{|A(T,n)|} \sum_{a \in A(T,n)} |J(T,H,a)| \leq n$$

and Lemma 23 we get

$$\boldsymbol{P}\left(\frac{1}{|A(T,n)|} \sum_{a \in A(T,n)} |J(T,H,a)| \geq \frac{1}{2} p\,|T|^{-2}\,n\right) \geq \frac{p\,|T|^{-2}}{2 - p\,|T|^{-2}}. \qquad \text{(C)}$$

Let H be a subgroup satisfying (C). Using the inequality of arithmetic and geometric means and (C) we get

$$\frac{1}{|A(T,n)|} \sum_{a \in A(T,n)} (1-r)^{-|J(T,H,a)|} \geq (1-r)^{-\frac{1}{|A(T,n)|} \sum_{a \in A(T,n)} |J(T,H,a)|} \geq (1-r)^{-\frac{1}{2}p|T|^{-2}n}$$

which by (A) gives

$$|A_H(T,n)| > \sum_{a \in A(T,n)} (1-r)^{-|J(T,H,a)|} \geq |A(T,n)|\,(1-r)^{-\frac{1}{2}p|T|^{-2}n}.$$

Using $|A_H(T,n)| = |A_H(S,kn)|$, $|A(T,n)| = |A(S,kn)|$, dividing by d^{kn} and taking the nk-th roots of both sides, we get the following.

For every integer $n > 1$, with probability at least $p\,|T|^{-2}/(2 - p\,|T|^{-2})$, our random H satisfies

$$(p_{nk}(\mathrm{Sch}(\Gamma/H, S)))^{1/nk} > (p_{nk}(\mathrm{Cay}(\Gamma, S)))^{1/nk}\,(1-r)^{-\frac{1}{2k}p|T|^{-2}}$$

Since

$$\lim_{n\to\infty} (p_{2nk}(\mathrm{Cay}(\Gamma, S)))^{1/2nk} = \rho(\mathrm{Cay}(\Gamma, S)$$

we get that there exists n for which

$$\rho(\mathrm{Sch}(\Gamma/H, S)) \geq (p_{nk}(\mathrm{Sch}(\Gamma/H, S)))^{1/nk} > \rho(\mathrm{Cay}(\Gamma, S)$$

with probability at least $p\,|T|^{-2}/(2 - p\,|T|^{-2})$. The theorem holds. □

5 Local and spectral approximation for linear groups

In this section we prove Proposition 4, Theorem 5 and Theorem 6. We start with an easy lemma on free groups.

Lemma 24 *Let H be an amenable invariant random subgroup of the countable non-Abelian free group F. Then $H = 1$ a.s.*

Proof. By the Nielsen-Schreier theorem, all subgroups of F are free. Since non-Abelian free groups are non-amenable, we get that H is cylic a.s. The group F has only countably many cyclic subgroups, and for every infinite cyclic subgroup C of F, the conjugacy class of C in F is infinite. Hence, every invariant measure supported on cyclic subgroups is supported on the trivial subgroup. □

The following lemma is a variant on the Alon-Boppana theorem [16].

Lemma 25 *Let Γ be an infinite group generated by the finite symmetric set S and let $H_n \leq \Gamma$ be a sequence of subgroups of finite index with $|\Gamma : H_n| \to \infty$ such that $H_n \to H$ in Chabauty topology. Then*

$$\liminf \rho_0(\mathrm{Sch}(\Gamma/H_n, S)) \geq \rho(\mathrm{Sch}(\Gamma/H, S)).$$

Proof. By Chabauty convergence, $\mathrm{Sch}(\Gamma/H_n, S)$ converges to $\mathrm{Sch}(\Gamma/H, S)$ in rooted convergence (see Section 3). Since $|\mathrm{Sch}(\Gamma/H_n, S)| \to \infty$, we get that H has infinite index in Γ. By the definition of the spectral radius as a norm, for any $\varepsilon > 0$ there exists a finitely supported function $f \in l^2(\Gamma/H)$ with zero sum such that $\langle f, f\rangle = 1$ and

$$\langle fM, f\rangle > \rho(\mathrm{Sch}(\Gamma/H, S)) - \varepsilon.$$

Here M denotes the Markov operator on $l^2(\Gamma/H)$.

Let $R > 0$ such that the ball B of radius R in $\mathrm{Sch}(\Gamma/H, S)$ centered at the root contains the support of f. By rooted convergence, for every large enough n, the R-ball B' in $\mathrm{Sch}(\Gamma/H, S)$ centered at the root is isomorphic to B. Let $f' \in l^2(\Gamma/H_n)$ be the same as f on B' (using the isomorphism between B and B') and 0 outside. Then $f' \in l^2_0(\Gamma/H_n)$, $\langle f', f'\rangle = 1$ and

$$\rho_0(\mathrm{Sch}(\Gamma/H_n, S)) \geq \langle f'M, f'\rangle = \langle fM, f\rangle > \rho(\mathrm{Sch}(\Gamma/H, S)) - \varepsilon.$$

This gives

$$\liminf \rho_0(\mathrm{Sch}(\Gamma/H_n, S)) \geq \rho(\mathrm{Sch}(\Gamma/H, S)) - \varepsilon$$

which proves the lemma. □

Proof of Proposition 4. By the compactness of the Chabauty topology, any subsequence of H_n has a Chabauty convergent subsequence K_n with limit K. Applying Lemma 25 gives

$$\liminf \rho_0(\mathrm{Sch}(\Gamma/K_n, S)) \geq \rho(\mathrm{Sch}(\Gamma/K, S)) \geq \rho(\mathrm{Cay}(\Gamma, S))$$

using $\rho(\mathrm{Sch}(\Gamma/K, S)) \geq \rho(\mathrm{Cay}(\Gamma, S))$. This proves the proposition. □

We are ready to prove Theorem 5. In the proof, we are using the following result from [3].

Theorem 26 *Let Γ be a finitely generated linear group and let H be an IRS of Γ that is amenable a.s. Then $H \leq A(\Gamma)$ a.s., where $A(\Gamma)$ denotes the amenable radical of Γ.*

Proof of Theorem 5. Assume by contradiction that H_n does not approximate Γ locally. Then by Lemma 16 there exists $1 \neq g \in \Gamma$, $c > 0$ and a subsequence K_n of H_n such that we have

$$|\mathrm{Fix}(\Gamma/K_n, g)| > c\,|\Gamma : K_n| \quad (n \geq 1)$$

where

$$\mathrm{Fix}(\Gamma/K_n, g) = \{K_n x \mid K_n x g = K_n x\}$$

is the set of H-cosets fixed by g. Note that the above condition is equivalent to $g \in H^x$. Let M_n denote a uniform random conjugate of K_n. Then the invariant random subgroup M_n equals the stabilizer of a uniform random vertex in $\mathrm{Sch}(\Gamma/K_n, S)$, so we have $g \in M_n$ with probability at least c ($n \geq 1$).

By passing to a subsequence of K_n, we can assume that M_n is weakly convergent. Let M be the limit of M_n. By weak convergence, we have $g \in M_n$ with probability at least c.

Let L be in the support of M. Then by weak convergence, there exists L_n in the support of M_n with

$$\lim_{n\to\infty} L_n = L$$

in the Chabauty topology. Using Lemma 25 we get

$$\rho(\mathrm{Sch}(\Gamma/L, S)) \leq \liminf \rho_0(\mathrm{Sch}(\Gamma/L_n, S)) \leq \rho(\mathrm{Cay}(\Gamma, S))$$

by the assumption of the theorem, since L_n is a conjugate of K_n and hence $\mathrm{Sch}(\Gamma/L_n, S)$ is isomorphic to $\mathrm{Sch}(\Gamma/K_n, S)$. So, M is an IRS of Γ satisfying $\rho(\mathrm{Sch}(\Gamma/M, S)) = \rho(\mathrm{Cay}(\Gamma, S))$ a.s., and so by Theorem 3, M is amenable a.s.

Since Γ is linear, by Theorem 26, every amenable IRS of Γ lies in the amenable radical of Γ. By the assumption of the theorem, this is trivial, so $M = 1$ a.s. But $g \in M$ with probability at least c, a contradiction. □

Now Theorem 6 follows fast.

Proof of Theorem 6. Let $\phi : F_S \to \Gamma$ be the canonical projection defined by S and let $K_n = \phi^{-1}(H_n)$. Then $\mathrm{Sch}(\Gamma/H_n, S)$ is isomorphic to $\mathrm{Sch}(F_S/K_n, S)$. Since free groups are linear, we can apply Theorem 5 to K_n and get that K_n locally approximates F_S. In particular, $\mathrm{Sch}(\Gamma/H_n, S)$ has essentially large girth. Let M_n denote a uniform random conjugate of K_n. Then M_n converges to the trivial group, but M_n contains the kernel of ϕ a.s. $(n \geq 1)$. This implies that ϕ is an isomorphism and hence Γ is a free group freely generated by S. □

Note that in the proof of Theorem 6 one can use the easier Lemma 24 instead of Theorem 26, which makes it independent from [3].

Acknowledgments. M.A. has been supported by the grant IEF-235545. Y.G. was partially supported by ISF grant 441/11 B.V. was supported by the NSERC Discovery Accelerator Grant and the Canada Research Chair program.

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