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Ehrhart theory of polytopes and Seiberg–Witten invariants of plumbed 3–manifolds

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Let M be a rational homology sphere plumbed 3–manifold associated with a connected negative-definite plumbing graph. We show that its Seiberg–Witten invariants equal certain coefficients of an equivariant multivariable Ehrhart polynomial. For this, we construct the corresponding polytope from the plumbing graph together with an action of $H_1(M,\mathbb{Z})$ and we develop Ehrhart theory for them. At an intermediate level we define the 'periodic constant' of multivariable series and establish their properties. In this way, one identifies the Seiberg–Witten invariant of a plumbed 3–manifold, the periodic constant of its 'combinatorial zeta function' and a coefficient of the associated Ehrhart polynomial. We make detailed presentations for graphs with at most two nodes. The two node case has surprising connections with the theory of affine monoids of rank two.

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1 Introduction

1.1

The main motivation of the present article is the combinatorial computation of the Seiberg–Witten invariants of a negative-definite plumbed 3–manifold. The final output is the identification of these invariants with certain coefficients of a multivariable equivariant Ehrhart polynomial.

Let Γ denote the connected negative-definite decorated plumbing graph with vertices $\mathcal{V}$, which determines the oriented plumbed 3–manifold $M = M(\Gamma)$. We assume that Γ is a tree, and all the plumbed surfaces have genus zero, that is, M is a rational homology sphere. We denote by $\mathfrak{sw}_\sigma(M)$ the Seiberg–Witten invariants of M indexed by the spin^c–structures σ of M.

In the last years several combinatorial expressions were established regarding the Seiberg–Witten invariants. In [49] Nicolaescu proved (based on the surgery formulas of Marcolli and Wang [28]) that they are equivalent with Turaev's torsion normalized

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by the Casson–Walker invariant. In terms of Γ, a combinatorial formula for the Casson–Walker invariant can be deduced from Lescop's book [27], while the Turaev's torsion is determined by Némethi and Nicolaescu [40] in terms of a Dedekind–Fourier sum.

For some special graphs, when the Heegaard Floer homology of Ozsváth and Szabó [51; 53; 52] is determined, we obtain the Seiberg–Witten invariant as the normalized Euler characteristic of the Heegaard Floer homology. They can be determined inductively by surgery formulae as well; see eg [51], Némethi [33] and Rustamov [57]. Braun and Némethi [11] provides a different type of surgery formula (which is not induced by an exact triangle, but involves the periodic constant of a series, more in the spirit of the present work). In parallel, one can rely on the lattice cohomology too (introduced in Némethi [33; 35]): in [37] Némethi proved that the Seiberg–Witten invariant is the normalized Euler characteristic of the lattice cohomology of M. Hence, the surgery formulae of Némethi [38], and closed formulae for specific families in Némethi [34], Némethi and Román [45] provide further examples.

1.2

The starting point of the present article is the result of Némethi [37], when the Seiberg–Witten invariant appears as the *periodic constant of a multivariable series*. Next we provide some details.

Let us consider the plumbed 4–manifold $\widetilde{X}$ associated with Γ. Its second homology L is freely generated by the 2–spheres $\{E_v\}_{v\in\mathcal{V}}$, and its second cohomology L' by the (anti)dual classes $\{E_v^*\}_{v\in\mathcal{V}}$; the intersection form $I=(\ ,\ )$ embeds L into L'. Set $x^2:=(x,x)$.

Let $K\in L'$ be the canonical class (given by the adjunction relations), $\widetilde{\sigma}_{\mathrm{can}}$ the canonical spin^c–structure on $\widetilde{X}$ with $c_1(\widetilde{\sigma}_{\mathrm{can}})=-K$, and $\sigma_{\mathrm{can}}\in \mathrm{Spin}^c(M)$ its restriction on M. Set $H:=H_1(M,\mathbb{Z})=L'/L$. Then $\mathrm{Spin}^c(M)$ is an H–torsor, with action denoted by $*$.

Next, consider the multivariable Taylor expansion $Z(\boldsymbol{t})=\sum p_{l'}\boldsymbol{t}^{l'}$ at the origin of

$$\prod_{v\in\mathcal{V}}(1-\boldsymbol{t}^{E_v^*})^{\delta_v-2},$$

where for any $l'=\sum_v l_vE_v\in L'$ we write $\boldsymbol{t}^{l'}=\prod_v t_v^{l_v}$, and δ_v is the valency of v. This lives in $\mathbb{Z}[\![L']\!]$, the submodule of formal power series $\mathbb{Z}[\![\boldsymbol{t}^{\pm 1/d}]\!]$ in variables $\{t_v^{\pm 1/d}\}_v$, where $d=\det(-I)$. It has a natural decomposition $Z(\boldsymbol{t})=\sum_{h\in H}Z_h(\boldsymbol{t})$, where $Z_h(\boldsymbol{t})=\sum_{[l']=h}p_{l'}\boldsymbol{t}^{l'}$ (where $[l']$ is the class of l'). Then $\mathfrak{sw}_{-h*\sigma_{\mathrm{can}}}(M)$ can be deduced from Z_h as follows [37].

Assume that $l'=\sum_v a_vE_v^*$ satisfies $a_v\geq -(E_v^2+1)$ for all $v\in\mathcal{V}$. Then

$$\sum_{l\in L,\, l\not\geq 0} p_{l'+l} = -\frac{(K+2l')^2+|\mathcal{V}|}{8} - \mathfrak{sw}_{[-l']*\sigma_{\mathrm{can}}}(M). \tag{1.2.1}$$

The left-hand side appears as a *counting function* of the coefficients of Z_h associated with a special *truncation*, while the right-hand side is a multivariable quadratic polynomial whose *free term* is the normalized Seiberg–Witten invariant.

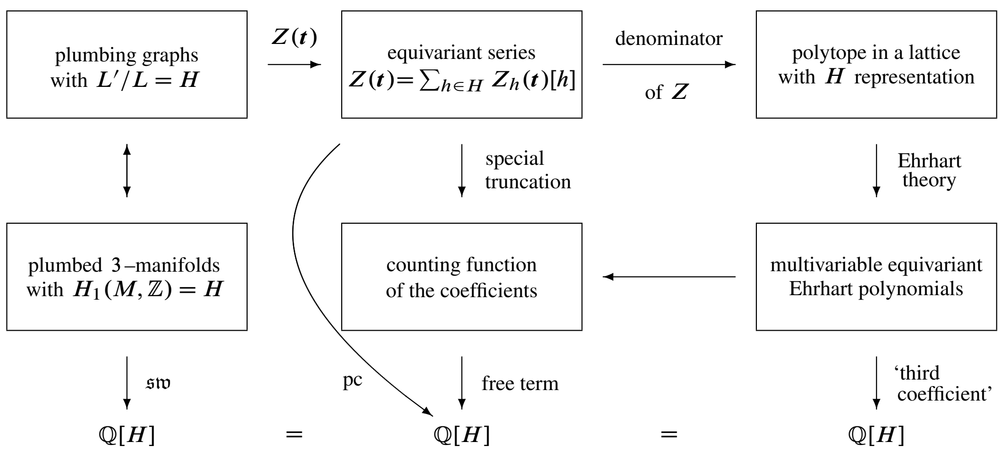
In order to guarantee the validity of the formula, the vector l' should sit in a special *chamber* described by the inequalities of the assumption. This, after we establish the necessary bridges, will read as follows: 'the third degree' coefficient of a multivariable Ehrhart polynomial associated with a certain polytope and specific chamber can be identified with the SW invariant.

In fact, the way how one recovers the needed information from the series can be done at several levels. The first one is entirely at the level of series (or Taylor expansions of rational functions). We develop a theory which associates with any series the counting function of its coefficients (given by a truncation of the monomials), like the right-hand side of (1.2.1). This is usually a piecewise quasipolynomial. Once we fix a chamber, the free term of the counting function is the so called 'periodic constant' (denoted by pc). In this terminology, the Seiberg–Witten invariant can be interpreted as the *multivariable periodic constant* $\mathrm{pc}(Z)$ of the series $Z(\boldsymbol{t})$, where the chosen chamber is described by the inequalities of the assumption (a part of the 'Lipman cone'). The 'periodicity' is related with the quasipolynomial behavior of the counting function. The 'periodic constant' of one variable series was introduced in Némethi and Okuma [43], Okuma [50], and it had several applications; see eg Némethi and Okuma [42; 43], Némethi [37], Braun and Némethi [11]. Here we create the general theory, which carries necessarily several difficult technical ingredients (eg one has to choose the 'right' truncation and summation procedure of the coefficients, which in the context of general series is not automatically motivated, and also it depends on the chamber decomposition of the space of exponents). The theory has some similarities with the theory of vector partition functions.

On the other hand, there is a more sophisticated way to generalize the identity (1.2.1) too.

From any Taylor expansion of a multivariable rational function with denominator of type $\prod_i(1-\boldsymbol{t}^{a_i})$ we construct a polytope situated in a lattice which carries also a representation of a finite abelian group H. Associated with these data we consider the equivariant multivariable Ehrhart piecewise quasipolynomials, whose existence, main properties (like the Ehrhart–MacDonald–Stanley type reciprocity law or chamber decompositions) will also be established. This applied to the series $Z(\boldsymbol{t})$ above, and to the quasipolynomial of those chambers which belong to the Lipman cone shows that the first three top-degree coefficients (at least) will carry geometrical/topological meaning, including the SW invariants of the link. (This coefficient identification, and in fact (1.2.1) too, supply an additional addendum to the intimate relationship between lattice point counting and the Riemann–Roch formula, exploited in global algebraic geometry by toric geometry.)

Here is a schematic picture of these connections and areas we target:



1.3

The number of terms in the denominator $\prod_i (1-t^{a_i})$ of the series equals the number of variables of the corresponding partition function (associated with vectors a_i), and it is also the rank of the lattice where the corresponding polytope sit. In the case of the series $Z(t)$ associated with plumbing graph, this is the number of *end vertices* of Γ. On the other hand, the number of variables of $Z(t)$ is the number $|\mathcal{V}|$ of vertices of Γ. Furthermore, in the Ehrhart theoretical part, the associated (nonconvex) polytope will be a union of $|\mathcal{V}|$ simplicial polytopes. Hence, the number of facets and the complexity of the polytope increases considerably with the number of vertices as well.

Nevertheless, Reduction Theorem 5.4.2 eliminates a part of this abundance of parameters: it says that from the periodic constant point of view, the number of variables of the series, and also the number of simplicial polytopes in the union, can be reduced to the number of *nodes* of the graph. Hence, in fact, the complexity level is measured by the number of nodes.

In the body of the article, besides the general theory, we make detailed computations for graphs with less than two nodes. Even in the special case of graphs without nodes (that is, the case of lens spaces) the description of the equivariant Ehrhart quasipolynomials is new. In the one node case (star-shaped graphs) we provide a detailed presentation of all the involved (SW and Ehrhart) invariants, and we establish closed formulae in terms of the Seifert invariants. Here we make connection with already known topological results regarding the Seiberg–Witten invariants of Seifert 3–manifolds, and also with analytic invariants of weighted homogeneous singularities.

In the two node case again we make complete presentations in terms of the analogs of the Seifert invariants of the chains and star-shaped subgraphs, including closed formulae for $\mathfrak{sw}(M)$. But, this case has a very interesting additional surprise in store. It turns out that the corresponding combinatorial series $Z(\boldsymbol{t})$ associated with Γ, reduced to the variables of the two nodes, is the Hilbert (characteristic) series of an affine monoid of rank two (and some of its modules). In particular, the Seiberg–Witten invariant appears as the periodic constant of Hilbert series associated with affine monoids (and certain modules indexed by H), and, in some sense, measures the nonnormality of these monoids.

1.4

It is important to emphasize that the origin (and main motivation) of the identity (1.2.1) was an analytic identity. Recall that the plumbed 3–manifolds M appear as links of complex normal surface singularities, and several of the above objects have their analytic counterparts. For example, the analogue of the series $Z(\boldsymbol{t})$ is the Hilbert series associated with the multivariable equivariant divisorial filtration of the local ring of the singular germ, and its equivariant periodic constants are the equivariant geometric genera. In the body of the paper we emphasize this parallelism as well, whenever the corresponding analytic invariants coincide with the topological ones. This happens eg in the case of star-shaped graphs and the weighted homogeneous analytic structures carried by them. For further relations with analytic structures (eg for the Seiberg–Witten Invariant Conjecture targeting these type of connections), see Némethi [31], Némethi and Nicolaescu [40].

The relevant terminology and additional connections with theory of complex singularities can be found in Arnold, Gusein-Zade and Varchenko [1], Cutkosky, Herzog and Reguera [18], Campillo, Delgado and Gusein-Zade [15], Eisenbud and Neumann [22], Némethi [32; 36; 39], its connection with Seiberg–Witten theory in Braun and Némethi [11; 10], Némethi [33; 34; 35; 37], Némethi and Nicolaescu [40; 41], Nicolaescu [49]. For some results in Ehrhart theory relevant to the present work see Barvinok [2], Barvinok and Pommersheim [3], Beck [4; 5; 6], Beck and Robins [8; 9], Beck, Diaz and Robins [7], Clauss and Loechner [17], Diaz and Robins [19], for partition functions see Brion and Vergne [12], Szenes and Vergne [60] and Sturmfels [59], while for affine monoids see Bruns and Gubeladze [13].

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2 Normal surface singularities: The main motivation

2.1 Surface singularities and their links and graphs

Let (X, o) be a complex normal surface singularity whose *link* M *is a rational homology sphere*. We denote the finite group $H_1(M, \mathbb{Z})$ by H. Let $\pi\colon \widetilde{X} \to X$ be a good resolution with dual graph Γ whose vertices are denoted by $\mathcal{V}$. Hence Γ is a tree and all the irreducible exceptional divisors have genus 0. Let δ_v be the valency of the vertex v. We distinguish the following subsets of vertices: the set of *nodes* $\mathcal{N} = \{v \in \mathcal{V} \mid \delta_v \geq 3\}$, and the set of *ends* $\mathcal{E} = \{v \in \mathcal{V} \mid \delta_v = 1\}$. If we delete from Γ the nodes and their adjacent edges we get the collection of (maximal) *chains* of the graph. A *leg* is a chain which is connected by only one node.

The number of vertices is denoted by $|\mathcal{V}|$ or s, while the number of nodes and ends is denoted by $|\mathcal{N}|$ and $|\mathcal{E}|$.

Set $L := H_2(\widetilde{X}, \mathbb{Z})$. It is freely generated by the classes of the irreducible exceptional curves $\{E_v\}_{v\in\mathcal{V}}$. L will also be identified with the group of integral cycles supported on $E = \pi^{-1}(o)$. We set $I_{vw} = (E_v, E_w)$. The vertex v of the graph is decorated by I_{vv}. The intersection matrix $I = \{I_{vw}\}$ is negative-definite, and any connected plumbing graph with negative-definite intersection form appears in this way for some singularity. The graph may also serve as the plumbing graph of the link $M = \partial\widetilde{X}$. In this case $\widetilde{X}$ is the plumbed 4–manifold associated with Γ, and one might consider this topological starting setup instead of the analytic one.

If L' denotes $H^2(\widetilde{X}, \mathbb{Z})$, then the intersection form provides an embedding $L \hookrightarrow L'$ with factor $H^2(\partial\widetilde{X}, \mathbb{Z}) \simeq H$; $[l']$ denotes the class of l'. The form $(\ ,\ )$ extends to L' (since $L' \subset L \otimes \mathbb{Q}$). The module L' over $\mathbb{Z}$ is freely generated by the (anti)duals $\{E^*_v\}_v$, where we prefer the convention $(E^*_v, E_w) = -1$ for $v = w$, and 0 otherwise. We write $\det(\Gamma) := \det(-I)$. The inverse of I has entries $(I^{-1})_{vw} = (E^*_v, E^*_w)$, all of them are negative. Furthermore, cf Eisenbud and Neumann [22, page 83 and Section 20],

(2.1.1) $-|H| \cdot (E^*_v, E^*_w)$ equals the determinant of the subgraph obtained from Γ by eliminating the shortest path connecting v and w.

The *canonical class* $K \in L'$ is defined by the *adjunction formulae*

$$(K + E_v, E_v) + 2 = 0 \quad \text{for all } v \in \mathcal{V}. \tag{2.1.2}$$

We set $\chi(l') := -(l', l' + K)/2$ for any $l' \in L'$. If $l \in L$ is effective then by Riemann–Roch theorem $\chi(l) = h^0(\mathcal{O}_l) - h^1(\mathcal{O}_l)$. The integer $\chi(l')$ has similar analytic interpretation via line bundles of $\widetilde{X}$; cf Némethi [34, 2.2.8]. The expression $K^2 + |\mathcal{V}|$ will appear in several formulae. One has the following combinatorial expression in terms of the graph; cf Némethi and Nicolaescu [40]:

$$K^2 + |\mathcal{V}| = \sum_{v\in\mathcal{V}} (E_v, E_v) + 3|\mathcal{V}| + 2 + \sum_{v,w\in\mathcal{V}} (2-\delta_v)(2-\delta_w) I^{-1}_{vw} \tag{2.1.3}$$

For $l_1, l_2 \in L \otimes \mathbb{Q}$ one writes $l_1 \geq l_2$ if $l_1 - l_2 = \sum r_v E_v$ with all $r_v \geq 0$. Denote by $\mathcal{S}'$ the Lipman cone $\{l' \in L' \mid (l', E_v) \leq 0 \text{ for all } v\}$. It is generated over $\mathbb{Z}_{\geq 0}$ by the elements E^*_v. Since *all the entries* of E^*_v are *strictly* positive, for any fixed $a \in L'$ one has

$$\{l' \in \mathcal{S}' \mid l' \ngeq a\} \quad \text{is finite.} \tag{2.1.4}$$

For any class $h \in H$ there exists a unique minimal element of $\{l' \in L' \mid [l'] = h\} \cap \mathcal{S}'$, cf Némethi [33, 5.4], it will be denoted by s_h. Furthermore, we set $\square = \big\{\sum_v l'_v E_v \in L' \mid 0 \leq l'_v < 1\big\}$ for the 'semiopen cube,' and for any $h \in H = L'/L$ we consider the unique representative $r_h \in \square$ with $[r_h] = h$. One has $s_h \geq r_h$, and usually $s_h \neq r_h$ (see eg Némethi [34, 4.5]). Moreover, using the generalized Laufer computation sequence of [34, 4.3.3] connecting $-r_h$ with $-s_h$ one gets

$$\chi(s_h) \leq \chi(r_h). \tag{2.1.5}$$

Denote by $\theta\colon H \to \widehat{H}$ the isomorphism $[l'] \mapsto e^{2\pi i(l',\cdot)}$ of H with its Pontrjagin dual $\widehat{H}$.

For more details on the resolution graphs see eg Némethi [32; 33; 34].

2.1.6 Spin^c–structures and the Seiberg–Witten invariant of M Let $\widetilde{\sigma}_{\text{can}}$ be the *canonical spin^c–structure* on $\widetilde{X}$; its first Chern class $c_1(\widetilde{\sigma}_{\text{can}})$ is $-K \in L'$; cf Gompf and Stipsicz [23, page 415]. The set of spin^c–structures $\text{Spin}^c(\widetilde{X})$ of $\widetilde{X}$ is an L'–torsor; if we denote the L'–action by $l' * \widetilde{\sigma}$, then $c_1(l' * \widetilde{\sigma}) = c_1(\widetilde{\sigma}) + 2l'$. Furthermore, all the spin^c–structures of M are obtained by restrictions from $\widetilde{X}$. $\text{Spin}^c(M)$ is an H–torsor, compatible with the restriction and the projection $L' \to H$. The *canonical spin^c–structure* σ_{can} of M is the restriction of $\widetilde{\sigma}_{\text{can}}$.

We denote the Seiberg–Witten invariant by $\mathfrak{sw}\colon \text{Spin}^c(M) \to \mathbb{Q}$, $\sigma \mapsto \mathfrak{sw}_\sigma$.

2.2 Motivation: $\mathfrak{sw}_\sigma(M)$ as the constant term of a 'combinatorial Hilbert series'

Consider the multivariable Taylor expansion $Z(\boldsymbol{t}) = \sum p_{l'} \boldsymbol{t}^{l'}$ at the origin of

$$\prod_{v \in \mathcal{V}} (1 - \boldsymbol{t}^{E_v^*})^{\delta_v - 2}, \tag{2.2.1}$$

where for any $l' = \sum_v l_v E_v \in L'$ we write $\boldsymbol{t}^{l'} = \prod_v t_v^{l_v}$ and δ_v is the valency of v as above. This lives in $\mathbb{Z}[\![L']\!]$, the submodule of formal power series $\mathbb{Z}[\![\boldsymbol{t}^{\pm 1/|H|}]\!]$ in variables $\{t_v^{\pm 1/|H|}\}_v$.

2.2.2 Theorem (Némethi [37]) *Fix some $l' \in L'$. Assume that for any $v \in \mathcal{V}$ the E_v^*–coordinate of l' is larger than or equal to $-(E_v^2 + 1)$. Then*

$$\sum_{l \in L, l \not\geq 0} p_{l'+l} = -\mathfrak{sw}_{[-l'] * \sigma_{\text{can}}}(M) - \frac{(K + 2l')^2 + |\mathcal{V}|}{8}, \tag{2.2.3}$$

where $*$ denotes the torsor action of H on $\text{Spin}^c(M)$. In particular,

$$-\mathfrak{sw}_{[-l'] * \sigma_{\text{can}}}(M) - \frac{K^2 + |\mathcal{V}|}{8}$$

appears as the constant term of a 'combinatorial multivariable Hilbert polynomial' (the right-hand side of (2.2.3)*).*

Since $Z(\boldsymbol{t})$ is supported on the Lipman cone, by (2.1.4) the sum (2.2.3) is finite.

Note also that the series $Z(\boldsymbol{t})$ decomposes into several series indexed by elements of H. Indeed, $Z(\boldsymbol{t}) = \sum_h Z_h(\boldsymbol{t})$, where $Z_h(\boldsymbol{t}) = \sum_{l' | [l'] = h} p_{l'} \boldsymbol{t}^{l'}$. The identity (2.2.3) involves only $Z_{[l']}$.

In fact, the above topological Theorem 2.2.2 was motivated by a similar theorem which targets the analytic invariants of the singularity. In order to have a complete picture and possibility to interpret the subsequent results via analytic invariants, we recall briefly this setup as well.

2.3 The analytic motivation: Multivariable Hilbert series of divisorial filtrations

One of the strongest analytic invariants of (X, o) is its *equivariant divisorial Hilbert series* $\mathcal{H}(\boldsymbol{t})$. This is defined as follows; for more details see Némethi [39; 36].

Fix a resolution π of (X, o) as in Section 2.1, let $c\colon (Y, o) \to (X, o)$ be the universal abelian cover of (X, o) with Galois group $H = H_1(M, \mathbb{Z})$, $\pi_Y\colon \widetilde{Y} \to Y$ the normalized pullback of π by c, and $\widetilde{c}\colon \widetilde{Y} \to \widetilde{X}$ the morphism which covers c. Then $\mathcal{O}_{Y,o}$ inherits the *divisorial multifiltration*:

$$\mathcal{F}(l') := \{f \in \mathcal{O}_{Y,o} \mid \operatorname{div}(f \circ \pi_Y) \geq \widetilde{c}^*(l')\}$$

Let $\mathfrak{h}(l')$ be the dimension of the $\theta([l'])$–eigenspace of $\mathcal{O}_{Y,o}/\mathcal{F}(l')$. Then the *equivariant divisorial Hilbert series* is

$$\mathcal{H}(\boldsymbol{t}) = \sum_{l'=\sum l_v E_v \in L'} \mathfrak{h}(l') t_1^{l_1} \cdots t_s^{l_s} = \sum_{l' \in L'} \mathfrak{h}(l') \boldsymbol{t}^{l'} \in \mathbb{Z}[\![L']\!].$$

In $\mathcal{H}(\boldsymbol{t})$ the exponents l' of the terms $\boldsymbol{t}^{l'}$ reflect the $L'/L \simeq H$ eigenspace decomposition too. For example, $\sum_{l \in L} \mathfrak{h}(l) \boldsymbol{t}^l$ corresponds to the H–invariants, hence it is the *Hilbert series* of $\mathcal{O}_{X,o}$ associated with the $\pi^{-1}(o)$–divisorial multifiltration; see eg Cutkosky, Herzog and Reguera [18], Campillo, Delgado and Gusein-Zade [15].

If l' is in the special 'Kodaira vanishing zone' $l' \in -K + \mathcal{S}'$, then by vanishing (of a certain first cohomology), and by Riemann–Roch, one obtains (see Némethi [39]) that the expression

$$\mathfrak{h}(l') + \frac{(K + 2l')^2 + |\mathcal{V}|}{8} \tag{2.3.1}$$

depends only on the class $[l'] \in L'/L$ *of* l'. The key bridge connecting $\mathcal{H}(\boldsymbol{t})$ with the topology of the link and with Γ is done by the series; cf Campillo, Delgado and Gusein-Zade [15; 25], Némethi [36; 39]:

$$\mathcal{P}(\boldsymbol{t}) = -\mathcal{H}(\boldsymbol{t}) \cdot \prod_v (1 - t_v^{-1}) \in \mathbb{Z}[\![L']\!]$$

Moreover, this identity can be 'inverted'; cf [39, (3.2.6)]:

$$\mathfrak{h}(l') = \sum_{l \in L, l \ngeq 0} \overline{p}_{l'+l}, \quad \text{where } \mathcal{P}(\boldsymbol{t}) = \sum_{l'} \overline{p}_{l'} \boldsymbol{t}^{l'}.$$

$\mathcal{P}$ is supported on $\mathcal{S}'$, cf [39, (3.2.2)], hence the sum is finite; cf (2.1.4). In particular,

$$\sum_{l \in L, l \ngeq 0} \overline{p}_{l'+l} = -\text{const}_{[-l']} - \frac{(K + 2l')^2 + |\mathcal{V}|}{8} \tag{2.3.2}$$

for any $l' \in -K + \mathcal{S}'$, where $\text{const}_{[-l']}$ depends only on the class $[-l']$ of $-l'$. The right-hand side can be interpreted as a 'multivariable Hilbert polynomial' of degree 2 associated with the series $\mathcal{H}(\boldsymbol{t})$, or with $\mathcal{P}(\boldsymbol{t})$. Its constant term is the (normalized) equivariant geometric genera of the universal abelian cover Y, that is (cf [39])

$$\dim(H^1(\widetilde{Y},\mathcal{O}_{\widetilde{Y}})_{\theta(h)}) = -\mathrm{const}_{[-r_h]} - \frac{(K+2r_h)^2+|\mathcal{V}|}{8}. \tag{2.3.3}$$

The point is that the *topological candidate* of $\mathcal{P}(\boldsymbol{t})$ is exactly $Z(\boldsymbol{t})$ from the previous subsection; they agree for several singularities; see eg [25], [36] and [39]. The identification of their constant terms (for 'nice' analytic structures) is the subject of the 'Seiberg–Witten invariant conjecture'; cf Némethi and Nicolaescu [40], Némethi [31; 34]. Hence, when $\mathcal{P}(\boldsymbol{t}) = Z(\boldsymbol{t})$, then $\mathrm{const}_{[-l']} = \mathfrak{sw}_{[-l']*\sigma_{\mathrm{can}}}(M)$ too, and (2.3.3) creates the bridge between the combinatorial/topological Seiberg–Witten theory and the analytic counterpart. The identity $\mathcal{P}(\boldsymbol{t}) = Z(\boldsymbol{t})$ is valid eg for splice quotient singularities [39], which include all the rational singularities (when the links M are L–spaces), minimally elliptic singularities, or weighted homogeneous singularities.

3 Equivariant multivariable Ehrhart theory

3.1 Preparatory results on Ehrhart theory

In this section we generalize the classical Ehrhart theory to the equivariant multivariable version, involving nonconvex polytopes, which will fit with our comparison with the equivariant multivariable series provided by plumbing graphs.

Let us start with a d–dimensional *rational lattice* $\mathcal{X} \subset \mathbb{Q}^d$ and a group homomorphism $\rho\colon \mathcal{X} \to \mathfrak{H}$ to a finite abelian group $\mathfrak{H}$. We consider a *rational vector-dilated polytope* with parameter $\boldsymbol{l} = (\boldsymbol{l}_1, \dots, \boldsymbol{l}_r)$, $\boldsymbol{l}_v \in \mathbb{Z}^{m_v}$,

$$P^{(\boldsymbol{l})} = \bigcup_{v=1}^{r} P_v^{(\boldsymbol{l}_v)}, \quad \text{where } P_v^{(\boldsymbol{l}_v)} = \{\boldsymbol{x} \in \mathbb{R}^d \mid A_v \boldsymbol{x} \le \boldsymbol{l}_v\}, \tag{3.1.1}$$

where A_v is an integral $m_v \times d$ matrix. If $\{A_{v,\lambda i}\}_{\lambda i}$ and $\{l_{v,\lambda}\}_\lambda$ are the entries of A_v and $\boldsymbol{l}_v$, then for any $\lambda = 1, \dots, m_v$ the inequality $A_v\boldsymbol{x} \le \boldsymbol{l}_v$ in (3.1.1) reads as $\sum_{i=1}^d x_i A_{v,\lambda i} \le l_{v,\lambda}$.

We will vary the parameter $\boldsymbol{l}$ in some 'chambers' (described below for the needed cases) such that the polytopes $P^{(\boldsymbol{l})}$ remain *combinatorially stable* (or preserve their *combinatorial type*) when $\boldsymbol{l}$ runs in the same chamber. This means that their face lattices are isomorphic. (This implies that they are connected by homeomorphisms, which preserve the stratification of the faces.) We also suppose that $P^{(\boldsymbol{l})}$ is homeomorphic to a d–dimensional manifold. Denote the set of all closed facets of $P^{(\boldsymbol{l})}$ by $\mathcal{F}$ and let $\mathcal{T}$ be a subset of $\mathcal{F}$, such that $\bigcup_{F^{(\boldsymbol{l})} \in \mathcal{T}} F^{(\boldsymbol{l})}$ is homeomorphic to a $(d-1)$–manifold.

Then we have the following generalization to the *equivariant version* of results of Stanley [58], McMullen [29] and Beck [4; 5].

3.1.2 Theorem *For any $h \in \mathfrak{H}$ and $\mathcal{T} \subset \mathcal{F}$ let*

$$\mathcal{L}_h(\boldsymbol{A}, \mathcal{T}, \boldsymbol{l}) := \text{cardinality of} \Bigg(\Bigg(P^{(\boldsymbol{l})} \setminus \bigcup_{F^{(\boldsymbol{l})} \in \mathcal{T}} F^{(\boldsymbol{l})} \Bigg) \cap \rho^{-1}(h) \Bigg). \tag{3.1.3}$$

(a) *If $\boldsymbol{l}$ moves in some region in such a way that $P^{(\boldsymbol{l})}$ stays combinatorially stable then the expression $\mathcal{L}_h(\boldsymbol{A}, \mathcal{T}, \boldsymbol{l})$ is a quasipolynomial in $\boldsymbol{l} \in \mathbb{Z}^{\sum m_v}$.*

(b) *For a fixed combinatorial type of $P^{(\boldsymbol{l})}$ and for a fixed $\mathcal{T}$, the quasipolynomials $\mathcal{L}_h(\boldsymbol{A}, \mathcal{T}, \boldsymbol{l})$ and $\mathcal{L}_{-h}(\boldsymbol{A}, \mathcal{F} \setminus \mathcal{T}, \boldsymbol{l})$ satisfy the Ehrhart–MacDonald–Stanley reciprocity law*

$$\mathcal{L}_h(\boldsymbol{A}, \mathcal{T}, \boldsymbol{l}) = (-1)^d \cdot \mathcal{L}_{-h}(\boldsymbol{A}, \mathcal{F} \setminus \mathcal{T}, \boldsymbol{l})|_{\text{replace } \boldsymbol{l} \text{ by } -\boldsymbol{l}}. \tag{3.1.4}$$

To avoid any confusion regarding the expression of (3.1.4) we note: the two quasipolynomials in (3.1.4) are associated with that domain of definition (chamber) which corresponds to the fixed combinatorial type. Usually for $-\boldsymbol{l}$ the combinatorial type of $P^{(\boldsymbol{l})}$ is different, hence the right-hand side of (3.1.4) *need not equal* $(-1)^d \cdot \mathcal{L}_{-h}(\boldsymbol{A}, \mathcal{F} \setminus \mathcal{T}, -\boldsymbol{l})$. This last expression is the value at $-\boldsymbol{l}$ of the quasipolynomial associated with the chamber which contains $-\boldsymbol{l}$.

For a reformulation of the above identity (3.1.4) in terms of the fixed chamber see Theorem 4.3.9(c).

Proof The statements for $\mathfrak{H} = 0$ are identical with those of Beck from [5]. Part (a) above for arbitrary $\mathfrak{H}$ can be proved identically as in [5] applied for the situation when the parameters $\boldsymbol{l}$ run in an overlattice of $\mathbb{Z}^{\sum m_v}$, instead of $\mathbb{Z}^{\sum m_v}$. Equivalently, one can apply Clauss and Loechner [17], which considers the nonequivariant case, but the integral parameters $\boldsymbol{l}$ of Beck are replaced by *rational affine parameters*.

For the convenience of the reader we provide the proof. First we notice that via standard additivity formulae, cf [5, Section 2], it is enough to prove the statement for each convex $P_v^{(\boldsymbol{l}_v)}$. But, considering $P_v^{(\boldsymbol{l}_v)}$ and $K := \ker(\rho)$, for any $\boldsymbol{r} \in \mathcal{X}$ one has the isomorphism

$$\{\boldsymbol{x} \in K + \boldsymbol{r} \mid \boldsymbol{A}_v \boldsymbol{x} \le \boldsymbol{l}_v\} \simeq \{\boldsymbol{y} \in K \mid \boldsymbol{A}_v \boldsymbol{y} \le \boldsymbol{l}_v - \boldsymbol{A}_v \boldsymbol{r}\}.$$

Hence [17, Theorem 2] (or [5] for an overlattice of $\mathbb{Z}^{\sum m_v}$) can be applied, which shows (a). Next, part (b) can also be reduced to [5]. Indeed, we can reduce the discussion again to $P_v^{(\boldsymbol{l}_v)}$. We drop the index v, we choose $\boldsymbol{r}_h \in \mathcal{X}$ with $\rho(\boldsymbol{r}_h) = h$, and we fix some $\boldsymbol{l}_0$. Then for $\boldsymbol{x} \in K \pm \boldsymbol{r}_h$ with $A\boldsymbol{x} \le \boldsymbol{l}_0$ we take $\boldsymbol{y} := \boldsymbol{x} \mp \boldsymbol{r}_y$ and $\boldsymbol{k} := \boldsymbol{l}_0 \mp A\boldsymbol{r}_h$, which satisfy $\boldsymbol{y} \in K$ and $A\boldsymbol{y} \le \boldsymbol{k}$. Therefore, using [5] for this polytope, we obtain

$$\begin{aligned}\mathcal{L}_h(A, \mathcal{T}, \boldsymbol{l}_0) = \mathcal{L}_0(A, \mathcal{T}, \boldsymbol{k}) &= (-1)^d \cdot \mathcal{L}_0(A, \mathcal{F} \setminus \mathcal{T}, \boldsymbol{k})|_{\text{replace } \boldsymbol{k} \text{ by } -\boldsymbol{k}} \\ &= (-1)^d \cdot \mathcal{L}_{-h}(A, \mathcal{F} \setminus \mathcal{T}, \boldsymbol{l}_0)|_{\text{replace } \boldsymbol{l}_0 \text{ by } -\boldsymbol{l}_0},\end{aligned}$$

where the second and the third term is associated with the lattice K. □

3.1.5 Definition The quasipolynomial $\mathcal{L}_h(A, \mathcal{T}, \boldsymbol{l})$ considered in Theorem 3.1.2, associated with a fixed combinatorial type of $P^{(\boldsymbol{l})}$, is called the *equivariant multivariable quasipolynomial* associated with the corresponding data.

If we vary $\boldsymbol{l}$ in $\mathbb{Z}^{\sum m_v}$ (hence we allow the variation of the combinatorial type) we obtain the *equivariant multivariable piecewise quasipolynomial* $\mathcal{L}_h(A, \mathcal{T}, \boldsymbol{l})$ (see also Theorem 4.3.9 and Corollary 4.3.11 below).

3.1.6 Remark Parallel to the collection $\{\mathcal{L}_h\}_h$ defined in (3.1.3) one can consider their Fourier transforms as well: for any character $\xi \in \widehat{\mathfrak{H}} = \mathrm{Hom}(\mathfrak{H}, S^1)$, one defines

$$\mathcal{L}_\xi(A, \mathcal{T}, \boldsymbol{l}) := \sum_{\boldsymbol{x} \in P^{(\boldsymbol{l})} \setminus \bigcup_{F^{(\boldsymbol{l})} \in \mathcal{T}} F^{(\boldsymbol{l})}} \xi^{-1}(\rho(\boldsymbol{x})), \tag{3.1.7}$$

which satisfies $\mathcal{L}_\xi = \sum_h \mathcal{L}_h \cdot \xi^{-1}(h)$, and $|\mathfrak{H}| \cdot \mathcal{L}_h = \sum_\xi \mathcal{L}_\xi \cdot \xi(h)$. Hence, the above properties of $\mathcal{L}_h$ can be obtained from similar properties of $\mathcal{L}_\xi$ as well. Hence, Theorem 3.1.2 can be deduced from Brion and Vergne [12, Section 4.3] too.

3.1.8 Remark In the sequel we will not consider polytopes with this high generality: our polytopes will be special ones associated with the denominators of type $\prod_i (1 - \boldsymbol{t}^{a_i})$ of multivariable rational functions, or their Taylor series. In order to avoid unnecessary technical details, the stability of the combinatorial type of $P^{(\boldsymbol{l})}$, and the corresponding chamber decomposition of $\mathbb{R}^{\sum m_v}$ will also be treated for this special polytopes; see Section 4.3.7.

4 Multivariable rational functions and their periodic constants

4.1 Historical remark: the one-variable case [43, 3.9; 50, 4.8(1)]

Let $S(t) = \sum_{l \ge 0} c_l t^l \in \mathbb{Z}[\![t]\!]$ be a formal power series. Suppose that for some positive integer p, the expression $\sum_{l=0}^{pn-1} c_l$ is a polynomial $P_p(n)$ in the variable n. Then the constant term $P_p(0)$ of $P_p(n)$ is independent of the 'period' p. We call $P_p(0)$ the *periodic constant* of S and denote it by $\mathrm{pc}(S)$. For example, if $l \mapsto Q(l)$ is a quasipolynomial and $S(t) := \sum_{l\geq 0} Q(l)t^l$, then one can take for p the period of Q, and one shows that $\mathrm{pc}(\sum_{l\geq 0} Q(l)t^l) = 0$.

Assume that $S(t)$ is the Hilbert series associated with a graded algebra/vector space $A = \bigoplus_{l\geq 0} A_l$ (ie $c_l = \dim A_l$), and the series S admits a Hilbert quasipolynomial $Q(l)$ (that is, $c_l = Q(l)$ for $l \gg 0$). Since the periodic constant of $\sum_l Q(l)t^l$ is zero, the periodic constant of $S(t)$ measures exactly the difference between $S(t)$ and its 'regularized series' $S_{\mathrm{reg}}(t) := \sum_{l\geq 0} Q(l)t^l$. That is: $\mathrm{pc}(S) = (S - S_{\mathrm{reg}})(1)$ collecting all the anomalies of the starting elements of S.

Note that $S_{\mathrm{reg}}(t)$ can be represented by a rational function of negative degree with denominator of type $A(t) = \prod_i (1-t^{a_i})$, and $S(t) - S_{\mathrm{reg}}(t)$ is a polynomial. Conversely, one has the following reinterpretation of the periodic constant; see Braun and Némethi [11, 7.0.2]. If $\sum_l c_l t^l$ is a rational function $B(t)/A(t)$ with $A(t) = \prod_i (1-t^{a_i})$, and one rewrites it as $C(t) + D(t)/A(t)$ with C and D polynomials and $D(t)/A(t)$ of negative degree, then $\mathrm{pc}(S) = C(1)$. From this fact one also gets that $\mathrm{pc}(S(t)) = \mathrm{pc}(S(t^N))$ for any $N \in \mathbb{Z}_{>0}$. We will refer to $C(t)$ as the *polynomial part* of S.

As an example, consider a subset $\mathcal{S} \subset \mathbb{Z}_{\geq 0}$ with finite complement. Then $S(t) = \sum_{s\in\mathcal{S}} t^s$ rewritten is $1/(1-t) - \sum_{s\notin\mathcal{S}} t^s$, hence $\mathrm{pc}(S) = -\#(\mathbb{Z}_{\geq 0} \setminus \mathcal{S})$. In particular, if $\mathcal{S}$ is the semigroup of a local irreducible complex plane curve singularity, then $-\mathrm{pc}(S)$ is the delta invariant of that germ. Our study below includes the generalization of this fact to surface singularities.

4.2 The setup for the multivariable generalization

4.2.1 We wish to extend the definition of the periodic constant to the case of Taylor expansions at the origin of multivariable rational functions of type

$$f(\boldsymbol{t}) = \frac{\sum_{k=1}^{r} \iota_k \boldsymbol{t}^{b_k}}{\prod_{i=1}^{d}(1-\boldsymbol{t}^{a_i})} \quad (\iota_k \in \mathbb{Z}). \tag{4.2.2}$$

Let us explain the notation. Let L be a lattice of rank s with fixed bases $\{E_v\}_{v=1}^s$. Let L' be an overlattice of it with same rank, $L \subset L' \subset L \otimes \mathbb{Q}$ with $|L'/L| = \mathfrak{d}$. Then, in (4.2.2), $\{b_k\}_{k=1}^r, \{a_i\}_{i=1}^d \in L'$ and for any $l' = \sum_v l'_v E_v \in L'$ we write $\boldsymbol{t}^{l'} = t_1^{l'_1} \dots t_s^{l'_s}$. We also assume that *all the coordinates $a_{i,v}$ of a_i are strict positive.* Hence, in general, the coefficients l'_v are not integral, and the Laurent expansion $Tf(\boldsymbol{t})$ of $f(\boldsymbol{t})$ at the origin is

$$Tf(\boldsymbol{t}) = \sum_{l'} p_{l'} \boldsymbol{t}^{l'} \in \mathbb{Z}[\![t_1^{1/\mathfrak{d}}, \dots, t_s^{1/\mathfrak{d}}]\!][t_1^{-1/\mathfrak{d}}, \dots, t_s^{-1/\mathfrak{d}}] := \mathbb{Z}[\![\boldsymbol{t}^{1/\mathfrak{d}}]\!][\boldsymbol{t}^{-1/\mathfrak{d}}].$$

We also consider the natural partial ordering of $L\otimes\mathbb{Q}$ (defined as in Section 2.1). If all vectors $b_k\geq 0$ then $Tf(\boldsymbol{t})$ is in $\sum_{l'} p_{l'}\boldsymbol{t}^{l'}\in\mathbb{Z}[\![\boldsymbol{t}^{1/\mathfrak{d}}]\!]$. Sometimes we will not make difference between f and Tf.

4.2.3 This will be extended to the following equivariant case. We fix a finite abelian group G, and for each $g\in G$ a series (or rational function) $Tf_g\in\mathbb{Z}[\![\boldsymbol{t}^{1/\mathfrak{d}}]\!][\boldsymbol{t}^{-1/\mathfrak{d}}]$ as in Section 4.2.1, and we set

$$Tf^e(\boldsymbol{t}):=\sum_{g\in G} Tf_g(\boldsymbol{t})\cdot[g]\in\mathbb{Z}[\![\boldsymbol{t}^{1/\mathfrak{d}}]\!][\boldsymbol{t}^{-1/\mathfrak{d}}][G].$$

In some cases this equivariant extension is given automatically in the context of Section 4.2.1. Indeed, if in Section 4.2.1 we set $H:=L'/L$, and for

$$Tf=\sum_{l'} p_{l'}\boldsymbol{t}^{l'} \quad\text{we define}\quad Tf_h:=\sum_{[l']=h} p_{l'}\boldsymbol{t}^{l'}, \tag{4.2.4}$$

we obtain a decomposition of Tf as a sum $\sum_h Tf_h\in\mathbb{Z}[\![\boldsymbol{t}^{1/\mathfrak{d}}]\!][\boldsymbol{t}^{-1/\mathfrak{d}}][H]$ (with $\mathfrak{d}=|H|$).

In our cases we always start with this group $L'/L=H$ (hence f determines its decomposition $\sum_h f_h$). Nevertheless, some alterations will appear. First, we might consider the nonequivariant case, hence we can forget the decomposition over H. Another case appears as follows. In order to simplify the rational function we will eliminate some of its variables (eg, we substitute $t_i=1$ for certain indices i), or we restrict f to a linear subspace V. Then, after this substitution, the restricted function $f|_{t_i=1}$ will not determine anymore the restrictions $(f_h)|_{t_i=1}$ of the 'old' components f_h. That is, the new pair of lattices $(L_V,L'_V)=(L\cap V,L'\cap V)$ and the 'old group' $H=L'/L$ become rather independent. In such cases we will keep the old group $H=L'/L$ (and the 'old' decomposition f_h) without asking any compatibility with L'_V/L_V.

4.2.5 Since all the coordinates $a_{i,v}$ of a_i are strict positive, for any $Tf(\boldsymbol{t})=\sum_{l'} p_{l'}\boldsymbol{t}^{l'}$ we get a well defined *counting function* of the coefficients,

$$l'\mapsto Q(l'):=\sum_{l''\ngeq l'} p_{l''}.$$

If $Tf=\sum_h Tf_h$, then each Tf_h determines a counting function Q_h defined in the same way.

If $H=L'/L$ and Tf decomposes into $\sum_h Tf_h$ under the law from (4.2.4), then

$$\sum_{l''\ngeq l'} p_{l''}\cdot[l'']=\sum_{h\in H} Q_h(l')[h]. \tag{4.2.6}$$

The definitions are motivated by formulae (2.2.3) and (2.3.2). The functions $Q_h(l')$ will be studied in the next subsections via Ehrhart theory.

4.3 Ehrhart quasipolynomials associated with denominators of rational functions

First we consider the case $d>0$, the special case $d=0$ will be treated in Section 4.3.19.

4.3.1 The polytope associated with $\{a_i\}_{i=1}^d$ In order to run the Ehrhart theory we have first to fix the lattice $\mathcal{X}$ and the representation $\rho\colon \mathcal{X}\to\mathfrak{H}$; cf Section 3. First, set $\mathcal{X}=\mathbb{Z}^d$ and $\alpha\colon \mathcal{X}\to L'$ given by $\alpha(\boldsymbol{x})=\sum_{i=1}^d x_ia_i\in L'$. In the sequel we consider two possibilities for $(\mathfrak{H},\rho)$ which basically will cover all the cases we wish to study (equivariant/nonequivariant cases combined with situations before or after the reduction of variables, see the comment in Section 4.2.3):

(a) $\mathfrak{H}=H=L'/L$ and ρ is the composition $\mathcal{X}\xrightarrow{\alpha}L'\to L'/L$

(b) $\mathfrak{H}=0$ and $\rho=0$

This choice has an effect on the equivariant decomposition $f^e=\sum_g f_g[g]$ of f too. In case (a) usually we have $G=H$ and the decomposition is given by (4.2.4). In case (b) we can take either $G=0$ (this can happen eg when we forget the decomposition in case (a), and we sum up all the components), or we can take any G (by specifying each f_g). In this latter case each fixed f_g behaves like a function in the nonequivariant case $G=0$, hence can be treated in the same way.

Since the case (b) follows from case (a) (by forgetting the extra information from $\mathfrak{H}$), in the sequel we provide the details for case (a). Hence let us assume $\mathfrak{H}=G=L'/L$.

Consider the matrix A with column vectors $|H|a_i$ and write A_v for its rows. Then the construction of (3.1.1) can be repeated (eventually completing each A_v to assure the inequalities $x_i\geq 0$ as well). For $l\in\sum_v l_vE_v\in L$ consider

$$(4.3.2)\qquad P_v^{\triangleleft}:=\Big\{\boldsymbol{x}\in(\mathbb{R}_{\geq 0})^d\ \Big|\ |H|\cdot\sum_i x_ia_{i,v}<l_v\Big\}\quad\text{and}\quad P^{\triangleleft}:=\bigcup_{v=1}^{s}P_v^{\triangleleft}.$$

The closure P_v of $P_v^{\triangleleft}$ is a dilated convex (simplicial) polytope depending on the one-dimensional parameter l_v. Moreover, $P^{\triangleleft}$ is described via the partial ordering of $L\otimes\mathbb{R}$ as the set $\{l\mid\sum_i x_ia_i\not\geq l/|H|\}$. Since $L'\subset L/|H|$, we can restrict ourself to the lattice L' (preserving all the general results of Section 3). Hence for any $l'\in L'$ we set

$$(4.3.3)\qquad P^{(l'),\triangleleft}:=\Big\{\boldsymbol{x}\in(\mathbb{R}_{\geq 0})^d\ \Big|\ \sum_i x_ia_i\not\geq l'\Big\},\quad P^{(l')}=\text{closure of }(P^{(l'),\triangleleft}).$$

The combinatorial type of $P^{(l')}$ might vary with l'. Nevertheless, by definition, the facets will be grouped for all different combinatorial types by the same principle: we consider the coordinate facets $F_i := P^{(l')} \cap \{x_i = 0\}$, $1 \le i \le d$, and we denote by $\mathcal{T}$ the collection of all other facets. Hence $P^{(l'),\triangleleft} = P^{(l')} \setminus \bigcup_{F^{(l')} \in \mathcal{T}} F^{(l')}$. The construction is motivated by the summation from (2.2.3) (although in the general statements the choice of $\mathcal{T}$ is irrelevant).

Then Theorem 2.2.2 and Section 4.1 lead to the next counting function defined in the group ring $\mathbb{Z}[H]$ of H:

$$\mathcal{L}^e(\boldsymbol{A}, \mathcal{T}, l') := \sum_{h \in H} \mathcal{L}_h(\boldsymbol{A}, \mathcal{T}, l') \cdot [h] := \sum 1 \cdot [l''] \in \mathbb{Z}[H], \tag{4.3.4}$$

where the last sum runs over $l'' \in (P^{(l')} \setminus \bigcup_{F^{(l')} \in \mathcal{T}} F^{(l')}) \cap L' = P^{(l'),\triangleleft} \cap L'$.

The corresponding nonequivariant counting function, corresponding to $G = 0$, is denoted by

$$\mathcal{L}_{\mathrm{ne}}(\boldsymbol{A}, \mathcal{T}, l') := \sum_{h \in H} \mathcal{L}_h(\boldsymbol{A}, \mathcal{T}, l') \in \mathbb{Z}.$$

Similarly, we set $\mathcal{L}^e(\boldsymbol{A}, \mathcal{F} \setminus \mathcal{T}, l')$ too. For both of them Theorem 3.1.2 applies.

By the very construction, we have the following identity. Consider the equivariant Taylor expansion at the origin of the function determined by the *denominator of f*, namely

$$\widetilde{f}^e(\boldsymbol{t}) = \frac{1}{\prod_{i=1}^d (1 - [a_i]\boldsymbol{t}^{a_i})}; \qquad T\widetilde{f}^e(\boldsymbol{t}) = \sum_{l''} \widetilde{p}_{l''} \boldsymbol{t}^{l''} \cdot [l''] \in \mathbb{Z}[\![\boldsymbol{t}^{1/|H|}]\!][H]. \tag{4.3.5}$$

Note that since all the $\{E_v\}$–coefficients of each a_i are strict positive, for any $l' \in L'$ the set $\{l'' \mid \widetilde{p}_{l''} \neq 0,\ l'' \not\geq l'\}$ is finite. Then, by the above construction,

$$\sum_{l'' \not\geq l'} \widetilde{p}_{l''} \cdot [l''] = \mathcal{L}^e(\boldsymbol{A}, \mathcal{T}, l'). \tag{4.3.6}$$

4.3.7 Combinatorial types, chambers Next, we wish to make precise the *combinatorial stability* condition. The result of Sturmfels [59], Brion and Vergne [12], Clauss and Loechner [17] and Szenes and Vergne [60] implies that $\mathcal{L}^e$ from (4.3.6) (that is, each $\mathcal{L}_h$) is a *piecewise quasipolynomial* on L': the parameter space $L \otimes \mathbb{R}$ decomposes into several chambers, the restriction of $\mathcal{L}^e$ on each chamber is a quasipolynomial and $\mathcal{L}^e$ is continuous. The chambers are described as follows.

Notice that the combinatorial type of $P^{(l')}$ in (4.3.3) vary in the same way as the closure of its *convex* complement in $\mathbb{R}^d_{\geq 0}$, namely

$$\Big\{x \in (\mathbb{R}_{\geq 0})^d \;\Big|\; \sum_i x_i a_i \geq l'\Big\}, \tag{4.3.8}$$

since both are determined by their common boundary $\mathcal{T}$. The inequalities of (4.3.8) can be viewed as a *vector partition* $\sum_i x_i a_i + \sum_v y_v(-E_v) = l'$, with $x_i \geq 0$ and $y_v \geq 0$. Hence, according to the above references, we have the following chamber decomposition of $L \otimes \mathbb{R}$.

Let $\boldsymbol{M}$ be the matrix with column vectors $\{a_i\}_{i=1}^d$ and $\{-E_v\}_{v=1}^s$. A subset σ of indices of columns is called *basis* if the corresponding columns form a basis of $L \otimes \mathbb{R}$; in this case we write $\mathrm{Cone}(\boldsymbol{M}_\sigma)$ for the positive closed cone generated by them. Then the chamber decomposition is the polyhedral subdivision of $L \otimes \mathbb{R}$ provided by the common refinement of the cones $\mathrm{Cone}(\boldsymbol{M}_\sigma)$, where σ runs all over the basis. *A chamber is a closed cone of the subdivision whose interior is nonempty.* Usually we denote them by $\mathcal{C}$, let their index set (collection) be $\mathfrak{C}$.

We will need the associated *disjoint* decomposition of $L \otimes \mathbb{R}$ with relative open cones as well. A typical element of this disjoint decomposition is the *relative interior of an intersection of type* $\bigcap_{\mathcal{C}\in\mathfrak{C}'} \mathcal{C}$, where $\mathfrak{C}'$ runs over the subsets of $\mathfrak{C}$. For these cones we use the notation $\mathcal{C}_{\mathrm{op}}$.

Each chamber $\mathcal{C}$ determines an open cone, namely its interior. And, conversely, each top dimensional open cone determines a chamber $\mathcal{C}$, namely its closure.

The next theorem is the direct consequence of Brion and Vergne [12, 4.4], Szenes and Vergne [60, 0.2] and 3.1.2 using the additivity of the Ehrhart quasipolynomial on the suitable convex parts of $P^{(l')}$. (We state it for our specific facet-collection $\mathcal{T}$, the case which will be used later, but it is true for any other facet-decomposition of the boundary whenever $\bigcup_{F^{(l')}\in\mathcal{T}} F^{(l')}$ is homeomorphic to a $(d-1)$–manifold.)

4.3.9 Theorem (a) *For each relative open cone $\mathcal{C}_{\mathrm{op}}$ of $L \otimes \mathbb{R}$, $P^{(l')}$ is combinatorially stable, that is, the polytopes $\{P^{(l')}\}_{l'\in\mathcal{C}_{\mathrm{op}}}$ have the same combinatorial type. Therefore, for any fixed $h \in H$, the restrictions $\mathcal{L}_h^{\mathcal{C}_{\mathrm{op}}}(\boldsymbol{A}, \mathcal{T})$ and $\mathcal{L}_h^{\mathcal{C}_{\mathrm{op}}}(\boldsymbol{A}, \mathcal{F}\setminus\mathcal{T})$ to $\mathcal{C}_{\mathrm{op}}$ of $\mathcal{L}_h(\boldsymbol{A}, \mathcal{T})$ and $\mathcal{L}_h(\boldsymbol{A}, \mathcal{F}\setminus\mathcal{T})$ respectively are quasipolynomials.*

(b) *These quasipolynomials have a continuous extension to the closure of $\mathcal{C}_{\mathrm{op}}$. Namely, if $\mathcal{C}'_{\mathrm{op}}$ is in the closure of $\mathcal{C}_{\mathrm{op}}$, then $\mathcal{L}_h^{\mathcal{C}'_{\mathrm{op}}}(\boldsymbol{A}, \mathcal{T})$ is the restriction to $\mathcal{C}'_{\mathrm{op}}$ of the (abstract) quasipolynomial $\mathcal{L}_h^{\mathcal{C}_{\mathrm{op}}}(\boldsymbol{A}, \mathcal{T})$. (Similarly for $\mathcal{L}_h^{\mathcal{C}_{\mathrm{op}}}(\boldsymbol{A}, \mathcal{F}\setminus\mathcal{T})$.)*

Specifically, for any chamber $\mathcal{C}$ there is a well defined quasipolynomial $\mathcal{L}_h^{\mathcal{C}}(\boldsymbol{A},\mathcal{T})$, defined as $\mathcal{L}_h^{\mathcal{C}_{\mathrm{op}}}(\boldsymbol{A},\mathcal{T})$, where $\mathcal{C}_{\mathrm{op}}$ is the interior of $\mathcal{C}$, which equals $\mathcal{L}_h(\boldsymbol{A},\mathcal{T})$ for all points of $\mathcal{C}$.

This also shows that for any two chambers $\mathcal{C}_1$ and $\mathcal{C}_2$ one has the continuity property

$$\mathcal{L}_h^{\mathcal{C}_1}(\boldsymbol{A},\mathcal{T})|_{\mathcal{C}_1\cap\mathcal{C}_2} = \mathcal{L}_h^{\mathcal{C}_2}(\boldsymbol{A},\mathcal{T})|_{\mathcal{C}_1\cap\mathcal{C}_2}. \tag{4.3.10}$$

(c) *$\mathcal{L}_h^{\mathcal{C}}(\boldsymbol{A},\mathcal{T})$ and $\mathcal{L}_{-h}^{\mathcal{C}}(\boldsymbol{A},\mathcal{F}\setminus\mathcal{T})$, as abstract quasipolynomials associated with a fixed chamber $\mathcal{C}$, satisfy the reciprocity*

$$\mathcal{L}_h^{\mathcal{C}}(\boldsymbol{A},\mathcal{T},l') = (-1)^d\cdot\mathcal{L}_{-h}^{\mathcal{C}}(\boldsymbol{A},\mathcal{F}\setminus\mathcal{T},-l').$$

We have the following consequences regarding the counting function $l'\mapsto Q_h(l')$ of $f^e(\boldsymbol{t})$ defined in (4.2.6):

4.3.11 Corollary (a) *Q_h is a piecewise quasipolynomial. Indeed, for any $h\in H$ and $l'\in L'$*

$$Q_h(l') = \sum_k \iota_k\cdot\mathcal{L}_{h-[b_k]}(\boldsymbol{A},\mathcal{T},l'-b_k). \tag{4.3.12}$$

In particular, the right-hand side of (4.3.12) *is independent of the representation of f as in* (4.2.2) *(that is, of the choice of $\{b_k,a_i\}_{k,i}$), it depends only on the rational function f.*

(b) *Fix a chamber $\mathcal{C}$ of $L\otimes\mathbb{R}$, cf Theorem 4.3.9, and for any $h\in H$ define the quasipolynomial*

$$\bar{Q}_h^{\mathcal{C}}(l') := \sum_k \iota_k\cdot\mathcal{L}_{h-[b_k]}^{\mathcal{C}}(\boldsymbol{A},\mathcal{T},l'-b_k). \tag{4.3.13}$$

Then the restriction of $Q_h(l')$ to $\bigcap_k(b_k+\mathcal{C})$ is a quasipolynomial, namely

$$Q_h(l') = \bar{Q}_h^{\mathcal{C}}(l') \quad \text{on} \quad \bigcap_k(b_k+\mathcal{C}). \tag{4.3.14}$$

Moreover, there exists $l'_*\in\mathcal{C}$ such that $l'_*+\mathcal{C}\subset\bigcap_k(b_k+\mathcal{C})$.

(Warning: $\mathcal{L}_{h-[b_k]}^{\mathcal{C}}(\boldsymbol{A},\mathcal{T},l'-b_k)\neq\mathcal{L}_{h-[b_k]}(\boldsymbol{A},\mathcal{T},l'-b_k)$ unless $l'-[b_k]\in\mathcal{C}$.)

(c) *For any fixed $h\in H$, the quasipolynomial $\bar{Q}_h^{\mathcal{C}}(l')$ satisfies the following property: For any $l'\in L'$ with $[l']=h$, and any $q\in\square$ (the semiopen unit cube), one has*

$$\bar{Q}_h^{\mathcal{C}}(l') = \bar{Q}_h^{\mathcal{C}}(l'-q). \tag{4.3.15}$$

In particular, by taking $l'=q=r_h$:

$$\bar{Q}_h^{\mathcal{C}}(r_h) = \bar{Q}_h^{\mathcal{C}}(0) \tag{4.3.16}$$

Proof For (a) use (4.3.3) and the fact that $b_k + \sum x_i a_i \not\geq l'$ if and only if $\sum x_i a_i \not\geq l' - b_k$. Since the coefficients of the Taylor expansion depend only on f, the second sentence follows too.

For (b) use part (a) and the fact that $\mathcal{C} \cap \bigcap_k (b_k + \mathcal{C})$ contains a set of type $l'_* + \mathcal{C}$.

For (c) consider those values l' in some $l'_* + \mathcal{C}$ for which all elements of type $l' - b_k$ and $l' - q - b_k$ are in $\mathcal{C}$. For these values l', (4.3.15) follows from the identity $P^{(l'),\triangleleft} \cap \rho^{-1}(h) = P^{(l'-q),\triangleleft} \cap \rho^{-1}(h)$ whenever $[l'] = h$. This is true since for any l'' with $[l''] = [l']$, $l'' \geq l'$ is equivalent with $l'' \geq l' - q$. Indeed, taking $y = l'' - l'$, this reads as follows: for any $y \in L$, $y \geq 0$ if and only if $y \geq -q$.

Now, if two quasipolynomials agree on $l'_0 + \mathcal{C}$ then they are equal. □

4.3.17 Remark Thanks to Szenes and Vergne [60, Theorem 0.2], the continuity property (4.3.10) has the following extension (coincidence of the quasipolynomials on neighboring strips). Set $\square(\boldsymbol{A}) := \sum_i [0,1) a_i$. Then for any two chambers $\mathcal{C}_1$ and $\mathcal{C}_2$, and $S := (-\square(\boldsymbol{A}) + \mathcal{C}_1) \cap (-\square(\boldsymbol{A}) + \mathcal{C}_2)$

$$\mathcal{L}_h^{\mathcal{C}_1}(\boldsymbol{A}, \mathcal{T})|_S = \mathcal{L}_h^{\mathcal{C}_2}(\boldsymbol{A}, \mathcal{T})|_S. \tag{4.3.18}$$

4.3.19 The $d = 0$ case All the above properties can be extended for $d = 0$ as well. Although the polytope constructed in (4.3.3) does not exist, we can look at the polynomial $f(\boldsymbol{t}) = \sum_k \iota_k \boldsymbol{t}^{b_k}$ itself. Then using the notation of (4.2.6) we set

$$\sum_{h \in H} Q_h(l')[h] = \sum_{l'' \not\geq l'} p_{l''} \cdot [l''] = \sum_{\{k \mid b_k \not\geq l'\}} \iota_k [b_k].$$

Moreover, we have the chamber decomposition of $L \otimes \mathbb{R}$ defined by $\{-E_v\}_{v=1}^s$ via the same principle as above. This means two chambers: $\mathcal{C}_0 := \mathbb{R}_{\geq 0}\langle -E_v \rangle$ and $\mathcal{C}_1$, the closure of the complement of $\mathcal{C}_0$ in $\mathbb{R}^s$. Then $Q_h(l') = \sum_{\{k \mid [b_k] = h\}} \iota_k$ on $\bigcap_k (b_k + \mathcal{C}_1)$ and 0 on $\bigcap_k (b_k + \mathcal{C}_0)$.

4.4 The definition of the multivariable equivariant periodic constant of a rational function

We consider the situation of Sections 4.2.1 and 4.3.1(a). For each $h \in H$ define $r_h \in L'$ as in Section 2.1.

4.4.1 Definition Let $\mathcal{K} \subset L' \otimes \mathbb{R}$ be a closed real cone whose affine closure aff($\mathcal{K}$) has positive dimension. For any $h \in H$ we assume that there exist

- $l'_* \in \mathcal{K}$,
- a sublattice $\widetilde{L} \subset L$ of finite index,

- a quasipolynomial $l' \mapsto \widetilde{Q}_h(l')$, defined on $\widetilde{L} \cap \mathrm{aff}(\mathcal{K})$ such that

$$Q_h(l') = \widetilde{Q}_h(l') \quad \text{for any} \quad \widetilde{L} \cap (l'_* + \mathcal{K}). \tag{4.4.2}$$

Then we define the *equivariant periodic constant* of f associated with $\mathcal{K}$ by

$$\mathrm{pc}^{e,\mathcal{K}}(f) = \sum_{h\in H} \mathrm{pc}_h^{\mathcal{K}}(f)\cdot[h] := \sum_{h\in H} \widetilde{Q}_h(0)\cdot[h] \in \mathbb{Z}[H], \tag{4.4.3}$$

and we say that f *admits a periodic constant in* $\mathcal{K}$. (Sometimes we will use the same notation for the real cone $\mathcal{K}$ and for its lattice points $\mathcal{K}\cap L'$ in L'.)

4.4.4 Remark The above definition is independent of the choice of the sublattice $\widetilde{L}$: it can be replaced by any sublattice of finite index. The advantage of such sublattices is that convenient restrictions of Q_h might have nicer forms which are easier to compute. The choice of $\widetilde{L}$ corresponds to the choice of p in Section 4.1, and it is responsible for the name 'periodic' in the name of $\mathrm{pc}^{e,\mathcal{K}}(f)$.

4.4.5 Proposition (a) *Consider the chamber decomposition of $L\otimes\mathbb{R}$ given by the denominator $\prod_i(1-\boldsymbol{t}^{a_i})$ of f as in Theorem 4.3.9. Then f admits a periodic constant in each chamber $\mathcal{C}$ and*

$$\mathrm{pc}_h^{\mathcal{C}}(f) = \overline{Q}_h^{\mathcal{C}}(r_h) = \overline{Q}_h^{\mathcal{C}}(0). \tag{4.4.6}$$

(b) *If two functions f_1 and f_2 admit periodic constant in some cone $\mathcal{K}$, then the same is true for $\alpha_1 f_1+\alpha_2 f_2$ and*

$$\mathrm{pc}^{\mathcal{K}}(\alpha_1 f_1+\alpha_2 f_2) = \alpha_1\,\mathrm{pc}^{\mathcal{K}}(f_1)+\alpha_2\,\mathrm{pc}^{\mathcal{K}}(f_2) \quad (\alpha_1,\alpha_2\in\mathbb{C}).$$

(c) *If f admits periodic constants in two (top dimensional) cones $\mathcal{K}_1$ and $\mathcal{K}_2$, and the interior* $\mathrm{int}(\mathcal{K}_1\cap\mathcal{K}_2)$ *of the intersection $\mathcal{K}_1\cap\mathcal{K}_2$ is nonempty, then* $\mathrm{pc}^{\mathcal{K}_1}(f)=\mathrm{pc}^{\mathcal{K}_2}(f)$.

In particular, if $\{\mathcal{C}_i\}_{i=1,2}$ are two chambers as in (a)*, and f admits a periodic constant in $\mathcal{K}$, and* $\mathrm{int}(\mathcal{C}_i\cap\mathcal{K})\neq\varnothing$ *($i=1,2$), then* $\mathrm{pc}^{\mathcal{C}_1}(f)=\mathrm{pc}^{\mathcal{C}_2}(f)$.

Proof For (a) use Corollary 4.3.11; (b) is clear. For (c) we can assume that $\mathcal{K}_2\subset\mathcal{K}_1$ (by considering $\mathcal{K}_i$ and $\mathcal{K}_1\cap\mathcal{K}_2$). Then if Q_h is quasipolynomial on $l'_1+\mathcal{K}_1$ (with $l'_1\in\mathcal{K}_1$), then $(l'_1+\mathcal{K}_2)\cap\mathcal{K}_2$ contains a set of type $l'_2+\mathcal{K}_2$ with $l'_2\in\mathcal{K}_2$, on which one can take the restriction of the previous quasipolynomial. □

4.4.7 Remark Note that in the rational presentation of f we might assume that $a_i \in L$ for all i. Indeed, take $o_i \in \mathbb{Z}_{>0}$ such that $o_i a_i \in L$, and amplify the fraction by $\prod_i (1-\boldsymbol{t}^{o_i a_i})/(1-\boldsymbol{t}^{a_i})$. Therefore, for each h we can write $f_h(\boldsymbol{t})$ in the form

$$f_h(\boldsymbol{t}) = \boldsymbol{t}^{r_h} \sum_k \iota_k \cdot \frac{\boldsymbol{t}^{\bar{b}_k}}{\prod_i (1-\boldsymbol{t}^{a_i})},$$

where $a_i, \bar{b}_k \in L$, hence $f_h(\boldsymbol{t})/\boldsymbol{t}^{r_h} \in \mathbb{Z}[\![\boldsymbol{t}]\!][\boldsymbol{t}^{-1}]$. Then if we consider the nonequivariant periodic constant $\mathrm{pc}^{\mathcal{C}}$ of $f_h(\boldsymbol{t})/\boldsymbol{t}^{r_h}$, (4.2.6), (4.3.14) and (4.4.6) imply that $\mathrm{pc}^{\mathcal{C}}_h(f(\boldsymbol{t})) = \mathrm{pc}^{\mathcal{C}}(f_h(\boldsymbol{t})/\boldsymbol{t}^{r_h})$ for all chambers $\mathcal{C}$ associated with $\{a_i\}_i$.

4.4.8 Example Assume that $L = L' = \mathbb{Z}$ and $\mathcal{K} = \mathbb{R}_{\geq 0}$, and consider $S(t)$ as in Section 4.1. If $S(t)$ admits a periodic constant in $\mathcal{K}$, then $\mathrm{pc}^{\mathcal{K}}(S) = \mathrm{pc}(S)$, where $\mathrm{pc}(S)$ is the periodic constant defined in Section 4.1.

4.4.9 Example (The $d = 0$ case) (a) Assume that $f(\boldsymbol{t}) = \sum_{k=1}^{r} \iota_k \boldsymbol{t}^{b_k}$. Using Section 4.3.19 (and its notation), we have both $\mathrm{pc}^{e,\mathcal{C}_0}(f) = 0$ and $\mathrm{pc}^{e,\mathcal{C}_1}(f) = \sum_{k=1}^{r} \iota_k [b_k] \in \mathbb{Z}[H]$.

(b) Assume that the rank is $s = 2$ and $f(\boldsymbol{t}) = \boldsymbol{t}^b/(1-\boldsymbol{t}^a)$, with both entries (a_1, a_2) of a positive. We assume that $a \in L$ while $b \in L'$. Again, for $h \neq [b]$ the counting function, hence its periodic constant too, is zero. Assume $h = [b]$, and write $b = (b_1, b_2)$. Then the denominator provides three chambers: $\mathcal{C}_0 := \mathbb{Z}_{\geq 0}\langle -E_1, -E_2\rangle$, $\mathcal{C}_1 := \mathbb{Z}_{\geq 0}\langle a, -E_2\rangle$, $\mathcal{C}_2 := \mathbb{Z}_{\geq 0}\langle a, -E_1\rangle$. Then the three quasipolynomials for $1/(1-\boldsymbol{t}^a)$ are $\mathcal{L}^{\mathcal{C}_0}_h = 0$, $\mathcal{L}^{\mathcal{C}_i}_h(n_1, n_2) = \lceil n_i/a_i \rceil$; hence $\mathrm{pc}^{\mathcal{C}_0}_h(f) = 0$, $\mathrm{pc}^{\mathcal{C}_i}_h(f) = \lceil -b_i/a_i \rceil$ $(i = 1, 2)$. In particular, $\mathrm{pc}^{\mathcal{C}}_h(f)$, in general, depends on the choice of $\mathcal{C}$.

(c) Assume that $L = L'$ and $f(t) = (t_1^{b_1} t_2^{b_2})/((1-t_1 t_2)(1-t_1^2 t_2))$. Then the chambers associated with the denominator are: $\mathcal{C}_0 := \mathbb{R}_{\geq 0}\langle -E_1, -E_2\rangle$, $\mathcal{C}_2 := \mathbb{R}_{\geq 0}\langle -E_1, (1,1)\rangle$, $\mathcal{C} := \mathbb{R}_{\geq 0}\langle (1,1), (2,1)\rangle$ and $\mathcal{C}_1 := \mathbb{R}_{\geq 0}\langle (2,1), -E_2\rangle$. Then, by a computation,

$$\begin{aligned}
\mathcal{L}^{\mathcal{C}_0} &= 0,\\
\mathcal{L}^{\mathcal{C}_2}(l_1, l_2) &= \frac{l_2^2}{2} + \frac{l_2}{2},\\
\mathcal{L}^{\mathcal{C}}(l_1, l_2) &= \frac{l_1^2}{2} + l_2^2 + \frac{l_1}{2} - l_1 l_2,\\
\mathcal{L}^{\mathcal{C}_1}(l_1, l_2) &= \frac{l_1^2}{4} + \frac{l_1}{2} + \frac{1+(-1)^{l_1+1}}{8}.
\end{aligned} \tag{4.4.10}$$

Hence, by Proposition 4.4.5 and (4.3.13), one has $\mathrm{pc}^{\mathcal{C}_*}(f) = \mathcal{L}^{\mathcal{C}_*}(-b_1, -b_2)$.

4.4.11 Example (Normal affine monoids) Consider the following objects (compare with Section 4.2.1): a lattice L with fixed bases $\{E_v\}_{v=1}^d$ (hence $s=d$) and with induced partial ordering $\leq$, $L'\subset L\otimes\mathbb{Q}$ an overlattice with finite abelian quotient $H:=L'/L$ and projection $\rho\colon L'\to H$. Furthermore, let $\{a_i\}_{i=1}^d$ be linearly independent vectors in L' with all their $\{E_v\}$–coordinates positive. Let $\mathcal{K}$ be the positive real cone generated by the vectors $\{a_i\}_i$, and consider the Hilbert series of $\mathcal{K}$,

$$f(\boldsymbol{t}) := \sum_{l'\in\mathcal{K}\cap L'} \boldsymbol{t}^{l'}.$$

Since $\mathcal{K}$ depends only on the rays generated by the vectors a_i, we can assume that $a_i\in L$ for all i.

Set $\square(A)=\sum_{i=1}^d[0,1)a_i$ as above, and consider the monoid $M:=\mathbb{Z}_{\geq 0}\langle a_i\rangle$; cf eg Bruns and Gubeladze [13, 2.C]. Then the normal affine monoid $\mathcal{K}\cap L'$ is a module over M and if we set $B:=\square(A)\cap L'$, [13, Proposition 2.43] implies that

$$\mathcal{K}\cap L' = \bigsqcup_{b\in B} b+M.$$

In particular, $f(\boldsymbol{t})$ equals $\sum_{b\in B}\boldsymbol{t}^b/\prod_{i=1}^d(1-\boldsymbol{t}^{a_i})$ and has the form considered in Section 4.2.

If the rank d is ≥ 3 then $\mathcal{K}$ usually is cut in more chambers. Indeed, take eg $d=3$, $a_i=(1,1,1)+E_i$ for $i=1,2,3$. Then $\mathcal{K}$ is cut in its barycentric subdivision. Nevertheless, if $d=2$ then $\mathcal{K}$ consists of a unique chamber and f admits a periodic constant in $\mathcal{K}$. Indeed, one has the following.

4.4.12 Lemma *If $d=2$ then $\mathrm{pc}_h^{\mathcal{K}}(f)=0$ for all $h\in H$.*

Proof It is elementary to see that $\mathcal{K}$ is one of the chambers (use the construction from Section 4.3.7). Take $B=\{b_k\}_k$, and write $f=\sum_k f_k$, where $f_k=\boldsymbol{t}^{b_k}/(1-\boldsymbol{t}^{a_1})(1-\boldsymbol{t}^{a_2})$. The only relevant classes $h\in H$ are given by $\{[b_i]\mid b_i\in B\}$, otherwise already the Ehrhart quasipolynomials are zero (since $a_i\in L$). Fix such a class $h=[b_i]$. Let $\mathcal{L}_h^{\mathcal{K}}(\mathcal{T})$ be the quasipolynomial associated with the chamber $\mathcal{K}$ and the denominator of f. Then, by (4.4.6) and (4.3.13), $\mathrm{pc}_h^{\mathcal{K}}(f_k)=\mathcal{L}_{[b_i-b_k]}^{\mathcal{K}}(\mathcal{T},-b_k)$. This, by the Reciprocity Law 4.3.9(c), equals $\mathcal{L}_{[b_k-b_i]}^{\mathcal{K}}(\mathcal{F}\setminus\mathcal{T},b_k)$. Again, since the denominator is a series in L, for $[b_k-b_i]\neq 0$ the series is zero; so we may assume $[b_k-b_i]=0$. But, since $b_k\in\mathcal{K}$, the value $\mathcal{L}_0^{\mathcal{K}}(\mathcal{F}\setminus\mathcal{T},b_k)$ of the quasipolynomial carries its geometric meaning, it is the cardinality of the set $\{m=n_1a_1+n_2a_2\mid n_1>0,\ n_2>0,\ m\not> b_k\}$. But since for any such m one has $m\geq a_1+a_2>b_k$, contradicting $m\not> b_k$, this set is empty. □

4.4.13 Example (General affine monoids of rank $d = 2$) Consider the situation of Example 4.4.11 with $d = 2$, and let N be a submonoid of $\widehat{N} = \mathcal{K} \cap L'$ of rank 2, and we also assume that $\widehat{N}$ is the normalization of N. Set

$$f(\boldsymbol{t}) := \sum_{l' \in N} \boldsymbol{t}^{l'}.$$

Then $f(\boldsymbol{t})$ is again of type (4.2.2). Indeed, by Bruns and Gubeladze [13, Proposition 2.35], $\widehat{N} \setminus N$ is a union of a finite family of sets of type (I) $b \in \widehat{N}$, or (II) $b + \mathbb{Z}\ell a_i$, where $b \in \widehat{N}$, $\ell \in \mathbb{Z}_{\geq 0}$, $i = 1$ or 2. Obviously, two sets of type (II) with different i–values might have an intersection point of type (I). In particular,

$$f(\boldsymbol{t}) = \sum_{l' \in \widehat{N}} \boldsymbol{t}^{l'} - \sum_i \frac{\boldsymbol{t}^{b_{i,1}}}{1 - \boldsymbol{t}^{k_{i,1}a_1}} - \sum_j \frac{\boldsymbol{t}^{b_{j,2}}}{1 - \boldsymbol{t}^{k_{j,2}a_2}} + \sum_k (\pm \boldsymbol{t}^{b_k}).$$

Note that the periodic constant of the first sum is zero by Lemma 4.4.12, and the others can easily be computed (even with closed formulae) via Example 4.4.9, parts (a) and (b).

The computation shows that the periodic constant carries information about the failure of normality of N (compare with the delta invariant computation from the end of Section 4.1).

The situation is similar when we consider a *semigroup* of $\widehat{N}$, that is, when we eliminate the neutral element of the above N (or, when we consider a module over the submonoid $N \subset \widehat{N}$).

4.4.14 Example (Reduction of variables) The next statement is an example when the number of variables of the function f can be reduced in the procedure of the periodic constant computation. (For another reduction result, see Reduction Theorem 5.4.2.) For simplicity we assume $L' = L$.

4.4.15 Proposition *Let $f(\boldsymbol{t}) = \boldsymbol{t}^b / \prod_{i=1}^d (1 - \boldsymbol{t}^{a_i})$ and assume $b = \sum_{v=1}^s b_v E_v \in \mathcal{C}$, where $\mathcal{C}$ is a chamber associated with the denominator.*

We consider the subset $\mathrm{Pos} := \{v \mid b_v > 0\}$ with cardinality p, and the projection $\mathbb{R}^s \to \mathbb{R}^p$, defined by $(r_v)_{v=1}^s \mapsto (r_v)_{v \in \mathrm{Pos}}$ and denoted by $v \mapsto v^\dagger$. Accordingly, we set a new function

$$f^\dagger(\boldsymbol{z}) := \frac{\boldsymbol{z}^{b^\dagger}}{\prod_{i=1}^d (1 - \boldsymbol{z}^{a_i^\dagger})}$$

in p variables, and a new chamber $\mathcal{C}^\dagger := \mathbb{R}_{\geq 0}\langle \{w_j^\dagger\}_j \rangle$, where w_j are the generators of $\mathcal{C} = \mathbb{R}_{\geq 0}\langle \{w_j\}_j \rangle$. Then $\mathrm{pc}^{\mathcal{C}}(f) = \mathrm{pc}^{\mathcal{C}^\dagger}(f^\dagger)$.

Proof This is a direct application of Theorem 3.1.2(b). Indeed, by the Ehrhart–MacDonald–Stanley reciprocity law, we get $\mathrm{pc}^{\mathcal{C}}(f) = \mathcal{L}^{\mathcal{C}}(\boldsymbol{A}_v, \mathcal{T}, -b) = (-1)^d \cdot \mathcal{L}^{\mathcal{C}}(\boldsymbol{A}, \mathcal{F} \setminus \mathcal{T}, b)$. Since $b \in \mathcal{C}$, by the very definition of $\mathcal{L}^{\mathcal{C}}(\boldsymbol{A}, \mathcal{F} \setminus \mathcal{T})$, this (modulo the sign) equals the number of integral points of $P^{(b)} \setminus \bigcup_{F^{(b)} \in \mathcal{F} \setminus \mathcal{T}} F^{(b)} \subset \mathbb{R}^d$. But, if $v \notin \mathrm{Pos}$, ie $b_v \leq 0$, then in (3.1.1) $P_v^{(b_v)}$ has only nonpositive integral points. Therefore we can omit these polytopes without affecting the periodic constant. Then, this fact and $b^\dagger \in \mathcal{C}^\dagger$ imply that $\mathrm{pc}^{\mathcal{C}}$ can be computed as $(-1)^d \mathcal{L}^{\mathcal{C}^\dagger}(\boldsymbol{A}^\dagger, \mathcal{F}^\dagger \setminus \mathcal{T}^\dagger, b^\dagger)$. □

4.4.16 Remark Under the conditions of Proposition 4.4.15 we have the following application of the statement from Remark 4.3.17 (based on Szenes and Vergne [60]): *Assume that* $b \in \square(\boldsymbol{A}) - \mathcal{C}$ *and* $b \geq 0$. *Then* $\mathrm{pc}^{\mathcal{C}}(f) = 0$. Indeed, $\mathrm{pc}^{\mathcal{C}}(f) = \mathcal{L}^{\mathcal{C}}(\boldsymbol{A}, \mathcal{T}, -b) = \mathcal{L}^{\mathcal{C}(-b)}(\boldsymbol{A}, \mathcal{T}, -b)$, where $\mathcal{C}(-b)$ is a chamber containing $-b$. But since $-b \leq 0$ one gets $\mathcal{L}^{\mathcal{C}(-b)}(\boldsymbol{A}, \mathcal{T}, -b) = 0$ by Proposition 4.4.15.

One of the key messages of the above examples (starting from 4.4.9) is the following: 'if b is small compared with the a_i, then the periodic constant is zero' (compare with Section 4.1 too).

4.5 The polynomial part of rational functions with $d = s = 2$

In this case $\mathrm{rank}(L) = 2$, and we have two vectors in the denominator of f, namely $a_i = (a_{i,1}, a_{i,2})$, $i = 1, 2$. We will order them in such a way that a_2 sits in the cone of a_1 and E_1, that is,

$$\det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} < 0.$$

The chamber decomposition will be the following: $\mathcal{C}_0 := \mathbb{R}_{\geq 0}\langle -E_1, -E_2 \rangle$, $\mathcal{C}_2 := \mathbb{R}_{\geq 0}\langle -E_1, a_1 \rangle$, $\mathcal{C} := \mathbb{R}_{\geq 0}\langle a_1, a_2 \rangle$ and $\mathcal{C}_1 := \mathbb{R}_{\geq 0}\langle a_2, -E_2 \rangle$ (the index choice is motivated by the formulae from Example 4.4.9(b)).

Our goal is to write any rational function (with denominator $(1 - \boldsymbol{t}^{a_1})(1 - \boldsymbol{t}^{a_2})$) as a sum of $f^+(\boldsymbol{t})$ and $f^-(\boldsymbol{t})$, such that $f^+ \in \mathbb{Z}[L']$ (the 'polynomial part of f'), and $\mathrm{pc}^{e,\mathcal{C}}(f^-) = 0$. This is a generalization of the decomposition in the one-variable case discussed in Section 4.1, and will be a major tool in the computation of the periodic constant in Section 7 for graphs with two nodes. The specific form of the decomposition is motivated by Examples 4.4.9(b) and 4.4.11.

As above, we set $\square(\boldsymbol{A}) = [0, 1)a_1 + [0, 1)a_2$ and for $i = 1, 2$ we also consider the strips

$$\Xi_i := \{b = (b_1, b_2) \in L \otimes \mathbb{R} \mid 0 \leq b_i < a_{i,i}\}.$$

4.5.1 Theorem (1) *Any function* $f(\boldsymbol{t}) = \big(\sum_{k=1}^r \iota_k \boldsymbol{t}^{b_k}\big)/\prod_{i=1}^2(1-\boldsymbol{t}^{a_i})$ *(with* $\iota_k \in \mathbb{Z}$*) can be written as a sum* $f(\boldsymbol{t}) = f^+(\boldsymbol{t}) + f^-(\boldsymbol{t})$*, where*

(a) $f^+(\boldsymbol{t})$ *is a finite sum* $\sum_\ell \kappa_\ell \boldsymbol{t}^{\beta_\ell}$*, with* $\kappa_\ell \in \mathbb{Z}$ *and* $\beta_\ell \in L'$*,*

(b) $f^-(\boldsymbol{t})$ *has the form*

$$f^-(\boldsymbol{t}) = \frac{\sum_{k=1}^r \iota_k \boldsymbol{t}^{b'_k}}{\prod_{i=1}^2(1-\boldsymbol{t}^{a_i})} + \frac{\sum_{i=1}^{n_1} \iota_{i,1}\boldsymbol{t}^{b_{i,1}}}{1-\boldsymbol{t}^{a_1}} + \frac{\sum_{i=1}^{n_2} \iota_{i,2}\boldsymbol{t}^{b_{i,2}}}{1-\boldsymbol{t}^{a_2}}, \tag{4.5.2}$$

with $b'_k \in L' \cap \square(A)$ *for all* k*, and* $b_{i,j} \in L' \cap \Xi_j$ *for any* i *and* $j = 1, 2$*, and* $\iota_k, \iota_{i,1}, \iota_{i,2} \in \mathbb{Z}$*.*

(2) *Consider a sum*

$$\Sigma(\boldsymbol{t}) := \frac{Q(\boldsymbol{t})}{\prod_{i=1}^2(1-\boldsymbol{t}^{a_i})} + \frac{Q_1(\boldsymbol{t})}{1-\boldsymbol{t}^{a_1}} + \frac{Q_2(\boldsymbol{t})}{1-\boldsymbol{t}^{a_2}} + f^+(\boldsymbol{t}), \tag{4.5.3}$$

where $Q(\boldsymbol{t}) := \sum_{k=1}^r \iota_k \boldsymbol{t}^{b'_k}$ *with* $b'_k \in L' \cap \square(A)$ *for all* k*;* $Q_j(\boldsymbol{t}) = \sum_{i=1}^{n_1} \iota_{i,j}\boldsymbol{t}^{b_{i,j}}$ *with* $b_{i,j} \in L' \cap \Xi_j$ *for any* i *and* $j = 1, 2$*; and finally* $f^+ \in \mathbb{Z}[L']$ *is a polynomial as in part* (a) *above.*

Then, if $\Sigma(\boldsymbol{t}) = 0$*, then* $Q(\boldsymbol{t}) = Q_1(\boldsymbol{t}) = Q_2(\boldsymbol{t}) = f^+(\boldsymbol{t}) = 0$*.*

In particular, the decomposition in part (1) *is unique.*

(3) *The periodic constant of* $f^-(\boldsymbol{t})$ *associated with the chamber* $\mathcal{C}$ *is zero. Hence, in the decomposition* (1) *one also has* $\mathrm{pc}^{e,\mathcal{C}}(f) = \mathrm{pc}^{e,\mathcal{C}}(f^+) = \sum_\ell \kappa_\ell[\beta_\ell] \in \mathbb{Z}[H]$*.*

Proof (1) For every $b_k \in L'$ we have a (unique) $b'_k \in L' \cap \square(A)$ such that $b_k - b'_k \in \mathbb{Z}\langle a_1, a_2\rangle$. Set $Q(\boldsymbol{t}) := \sum_{k=1}^r \iota_k \boldsymbol{t}^{b'_k}$. Then $f(\boldsymbol{t}) - Q(\boldsymbol{t})/\prod_{i=1}^2(1-\boldsymbol{t}^{a_i})$ is a sum of terms of type $\boldsymbol{t}^{b'}(\boldsymbol{t}^{k_1a_1+k_2a_2}-1)/\prod_{i=1}^2(1-\boldsymbol{t}^{a_i})$. This decomposes as a sum with terms of type $\boldsymbol{t}^c/(1-\boldsymbol{t}^{a_i})$. Then for every such expression, there exists $c_i \in \Xi_i$ such that $(\boldsymbol{t}^c - \boldsymbol{t}^{c_i})/(1-\boldsymbol{t}^{a_i})$ is as in (a).

(2) Part (2) is again elementary. First we show that $Q(\boldsymbol{t}) = 0$. For any $b' \in L' \cap \square(A)$ consider $\Xi_{b'} := b' + \mathbb{Z}\langle a_1, a_2\rangle$. For any $P(\boldsymbol{t}) = \sum \iota_k \boldsymbol{t}^{c'_k}$ write $P_{b'}(\boldsymbol{t}) = \sum_{c_k \in \Xi_{b'}} \iota_k \boldsymbol{t}^{c'_k}$ for its part supported on $\Xi_{b'}$. This decomposition can be done for Q, Q_1, Q_2 and f^+, hence for $\Sigma(\boldsymbol{t})$. Note that it is enough to prove (2) for such $\Sigma_{b'}(\boldsymbol{t})$, for a fixed b'. Hence, we can assume that $\Sigma(\boldsymbol{t})$ is supported on some $\Xi_{b'}$, $b' \in L' \cap \square(A)$. Since $\Xi_{b'} \cap \square(A) = \{b'\}$, in this case $Q(\boldsymbol{t}) = \iota \boldsymbol{t}^{b'}$. Multiplying $\Sigma(\boldsymbol{t})$ by $\prod_{i=1}^2(1-\boldsymbol{t}^{a_i})$ and substituting $t_1 = t_2 = 1$ we get $\iota = 0$. Hence $Q(\boldsymbol{t}) = 0$.

Next, consider the identity $(1-\boldsymbol{t}^{a_2})Q_1(\boldsymbol{t}) + (1-\boldsymbol{t}^{a_1})Q_2(\boldsymbol{t}) + \prod_{i=1}^2(1-\boldsymbol{t}^{a_i}) \cdot f^+(\boldsymbol{t}) = 0$. Since $\mathbb{Z}[t_1, t_2]$ is UFD and the polynomials $1-\boldsymbol{t}^{a_1}$ and $1-\boldsymbol{t}^{a_2}$ are relative primes, we get that $1-\boldsymbol{t}^{a_i}$ divides $Q_i(\boldsymbol{t})$. This together with the support assumption of Q_i implies $Q_i=0$.

(3) The vanishing of the periodic constant of the first fraction of f^- follows from the proof of Lemma 4.4.12. The vanishing of $\mathrm{pc}^{e,\mathcal{C}}$ of the other two fractions follows from Example 4.4.9(b). For the last expression see Example 4.4.9(a). □

4.5.4 Remark (a) The first part of the proof provides an algorithm how one finds the decomposition.

(b) Since $\mathrm{pc}^{e,\mathcal{C}}(f^-)=0$ by (3), the above decomposition $f=f^++f^-$ is well-suited for computing the periodic constant of f associated with chamber $\mathcal{C}$ via f^+.

5 The case of rational functions associated with plumbing graphs

5.1 A 'classical' connection between polytopes and gauge invariants (and its limits)

In the literature of normal surface singularities there is a sequence of results which connect the topology of the link with the number of lattice points in a certain polytope. Here are some details.

The first step is based on the theory of hypersurface singularities with Newton nondegenerate principal part; see eg Arnold, Gusein-Zade and Varchenko [1]. According to this, for such a germ one defines the Newton polytope Γ_N^- using the nontrivial monomials of the defining equation of the germ, and one proves that several invariants of the germ can be recovered from Γ_N^-; see eg Braun and Némethi [10]. For example, by a result of Merle and Teissier [30], the geometric genus p_g equals the number of lattice points in $((\mathbb{Z}_{>0})^3\cap\Gamma_N^-)$. The second step is provided by Laufer–Durfee formula, which determines the signature of the Milnor fiber σ as $-8p_g-K^2-|\mathcal{V}|$; see Durfee [21]. Finally, there is a conjecture of Neumann and Wahl [47], formulated for hypersurfaces with integral homology sphere links, and proved eg for Brieskorn, suspension [47] and splice quotient (Némethi and Okuma [43]) singularities, according to which $\sigma/8=\lambda(M)$, the Casson invariant of the link. Therefore, if all these steps run, eg in the Brieskorn case, then the Casson invariant of the link, normalized by $K^2+|\mathcal{V}|$, can be expressed as the number of lattice points of a polytope associated with the equation of the germ.

(For the computations of the lattice points in the case of simplicial polytopes in terms of Dedekind sums, see eg Barvinok and Pommersheim [3], Beck [4], Beck and Robins [8], Diaz and Robins [19] and the citations therein; for its relation with the Riemann–Roch formulae, see eg Capell and Shaneson [16], Kantor and Khovanskii [26], Pommersheim [56] or literature of classical toric geometry; while for the relation of Dedekind sums with the Casson invariant, see the classical book Lescop [27].)

The above correspondence has several deficiencies. First, even in simple cases, we do not know how to extend the correspondence to the equivariant case (that is, how to express the equivariant geometric genus from Γ_N^-). Second, the expected generalization, the Seiberg–Witten invariant conjecture (see Section 2.3), which aims to identify the Seiberg–Witten invariant of the link with p_g (or σ) is still open for Newton nondegenerate germs. Finally, this family of germs is rather restrictive. (Additionally, as a general fact about lattice point computations, in the literature there are very few explicit formulae for the Ehrhart polynomial of nonsimplicial polyhedra.)

The present article defines another polytope, which carries an action of the group H, and its Ehrhart invariants determine the Seiberg–Witten invariant *in any case*. It is not described from the equations of the germ, but from its multivariable 'zeta function' $Z(\boldsymbol{t})$. Furthermore, the polytope is a union of several simplices, and those coefficients of the Ehrhart polynomial which carry the information about the Seiberg–Witten invariant will be determined.

5.2 The new construction: Applications of Section 4

Consider the topological setup of a surface singularity, as in Section 2.1. The lattice L has a canonical basis $\{E_v\}_{v\in\mathcal{V}}$ corresponding to the vertices of the graph Γ. We investigate the periodic constant of the rational function $Z(\boldsymbol{t})$, defined in Section 2.2 from Γ. Since $Z(\boldsymbol{t})$ has the form (4.2.2), all the results of Section 4 can be applied. In particular, if $\mathcal{E} = \{v \in \mathcal{V} \mid \delta_v = 1\}$ denotes the set of *ends* of the graph, then $\boldsymbol{A}$ has column vectors $a_v = E_v^*$ for $v \in \mathcal{E}$. Hence, the rank of the lattice/space where the polytopes $P^{(l')} = \bigcup_v P_v$ sit is $d = |\mathcal{E}|$, and the convex polytopes $\{P_v\}$ are indexed by $\mathcal{V}$. Furthermore, the dilation parameter l' of the polytopes runs in a $|\mathcal{V}|$–dimensional space. In the sequel we will drop the symbol $\boldsymbol{A}$ from $\mathcal{L}_h^{\mathcal{C}}(\boldsymbol{A}, \mathcal{T}, l')$.

(The construction has some analogies with the construction of the splice quotient singularities; cf Neumann and Wahl [48]: in that case the equations of the universal abelian cover of the singularity are written in $\mathbb{C}^d$, together with an action of H. Nevertheless, in the present situation, we are not obstructed with the semigroup and congruence relations present in that theory.)

In this new construction, a crucial additional ingredient comes from singularity theory, namely, Theorem 2.2.2 (in fact, this is the main starting point and motivation of the whole approach). This, combined with facts from Section 4, gives the following.

5.2.1 Corollary *Let $\mathcal{S} = \mathcal{S}_{\mathbb{R}}$ be the (real) Lipman cone $\{x \in \mathbb{R}^{|\mathcal{V}|} \mid (x, E_v) \leq 0$ for all $v\}$.*

(a) *The rational function $Z(\boldsymbol{t})$ admits a periodic constant in the cone $\mathcal{S}$, which equals the normalized Seiberg–Witten invariant*

$$\mathrm{pc}_h^{\mathcal{S}}(Z) = -\frac{(K+2r_h)^2 + |\mathcal{V}|}{8} - \mathfrak{sw}_{-h*\sigma_{\mathrm{can}}}(M). \tag{5.2.2}$$

(b) *Consider the chamber decomposition associated with the denominator of $Z(\boldsymbol{t})$ as in Theorem 4.3.9, and let $\mathcal{C}$ be a chamber such that $\mathrm{int}(\mathcal{C} \cap \mathcal{S}) \neq \varnothing$. Then $Z(\boldsymbol{t})$ admits a periodic constant in $\mathcal{C}$, which equals both $\mathrm{pc}_h^{\mathcal{S}}(Z)$ (satisfying (5.2.2)) and also*

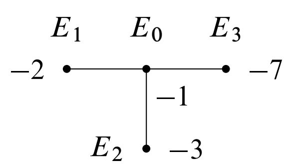
$$\mathrm{pc}_h^{\mathcal{C}}(Z) = \sum_k \iota_k \cdot \mathcal{L}^{\mathcal{C}}_{h-[b_k]}(\mathcal{T}, -b_k) = \sum_k \iota_k \cdot \mathcal{L}^{\mathcal{C}}_{[b_k]-h}(\mathcal{F} \setminus \mathcal{T}, b_k). \tag{5.2.3}$$

In particular, $\mathrm{pc}_h^{\mathcal{C}}(Z)$ does not depend on the choice of $\mathcal{C}$ (under the above assumption).

Proof Write $l' = \tilde{l} + r_h$ with $\tilde{l} \in L$ in (2.2.3). Since we have $\sum_{l \in L, l \not\geq 0} p_{l'+l} = \sum_{l'' \not\geq \tilde{l}, [l'']=h} p_{l''}$, (a) follows from Theorem 2.2.2. For (b) use Corollary 4.3.11 and Proposition 4.4.5. □

We note that the Lipman cone $\mathcal{S}$ can indeed be cut in several chambers (of the denominator of Z). This can happen even in the simple case of Brieskorn germs. Below we provide such an example together with several exemplifying details of the construction.

5.2.4 Example (Lipman cone cut in several chambers) Consider the 3–manifold $S^3_{-1}(T_{2,3})$ (where $T_{2,3}$ is the right-handed trefoil knot), or, equivalently, the link of the hypersurface singularity $z_1^2 + z_2^3 + z_3^7 = 0$. Its plumbing graph Γ and matrix $-I^{-1}$ are:



$$-I^{-1} = \begin{pmatrix} 42 & 21 & 14 & 6 \\ 21 & 11 & 7 & 3 \\ 14 & 7 & 5 & 2 \\ 6 & 3 & 2 & 1 \end{pmatrix}$$

where the row/column vectors of $-I^{-1}$ are E_0^*, E_1^*, E_2^* and E_3^* in the $\{E_v\}$ basis. The polytopes defined in (3.1.1), or in (4.3.2), with parameter $l = (l_0, l_1, l_2, l_3) \subset \mathbb{Z}^4$, sit in $\mathbb{R}^3$. Let u_1, u_2, u_3 be the basis of $\mathbb{R}^3$. Then the polytopes are the following convex closures:

$$\begin{aligned}
P_0^{(l)} &= \mathrm{conv}(0, (l_0/21)u_1, (l_0/14)u_2, (l_0/6)u_3) \\
P_1^{(l)} &= \mathrm{conv}(0, (l_1/11)u_1, (l_1/7)u_2, (l_1/3)u_3) \\
P_2^{(l)} &= \mathrm{conv}(0, (l_2/7)u_1, (l_2/5)u_2, (l_2/2)u_3) \\
P_3^{(l)} &= \mathrm{conv}(0, (l_3/3)u_1, (l_3/2)u_2, (l_3/1)u_3)
\end{aligned}$$

Since $E_0^* + \varepsilon(-E_0)$ is in the interior of the (real) Lipman cone for $0 < \varepsilon \ll 1$, we get that the Lipman cone is cut in several chambers. The periodic constant can be computed with any of them. In fact, by the continuity of the quasipolynomials associated with the chambers, any quasipolynomial associated with a chamber which contains any ray in the Lipman cone, even if it is situated at its boundary, provides the periodic constant. One such degenerated polytope provided by a ray on the boundary of $\mathcal{S}$ is of special interest. Namely, if we take $l = \lambda E_0^* \in \mathcal{S}$ for $\lambda > 0$, then $P^{(l)} = \bigcup_{v=0}^{3} P_v^{(l)}$ is the same as $P_0^{(l)} = \mathrm{conv}(0, 2\lambda u_1, 3\lambda u_2, 7\lambda u_3)$. Moreover, if $\mathcal{C}$ is any chamber which contains the ray $\mathbb{R}_{\geq 0} E_0^*$ at its boundary, then for any $l = \lambda E_0^*$ one has $\mathcal{L}^{\mathcal{C}}(\boldsymbol{A}, \mathcal{T}, l) = \mathcal{L}(\widetilde{P}_0, \mathcal{T}, \lambda)$, where the last is the classical Ehrhart polynomial of the tetrahedron $\widetilde{P}_0 := \mathrm{conv}(0, 2u_1, 3u_2, 7u_3)$. Here we witness an additional coincidence of $\widetilde{P}_0$ with the Newton polytope Γ_N^- of the equation $z_1^2 + z_2^3 + z_3^7$.

We compute $\mathcal{L}(\widetilde{P}_0, \mathcal{T}, \lambda)$ as follows. From (2.3.2)–(2.3.3) and Corollary 4.3.11, we get that

$$\text{(5.2.5)} \quad \chi(\lambda E_0^*) + \text{geometric genus of } \{z_1^2 + z_2^3 + z_3^7 = 0\} \\ = \mathcal{L}(\widetilde{P}_0, \mathcal{T}, \lambda) - \mathcal{L}(\widetilde{P}_0, \mathcal{T}, \lambda - 1).$$

Since this geometric genus is 1, and the free term of $\mathcal{L}(\widetilde{P}_0, \mathcal{T}, \lambda)$ is zero (since for $\lambda = 0$ the zero polytope with boundary conditions contains no lattice point), and $-K = 2E_0 + E_1 + E_2 + E_3$, we get that $\mathcal{L}(\widetilde{P}_0, \mathcal{T}, \lambda) = 7\lambda^3 + 10\lambda^2 + 4\lambda$. We emphasize that a formula as in (5.2.5), realizing a bridge between the Riemann–Roch expression χ (supplemented with the geometric genus) and the Ehrhart polynomial of the Newton diagram, was not known for Newton nondegenerate germs.

In the sequel we will provide several examples, when the Newton polytope is not even defined.

5.3 Example: The case of lens spaces

5.3.1 As we will see in Reduction Theorem 5.4.2, the complexity of the problem depends basically on the number of nodes of Γ. In this subsection we treat the case when there are no nodes at all, that is M is a lens space. In this case the numerator of the rational function $f(\boldsymbol{t})$ is 1, hence everything is described by the 2–dimensional polytopes determined by the denominator. In the literature there are several results about lens spaces fitting in the present program, here we collect the relevant ones completing with the new interpretations. This subsection also serves as a preparatory part, or model, for the study of chains of arbitrary graphs.

5.3.2 The setup Assume that the plumbing graph is

with all $k_v \geq 2$, and p/q is expressed via the (Hirzebruch, or negative) continued fraction

$$[k_1, \dots, k_s] = k_1 - 1/(k_2 - 1/(\cdots - 1/k_s)\cdots). \tag{5.3.3}$$

Then M is the lens space $L(p,q)$. We also define q' by $q'q \equiv 1 \pmod p$, and $0 \leq q' < p$. Furthermore, we set $g_v = [E_v^*] \in H$. Then g_s generates $H = \mathbb{Z}_p$, and any element of H can be written as ag_s for some $0 \leq a < p$. Recall the definitions of r_h and s_h from Section 2.1 as well.

From the analytic point of view (X,o) is a cyclic quotient singularity $(\mathbb{C}^2,o)/\mathbb{Z}_p$, where the action is $\xi * (x,y) = (\xi x, \xi^q y)$ (here ξ runs over p–roots of unity).

5.3.4 The Seiberg–Witten invariant Since (X,o) is rational, in this case $Z(\boldsymbol{t}) = P(\boldsymbol{t})$; cf Section 2.3. Moreover, in (2.3.3) $H^1(\mathcal{O}_{\widetilde{Y}}) = 0$, hence

$$\mathfrak{sw}_{-h*\sigma_{\mathrm{can}}}(M) = -\frac{(K+2r_h)^2 + |\mathcal{V}|}{8} = -\frac{K^2 + |\mathcal{V}|}{8} + \chi(r_h). \tag{5.3.5}$$

On the other hand, in Némethi [33; 35] a similar formula is proved for the SW invariant: one only has to replace in (5.3.5) $\chi(r_h)$ by $\chi(s_h)$. In particular, for lens spaces, and for any $h \in H$ one has

$$\chi(r_h) = \chi(s_h). \tag{5.3.6}$$

(Note that, in general, for other links, $\chi(r_h) > \chi(s_h)$ might happen; see Example 6.4.8. Here, (5.3.6) follows from the vanishing of the geometric genus of the universal abelian cover of (X,o).)

In general, the coefficients of the representatives s_{ag_s} and r_{ag_s} $(0 \leq a < p)$ are rather complicated arithmetical expressions; for s_{ag_s} see Némethi [33, 10.3] (where g_s is defined with opposite sign). The value $\chi(s_{ag_s})$ is computed in [33, 10.5.1] as

$$\chi(s_{ag_s}) = \frac{a(1-p)}{2p} + \sum_{j=1}^{a} \left\{ \frac{jq'}{p} \right\}. \tag{5.3.7}$$

For completeness of the discussion we also recall that $K = E_1^* + E_s^* - \sum_v E_v$ and

$$(K^2 + |\mathcal{V}|)/4 = (p-1)/(2p) - 3 \cdot \boldsymbol{s}(q,p), \tag{5.3.8}$$

cf [33, 10.5], where $\boldsymbol{s}(q,p)$ denotes the Dedekind sum

$$\boldsymbol{s}(q,p) = \sum_{l=0}^{p-1} \left(\left(\frac{l}{p}\right)\right)\left(\left(\frac{ql}{p}\right)\right), \quad \text{where } ((x)) = \begin{cases} \{x\} - 1/2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

In particular, $\mathfrak{sw}_{-h*\sigma_{\text{can}}}(M)$ is determined via the formulae (5.3.5)–(5.3.8).

The nonequivariant picture looks as follows: $\sum_h \mathfrak{sw}_{-h*\sigma_{\text{can}}} = \lambda$, the Casson–Walker invariant of M, hence (5.3.5) gives

$$\lambda = -p(K^2 + |\mathcal{V}|)/8 + \textstyle\sum_h \chi(r_h).$$

This is compatible with (5.3.8) and formulae $\lambda(L(p,q)) = p \cdot \boldsymbol{s}(q,p)/2$ and $\sum_h \chi(r_h) = (p-1)/4 - p \cdot \boldsymbol{s}(q,p)$; cf [33, 10.8].

5.3.9 The polytope and its quasipolynomial We compare the above data with Ehrhart theory. In this case $Z(\boldsymbol{t}) = (1 - \boldsymbol{t}^{E_1^*})^{-1}(1 - \boldsymbol{t}^{E_s^*})^{-1}$. The vectors $a_1 = E_1^*$ and $a_s = E_s^*$ determine the polytopes $P^{(l')}$ and a chamber decomposition.

For $1 \le v \le w \le s$ let n_{vw} denote the numerator of the continued fraction $[k_v, \dots, k_w]$ (or, the determinant of the corresponding bamboo subgraph). For example, $n_{1s} = p$, $n_{2s} = q$ and $n_{1,s-1} = q'$. We also set $n_{v+1,v} := 1$. Then $pE_1^* = \sum_{v=1}^{s} n_{v+1,s} E_v$ and $pE_s^* = \sum_{v=1}^{s} n_{1,v-1} E_v$.

In particular, for any $l' = \sum_v l'_v E_v \in \mathcal{S}'$, the (nonconvex) polytopes are

$$P^{(l')} = \bigcup_{v=1}^{s} \{(x_1, x_s) \in \mathbb{R}^2_{\ge 0} \mid x_1 n_{v+1,s} + x_s n_{1,v-1} \le p l'_v\} \subset \mathbb{R}^2_{\ge 0}. \tag{5.3.10}$$

The representation $\mathbb{Z}^2 \xrightarrow{\rho} \mathbb{Z}_p$ is $(x_1, x_s) \mapsto (qx_1 + x_s) g_s$.

Though $P^{(l')}$ is a plane polytope, the direct computation of its equivariant Ehrhart multivariable polynomial (associated with a chamber, or just with the Lipman cone) is still highly nontrivial. Here we will rely again on Theorem 2.2.2. On a subset of type $l'_0 + \mathcal{S}'$ the identity (2.2.3) provides the counting function. The right-hand side of (2.2.3) depends on all the coordinates of l', hence all the triangles P_v contribute in $P^{(l')}$. Since this can happen only in a unique combinatorial way, we get that there is a chamber $\mathcal{C}$ which contains the Lipman cone. Let $\mathcal{L}^{e,\mathcal{C}}$ be its quasipolynomial, and $\mathcal{L}^{e,\mathcal{S}}$ its restriction to $\mathcal{S}$. Since the numerator of $Z(\boldsymbol{t})$ is 1, $\bar{Q}^{\mathcal{C}}_h = \mathcal{L}^{\mathcal{C}}_h$. Since this agrees with the right-hand side of (2.2.3) on a cone of type $l'_0 + \mathcal{S}'$, and the Lipman cone is in $\mathcal{C}$, we get that

$$Q_h(l') = \bar{Q}^{\mathcal{C}}_h(l') = \mathcal{L}^{\mathcal{S}}_h(l') = -\mathfrak{sw}_{-h*\sigma_{\mathrm{can}}}(M) - \frac{(K+2l')^2 + |\mathcal{V}|}{8} \tag{5.3.11}$$

for any $l' \in (r_h + L) \cap \mathcal{S}'$ and $h \in H$. Using the identity (5.3.5), this reads as

$$\mathcal{L}^{\mathcal{S}}_h(\mathcal{T}, l') = \chi(l') - \chi(r_h), \quad l' \in (r_h + L) \cap \mathcal{S}'. \tag{5.3.12}$$

Note that for any fixed h and any l' there exists a unique $q = q_{l',h} \in \square$ such that $l' + q := l'' \in r_h + L$. Indeed, take for q the representative of $r_h - l'$ in $\square$. Then (4.3.15) and (5.3.12) imply

$$\mathcal{L}^{\mathcal{S}}_h(\mathcal{T}, l') = \mathcal{L}^{\mathcal{S}}_h(\mathcal{T}, l'') = \chi(l' + q_{l',h}) - \chi(r_h). \tag{5.3.13}$$

This formula emphasizes the periodic behavior of $\mathcal{L}^{\mathcal{S}}_h(\mathcal{T}, l')$ as well.

If l' is an element of L then $q_{l',h} = r_h$, hence (5.3.13) gives in this case

$$\mathcal{L}^{\mathcal{S}}_h(\mathcal{T}, l) = \chi(l + r_h) - \chi(r_h) = \chi(l) - (l, r_h) \quad \text{for } l \in L \cap \mathcal{S}. \tag{5.3.14}$$

In particular, $\mathrm{pc}(\mathcal{L}^{\mathcal{S}}_h(\mathcal{T})) = \chi(r_h) - \chi(r_h) = 0$ (a fact compatible with $H^1(\mathcal{O}_{\tilde{Y}}) = 0$).

Even the nonequivariant case looks rather interesting. Let $\mathcal{L}^{\mathcal{S}}_{\mathrm{ne}}(\mathcal{T}) = \sum_{h\in H} \mathcal{L}^{\mathcal{S}}_h(\mathcal{T})$ be the Ehrhart polynomial of $P^{(l')}$ (with boundary condition $\mathcal{T}$), where we count all the lattice points independently of their class in H. Then, (5.3.14) gives for $l \in L \cap \mathcal{S}$

$$\mathcal{L}^{\mathcal{S}}_{\mathrm{ne}}(\mathcal{T}, l) = p \cdot \chi(l) - (l, \textstyle\sum_h r_h) = -p \cdot (l,l)/2 - p \cdot (l, K)/2 - (l, \textstyle\sum_h r_h). \tag{5.3.15}$$

In fact, $\sum_h r_h$ can explicitly be computed. Indeed, set $d_v = \gcd(p, n_{1,v-1})$ and $p_v = p/d_v$. Then one checks that $aE^*_s = \sum_v n_{1,v-1}\frac{a}{p}E_v$, $r_h = \sum_v \{n_{1,v-1}\frac{a}{p}\}E_v$ and $\sum_h r_h = \sum_v d_v \frac{p_v - 1}{2} E_v$.

The coefficients of the polynomial $\mathcal{L}^{\mathcal{S}}_{\mathrm{ne}}(\mathcal{T}, l)$ can be compared with the coefficients given by general theory of Ehrhart polynomials applied for $P^{(l)}$. For example, the leading coefficient gives

$$-p \cdot (l,l)/2 = \text{Euclidean area of } P^{(l)}.$$

Knowing that in $P^{(l)}$ all the P_v's contribute, and it depends on s parameters, and the intersection of their boundary is messy, the simplicity and conceptual form of (5.3.15) is rather remarkable.

5.4 Reduction of the variables of $Z(\boldsymbol{t})$

Let $\mathcal{N}$ denote the set of nodes as above. Let $\mathcal{S}_{\mathcal{N}}$ be the positive cone $\mathbb{R}_{\geq 0}\langle E_n^*\rangle_{n\in\mathcal{N}}$ generated by the dual base elements indexed by $\mathcal{N}$, and $V_{\mathcal{N}} := \mathbb{R}\langle E_n^*\rangle_{n\in\mathcal{N}}$ be its supporting linear subspace in $L\otimes\mathbb{R}$. Clearly $\mathcal{S}_{\mathcal{N}}\subset\mathcal{S}$. Furthermore, consider $L_{\mathcal{N}} := \mathbb{Z}\langle E_n\rangle_{n\in\mathcal{N}}$ generated by the node base elements, and the projection $pr_{\mathcal{N}}\colon L\otimes\mathbb{R}\to L_{\mathcal{N}}\otimes\mathbb{R}$ on the node coordinates.

5.4.1 Lemma *The restriction of $pr_{\mathcal{N}}$ to $V_{\mathcal{N}}$, namely $pr_{\mathcal{N}}\colon V_{\mathcal{N}}\to L_{\mathcal{N}}\otimes\mathbb{R}$, is an isomorphism.*

Proof Follows from the negative-definiteness of the intersection form of the plumbing, which guarantees that any minor situated centrally on the diagonal is nondegenerate. □

Our goal is to prove that restricting the counting function to the subspace $V_{\mathcal{N}}$, the nonnode variables of $Z(\boldsymbol{t})$ and $Q(l')$ became nonvisible, hence they can be eliminated. This fact will provide a remarkable simplification in the periodic constant computation. But, *before* any elimination-substitution, we have first to decompose our series $Z(\boldsymbol{t})$ into $\sum_{h\in H} Z_h(\boldsymbol{t})[h]$ if we wish to preserve the information about all the H invariants; cf the comment at the end of Section 4.2.3.

5.4.2 Reduction Theorem (a) *The restriction of $\mathcal{L}_h(\boldsymbol{A},\mathcal{T},l')$ to $V_{\mathcal{N}}$ depends only on those coordinates which are indexed by the nodes (that is, it depends only on $pr_{\mathcal{N}}(l')$ whenever $l'\in V_{\mathcal{N}}$).*

(b) *The same is true for the counting function Q_h associated with $Z_h(\boldsymbol{t})$ as well. In other words, if we consider the restriction*

$$Z_h(\boldsymbol{t}_{\mathcal{N}}) := Z_h(\boldsymbol{t})|_{t_v=1 \text{ for all } v\notin\mathcal{N}}$$

then for any $l'\in L_{\mathcal{N}}$, the counting functions $\sum_{l''\not\geq l'} p_{l''}[l'']$ of $Z_h(\boldsymbol{t})$ and $Z_h(\boldsymbol{t}_{\mathcal{N}})$ are the same.

(c) *Consider the chamber decomposition of $\mathcal{S}_{\mathcal{N}}$ by intersections of type $\mathcal{C}_{\mathcal{N}} := \mathcal{C}\cap\mathcal{S}_{\mathcal{N}}$, where $\mathcal{C}$ denotes a chamber (of Z) such that $\operatorname{int}(\mathcal{C}\cap\mathcal{S})\neq\varnothing$, and the intersection of $\mathcal{C}$ with the relative interior of $\mathcal{S}_{\mathcal{N}}$ is also nonempty. Then*

$$\mathrm{pc}^{\mathcal{C}}(Z_h(\boldsymbol{t})) = \mathrm{pc}^{\mathcal{C}_{\mathcal{N}}}(Z_h(\boldsymbol{t}_{\mathcal{N}})). \tag{5.4.3}$$

The theorem applies as follows. Assume that we are interested in the computation of $\mathrm{pc}^{\mathcal{C}}_h(Z(\boldsymbol{t}))$ for some chamber $\mathcal{C}$ (eg when $\mathcal{C}\subset\mathcal{S}$; cf Corollary 5.2.1). Assume that $\mathcal{C}$ intersects the relative interior of $\mathcal{S}_{\mathcal{N}}$. Then, the restriction to $\mathcal{C}\cap\mathcal{S}_{\mathcal{N}}$ of the quasipolynomial associated with $\mathcal{C}$ has two properties: it still preserves sufficient information to determine $\mathrm{pc}^{\mathcal{C}}_h(Z(\boldsymbol{t}))$ (via the periodic constant of the restriction; see (5.4.3)), but it also has the advantage that for these dilation parameters l' the union $\bigcup_{v\in\mathcal{V}} P_v^{(l'),\triangleleft}$ equals the union of significantly less polytopes, namely $\bigcup_{n\in\mathcal{N}} P_v^{(l'),\triangleleft}$.

For example, when we have only one node, one has to handle one simplex instead of $|\mathcal{V}|$ many.

Proof (a) We show that for any $l'\in V_{\mathcal{N}}$ one has the inclusions

$$P_v^{(l'),\triangleleft}\subset\bigcup_{n\in\mathcal{N}} P_n^{(l'),\triangleleft}\quad\text{for any } v\notin\mathcal{N}. \tag{5.4.4}$$

We consider two cases. First we assume that v is on a leg (chain) connecting an end $e(v)\in\mathcal{E}$ with a node $n(v)$ (where $e(v)=v$ is also possible). Then, clearly, (5.4.4) follows from

$$P_v^{(l'),\triangleleft}\subset P_{n(v)}^{(l'),\triangleleft}\quad\text{for any } l'\in\mathcal{S}_{\mathcal{N}}. \tag{5.4.5}$$

Let $E^*_{uv}=(E^*_u)_v=-(E^*_u,E^*_v)$ be the v–cordinate of E^*_u. Note that $E^*_{uv}=E^*_{vu}$. Using the definition of the polytopes, (5.4.5) is equivalent with the implication (cf Section 4.3.1)

$$\Big(\sum_{e\in\mathcal{E}} x_eE^*_{ve}<l'_v\Big)\Longrightarrow\Big(\sum_{e\in\mathcal{E}} x_eE^*_{n(v)e}<l'_{n(v)}\Big)\quad\text{for any } l'\in\mathcal{S}_{\mathcal{N}},\ x_e\geq 0. \tag{5.4.6}$$

Let $\mathcal{W}$ be the set of vertices on this leg (including $e(v)$ but not $n(v)$). Then, one verifies that there exist positive rational numbers α and $\{\alpha_w\}_{w\in\mathcal{W}}$, such that

$$E^*_v=\alpha E^*_{n(v)}+\sum_{w\in\mathcal{W}}\alpha_wE_w. \tag{5.4.7}$$

The numbers α and $\{\alpha_w\}_{w\in\mathcal{W}}$ can be determined from the linear system obtained by intersecting the identity (5.4.7) by $\{E_w\}_w$ and $E_{n(v)}$. Intersecting (5.4.7) by E^*_e ($e\in\mathcal{E}$), we get that $E^*_{ve}=\alpha E^*_{n(v)e}$ for any $e\neq e(v)$, and $E^*_{v,e(v)}=\alpha E^*_{n(v),e(v)}+\alpha_{e(v)}$. Hence

$$\sum_{e\in\mathcal{E}} x_eE^*_{ve}=\alpha\sum_{e\in\mathcal{E}} x_eE^*_{n(v)e}+x_{e(v)}\alpha_{e(v)}. \tag{5.4.8}$$

On the other hand, intersecting (5.4.7) with E_n^*, for $n \in \mathcal{N}$, we get $E_{vn}^* = \alpha E_{n(v)n}^*$. Since l' is a linear combination of E_n^*'s, we get that

$$-l'_v = (l', E_v^*) = \alpha(l', E_{n(v)}^*) = -\alpha l'_{n(v)}. \tag{5.4.9}$$

Since $x_{e(v)}\alpha_{e(v)} \geq 0$, (5.4.8) and (5.4.9) imply (5.4.6). This ends the proof of this case.

Next, we assume that v is on a chain connecting two nodes $n(v)$ and $m(v)$. Let $\mathcal{W}$ be the set of vertices on this bamboo (not including $n(v)$ and $m(v)$). Then we will show that

$$P_v^{(l'),\lhd} \subset P_{n(v)}^{(l'),\lhd} \cup P_{m(v)}^{(l'),\lhd} \quad \text{for any } l' \in \mathcal{S}_{\mathcal{N}}. \tag{5.4.10}$$

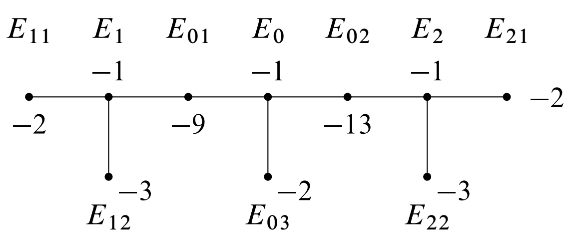
This follows as above from the existence of positive rational numbers α, β and $\{\alpha_w\}_{w\in\mathcal{W}}$ with

$$E_v^* = \alpha E_{n(v)}^* + \beta E_{m(v)}^* + \sum_{w\in\mathcal{W}} \alpha_w E_w. \tag{5.4.11}$$

(b) Part (b) follows from (a) and from the fact that all b_k entries in the numerator of $Z(\boldsymbol{t})$ belong to $\mathcal{S}_{\mathcal{N}}$.

(c) If $\mathrm{pc}^{\mathcal{C}}(Z_h(\boldsymbol{t}))$ is computed as $\widetilde{Q}_h(0)$ for some quasipolynomial $\widetilde{Q}_h$ defined on $\widetilde{L} \subset L$, then part (b) shows that $\mathrm{pc}^{\mathcal{C}_{\mathcal{N}}}(Z_h(\boldsymbol{t}_{\mathcal{N}}))$ can be computed as $(\widetilde{Q}_h|_{\widetilde{L}\cap\mathcal{S}_{\mathcal{N}}})(0)$, which equals $\widetilde{Q}_h(0)$. □

5.4.12 Example Consider the following graph Γ:



By Reduction Theorem 5.4.2 we are interested only in those polytopes $P_v \subset \mathbb{R}^5$ which are associated to the nodes E_1, E_2 and E_0. Let $l \in \mathcal{S}_{\mathcal{N}}$, ie $l = \lambda_1 E_1^* + \lambda_2 E_2^* + \lambda_0 E_0^*$. Then one can verify that $\mathcal{S}_{\mathcal{N}}$ is divided by the plane $\lambda_1 = (13/9)\lambda_2$. Hence, in general $\mathcal{S}_{\mathcal{N}}$ can also be divided into several chambers. (On the other hand, for graphs with at most two nodes this does not happen.)

6 The one-node case, star-shaped plumbing graphs

6.1 Seifert invariants and other notation

Assume that the graph is star-shaped with d legs. Each leg is a chain with normalized Seifert invariant (α_i, ω_i), where $0 < \omega_i < \alpha_i$, $\gcd(\alpha_i, \omega_i) = 1$. We also use ω_i' satisfying $\omega_i \omega_i' \equiv 1 \pmod{\alpha_i}$, $0 < \omega_i' < \alpha_i$.

If we consider the Hirzebruch/negative continued fraction expansion, cf (5.3.3),

$$\alpha_i/\omega_i = [b_{i1}, \dots, b_{i\nu_i}] = b_{i1} - 1/(b_{i2} - 1/(\cdots - 1/b_{i\nu_i})\cdots) \quad (b_{ij} \geq 2),$$

then the i^{th} leg has ν_i vertices, say $v_{i1}, \dots, v_{i\nu_i}$, with decorations $-b_{i1}, \dots, -b_{i\nu_i}$, where v_{i1} is connected by the central vertex. The corresponding base elements in L are $\{E_{ij}\}_{j=1}^{\nu_i}$. Let b be the decoration of the central vertex; this vertex also defines $E_0 \in L$. The plumbed 3–manifold M associated with such a star-shaped graph has a Seifert structure. We will assume that M is a rational homology sphere, or, equivalently, the central vertex has genus zero.

The classes in $H = L'/L$ of the dual base elements are denoted by $g_{ij} = [E_{ij}^*]$ and $g_0 = [E_0^*]$. For simplicity we also write $E_i := E_{i\nu_i}$ and $g_i := g_{i\nu_i}$. A possible presentation of H is

$$H = \left\langle g_0, g_1, \dots, g_d \,\middle|\, -b \cdot g_0 = \sum_{i=1}^{d} \omega_i \cdot g_i;\ g_0 = \alpha_i \cdot g_i\ (1 \leq i \leq d) \right\rangle; \tag{6.1.1}$$

cf Neumann [46] (or use $\sum_k I_{ik} g_k$ repeatedly; see also (6.1.3)). The orbifold Euler number of M is defined as $e = b + \sum_i \omega_i/\alpha_i$. The negative-definiteness of the intersection form implies $e < 0$. We write $\alpha := \operatorname{lcm}(\alpha_1, \dots, \alpha_d)$, $\mathfrak{d} := |H|$ and $\mathfrak{o}$ for the order of g_0 in H. One has (see eg [46])

$$\mathfrak{d} = \alpha_1 \cdots \alpha_d |e|, \quad \mathfrak{o} = \alpha |e|. \tag{6.1.2}$$

Each leg has similar invariants as the graph of a lens space, cf Section 5.3, and we can introduce similar notation. For example, the determinant of the i^{th} leg is α_i. We write $n^i_{j_1 j_2}$ for the determinant of the subchain of the i^{th} leg connecting the vertices v_{ij_1} and v_{ij_2} (including these vertices too). Then, using the correspondence between intersection pairing of the dual base elements and the determinants of the subgraphs, cf (2.1.1) or Némethi [33, 11.1], one has

$$\begin{aligned} &(E_0^*, E_{ij}^* - n^i_{j+1,\nu_i} E_{i\nu_i}^*) = 0, && g_{ij} = n^i_{j+1,\nu_i} g_{i\nu_i}\ (1 \leq i \leq d, 1 \leq j \leq \nu_i), \\ &(E_i^*, E_0^*) = \tfrac{1}{\alpha_i e}, && (E_0^*, E_0^*) = \tfrac{1}{e}. \end{aligned} \tag{6.1.3}$$

The second equation also explains why we do not need to insert the generators g_{ij} ($j < \nu_i$) in (6.1.1).

For any $l' \in L'$ we set $\widetilde{c}(l') := -(E_0^*, l')$, the E_0–coefficient of l'. Furthermore, if $l' = c_0 E_0^* + \sum_{i,j} c_{ij} E_{ij}^* \in L'$, then we define its *reduced transform* by

$$l'_{\text{red}} := c_0 E_0^* + \sum_{i,j} c_{ij} \cdot n^i_{j+1,\nu_i} E_i^*.$$

By (6.1.3) we get $[l'] = [l'_{\text{red}}]$ in H, $\widetilde{c}(l') = \widetilde{c}(l'_{\text{red}})$, and if $l'_{\text{red}} = \sum_{i=0}^d c_i E_i^*$, then $\widetilde{c}(l'_{\text{red}})$ is

$$\widetilde{c} := \frac{1}{|e|} \cdot \left(c_0 + \sum_{i=1}^d \frac{c_i}{\alpha_i} \right). \tag{6.1.4}$$

If $h \in H$, and $l'_h \in L'$ is any of its lifting (that is, $[l'_h] = h$), then $l'_{h,\text{red}}$ is also a lifting of the same h with $\widetilde{c}(l'_h) = \widetilde{c}(l'_{h,\text{red}})$. In general, $\widetilde{c} = \widetilde{c}(l'_h)$ depends on the lifting, nevertheless replacing l'_h by $l'_h \pm E_0$ we modify $\widetilde{c}$ by ± 1, hence we can always achieve $\widetilde{c} \in [0, 1)$, where it is determined uniquely by h. For example, since $r_h \in \square$, its E_0–coefficient $\widetilde{c}(r_h)$ is in $[0, 1)$.

Finally, we consider

$$\gamma := \frac{1}{|e|} \cdot \left(d - 2 - \sum_{i=1}^d \frac{1}{\alpha_i} \right). \tag{6.1.5}$$

It has several 'names.' Since the canonical class is given by $K = -\sum_v E_v + \sum_v (\delta_v - 2) E_v^*$, by (6.1.3) we get that the E_0 coefficient of $-K$ is $(K, E_0^*) = \gamma + 1$. The number $-\gamma$ is sometimes called the 'log discrepancy' of E_0, γ the 'exponent' of the weighted homogeneous germ (X, o), and $\mathfrak{o}\gamma$ is the Goto–Watanabe a–invariant of the universal abelian cover of (X, o) (see Goto and Watanabe [24, (3.1.4)], Bruns and Herzog [14, (3.6.13)]) while in Neumann [46] $e\gamma$ appears as an orbifold Euler characteristic.

6.2 The function Z

By the reduction (5.4.3), for the periodic constant computation, we can reduce ourself to the variable of the single node, it will be denoted by t.

First we analyze the equivariant rational function associated with the denominator of Z^e,

$$Z^{/H}(t) = \prod_{i=1}^d (1 - t^{-(E_i^*, E_0^*)}[g_i])^{-1} = \sum_{x_1,\dots,x_d \ge 0} t^{\sum_i x_i/(\alpha_i |e|)} \left[\sum_i x_i g_i \right] \in \mathbb{Z}[\![t^{1/\mathfrak{o}}]\!][H].$$

The right-hand side of the above expression can be transformed as follows; cf Némethi and Nicolaescu [41, Section 3]. If we fix a lift $\sum_{i=0}^d c_i E_i^*$ of h as above, then using the presentation (6.1.1) one gets that $\sum_{i=1}^d x_i g_i$ equals h if and only if there exist $\ell, \ell_1, \dots, \ell_d \in \mathbb{Z}$ such that

(a) $-c_0 = \ell_1 + \cdots + \ell_d - \ell b$,

(b) $x_i - c_i = -\omega_i \ell - \alpha_i \ell_i \quad (i = 1, \dots, d)$.

Since $x_i \geq 0$, from (b) we get $\widetilde{\ell}_i := \lfloor \frac{c_i - \omega_i \ell}{\alpha_i} \rfloor - \ell_i \geq 0$. Moreover, if we set for $\boldsymbol{c} = (c_0, c_1, \dots, c_d)$

$$N_{\boldsymbol{c}}(\ell) := 1 + c_0 - \ell b + \sum_{i=1}^d \left\lfloor \frac{c_i - \omega_i \ell}{\alpha_i} \right\rfloor, \tag{6.2.1}$$

then the number of realizations of $h = \sum_i c_i g_i$ in the form $\sum_i x_i g_i$ is given by the number of integers $(\widetilde{\ell}_1, \dots, \widetilde{\ell}_d)$ satisfying $\widetilde{\ell}_i \geq 0$ and $\sum_i \widetilde{\ell}_i = N_{\boldsymbol{c}}(\ell) - 1$. This is $\binom{N_{\boldsymbol{c}}(\ell)+d-2}{d-1}$. Moreover, the nonnegative integer $\sum_i x_i/(\alpha_i |e|)$ equals $\ell + \widetilde{c}$. Therefore,

$$Z_h^{/H}(t) = \sum_{\ell \geq -\widetilde{c}} \binom{N_{\boldsymbol{c}}(\ell) + d - 2}{d - 1} t^{\ell + \widetilde{c}}. \tag{6.2.2}$$

This expression is independent of the choice of $\boldsymbol{c} = \{c_i\}_{i=0}^d$. Similarly, for any function ϕ, the expression $\sum_{\ell \geq -\widetilde{c}} \phi(N_{\boldsymbol{c}}(\ell)) t^{\ell + \widetilde{c}}$ is independent of the choice of $\boldsymbol{c}$, it depends only on $h = \sum_i c_i g_i$.

Furthermore, one checks that $N_{\boldsymbol{c}}(\ell) \leq 1 + (\ell + \widetilde{c})|e|$, hence if $\ell + \widetilde{c} < 0$ then $N_{\boldsymbol{c}}(\ell) \leq 0$, therefore $\binom{N_{\boldsymbol{c}}(\ell)+d-2}{d-1} = 0$ as well. Hence, in (6.2.2) the inequality $\ell + \widetilde{c} \geq 0$ below the sum, in fact, is not restrictive.

Next, we consider the numerator $(1 - [g_0] t^{1/|e|})^{d-2}$ of $Z^e(t)$. A similar computation as above done for $Z^e(t)$ (see Neumann [46], Némethi and Nicolaescu [41, Section 3]), or by multiplying (6.2.2) by the numerator and using $\sum_{k=0}^{d-2} (-1)^k \binom{d-2}{k} \binom{N-k+d-2}{d-1} = \binom{N}{1}$, gives

$$Z_h(t) = \sum_{\ell \geq -\widetilde{c}} \max\{0, N_{\boldsymbol{c}}(\ell)\} t^{\ell + \widetilde{c}}. \tag{6.2.3}$$

In order to compute the periodic constant of $Z_h(t)$ we decompose $Z_h(t)$ into its 'polynomial and negative degree parts'; cf Section 4.1. Namely, we write $Z_h(t) =$

$Z_h^+(t) + Z_h^-(t)$, where

$$Z_h^+(t) = \sum_{\ell \geq -\widetilde{c}} \max\{0, -N_{\boldsymbol{c}}(\ell)\} t^{\ell+\widetilde{c}} \quad \text{(finite sum with positive exponents)}, \tag{6.2.4}$$
$$Z_h^-(t) = \sum_{\ell \geq -\widetilde{c}} N_{\boldsymbol{c}}(\ell) t^{\ell+\widetilde{c}}.$$

In Z_h^- it is convenient to fix a choice with $\widetilde{c} \in [0,1)$, hence the summation is over $\ell \geq 0$. Then a computation (left to the reader) shows that it is a rational function of negative degree

$$Z_h^-(t) = \left(\frac{1 - e\widetilde{c}}{1-t} - \frac{e \cdot t}{(1-t)^2} - \sum_{i=1}^{d} \sum_{r_i=0}^{\alpha_i - 1} \left\{ \frac{c_i - \omega_i r_i}{\alpha_i} \right\} t^{r_i} \cdot \frac{1}{1 - t^{\alpha_i}} \right) \cdot t^{\widetilde{c}}. \tag{6.2.5}$$

(This expression can be compared with the Laurent expansion of Z_h at $t = 1$ which was already considered in the literature. Dolgachev [20], Pinkham [54], Neumann [46] and Wagreich [61] determine the first two terms (the pole part), while Némethi and Nicolaescu [41], Némethi [33] determine the third terms as well. Nevertheless the above $Z_h^+ + Z_h^-$ decomposition does not coincide with the 'pole+regular part' decomposition of the Laurent expansion terms, and focuses on different aspects.)

Since the degree of Z_h^- is negative (or by a direct computation) $\mathrm{pc}(Z_h^-) = 0$; cf Section 4.1.

On the other hand, since $e < 0$, in $Z_h^+(t)$ the sum is finite. (The degree of Z_0^+ is less than or equal to γ; see eg Némethi and Okuma [44]. Since $N_{\boldsymbol{c}(r_{h,\mathrm{red}})}(\ell) \geq N_0(\ell)$, the degree of Z_h^+ is less than or equal to $\gamma + \widetilde{c}(r_h)$ too.) By Section 4.1,

$$\mathrm{pc}(Z_h) = Z_h^+(1) = \sum_{\ell \geq -\widetilde{c}} \max\{0, -N_{\boldsymbol{c}}(\ell)\} \tag{6.2.6}$$

for *any* lifting $\boldsymbol{c}$ of $h = \sum_i c_i g_i$. In this sum the bound $\ell \geq -\widetilde{c}$ is really restrictive.

We also consider the nonequivariant version, the projection of $Z^e \in \mathbb{Z}[\![t^{1/o}]\!][H]$ into $\mathbb{Z}[\![t^{1/o}]\!]$,

$$Z_{\mathrm{ne}}(t) = \sum_h Z_h(t) = \frac{(1 - t^{1/|e|})^{d-2}}{\prod_{i=1}^{d} (1 - t^{1/(|e|\alpha_i)})} \in \mathbb{Z}[\![t^{1/o}]\!].$$

We can get its $Z_{\mathrm{ne}}^+ + Z_{\mathrm{ne}}^-$ decomposition either by summation of Z_h^+ and Z_h^-, or as follows. Write

$$Z_{\mathrm{ne}}(t) = \frac{1}{(1 - t^{1/|e|})^2} \prod_{i=1}^{d} \frac{1 - t^{1/|e|}}{1 - t^{1/(|e|\alpha_i)}} = \frac{1}{(1 - t^{1/|e|})^2} \sum_{\substack{0 \leq x_i < \alpha_i \\ 0 \leq i \leq d}} t^{\frac{1}{|e|} \cdot S(x)}, \tag{6.2.7}$$

where $S(x) := \sum_i \frac{x_i}{\alpha_i}$. Then its decomposition into $Z^+_{\mathrm{ne}}(t) + Z^-_{\mathrm{ne}}(t)$ is

$$Z^-_{\mathrm{ne}}(t) = \sum_{\substack{0\le x_i<\alpha_i\\ 0\le i\le d}} t^{\frac{1}{|e|}\cdot\{S(x)\}}\cdot\Big(\frac{1}{(1-t^{1/|e|})^2} - \frac{\lfloor S(x)\rfloor}{1-t^{1/|e|}}\Big), \tag{6.2.8}$$

$$Z^+_{\mathrm{ne}}(t) = \sum_{\substack{0\le x_i<\alpha_i\\ 0\le i\le d}} t^{\frac{1}{|e|}\cdot\{S(x)\}}\cdot\frac{t^{\frac{1}{|e|}\cdot\lfloor S(x)\rfloor} - \lfloor S(x)\rfloor t^{\frac{1}{|e|}} + \lfloor S(x)\rfloor - 1}{(1-t^{1/|e|})^2}. \tag{6.2.9}$$

After dividing in $Z^+_{\mathrm{ne}}(t)$ (or by L'Hospital rule), we get

$$\mathrm{pc}(Z_{\mathrm{ne}}) = Z^+_{\mathrm{ne}}(1) = \frac{1}{2}\cdot\sum_{\substack{0\le x_i<\alpha_i\\ 0\le i\le d}} \lfloor S(x)\rfloor\cdot\lfloor S(x)-1\rfloor. \tag{6.2.10}$$

6.3 Analytic interpretations

Rational homology sphere negative-definite Seifert 3–manifolds can be realized analytically as links of weighted homogeneous singularities, or by equisingular deformations of weighted homogeneous singularities provided by splice quotient equations; see Neumann [46], Neumann and Wahl [48].

Consider the smooth germ at the origin of $\mathbb{C}^d$ having coordinate ring $\mathbb{C}\{z\} = \mathbb{C}\{z_1,\dots,z_d\}$, where z_i corresponds to the i^{th} end. Then H acts on it by the diagonal action $h*z_i = \theta(g_i)(h)z_i$. Similarly, we can introduce a multidegree $\deg(z_i)=E^*_i\in L'$, hence the Poincaré series of $\mathbb{C}\{z\}$ associated with this multidegree is $\prod_i(1-\boldsymbol{t}^{E^*_i})^{-1}$. Moreover, considering the action of H on it, $\widetilde{Z}(\boldsymbol{t}) = \prod_i(1-[g_i]\boldsymbol{t}^{E^*_i})^{-1}$ is the equivariant Poincaré series of $\mathbb{C}^d$, the invariant part $\widetilde{Z}_0(\boldsymbol{t})$ being the Poincaré series of the corresponding quotient space $\mathbb{C}^d/H$.

In $\mathbb{C}^d$ one can consider the 'splice equations' as follows. Consider a matrix $\{\lambda_{ij}\}_{ij}$ of full rank and of size $d\times(d-2)$. Then the equations $\sum_{i=1}^d \lambda_{ij}z_i^{\alpha_i} = 0$, for $j=1,\dots,d-2$, determine in $\mathbb{C}^d$ an isolated complete intersection singularity (Y,o) on which the group H acts as well. Then $(X,o)=(Y,o)/H$ is a normal surface singularity with the topological type of the Seifert manifold we started with. The equivariant Poincaré series of (Y,o) is $Z(\boldsymbol{t})$ [46]. For (X,o) Braun and Némethi [11] proves the identity $P(\boldsymbol{t}) = Z(\boldsymbol{t})$ mentioned in Section 2.3, hence $Z(\boldsymbol{t})$ is also the Poincaré series of the equivariant divisorial filtration associated with all the vertices.

Reduction Theorem 5.4.2 reduces the filtration to the $\mathbb{Z}$–filtration: the divisorial filtration associated with the central vertex. In the weighted homogeneous case this filtration is also induced by the weighted homogeneous equations. Then, $Z^{/H}(t)$ is the

Poincaré series of $\mathbb{C}^d/H$, $Z(t)$ is the equivariant Poincaré series of Y, hence $Z_0(t)$ is the Poincaré series of X; cf Dolgachev [20], Neumann [46], Pinkham [54].

By Section 2.3, $\{\mathrm{pc}(Z_h)\}_{h\in H}$ are the equivariant geometric genera of the universal abelian cover Y of X, hence $\mathrm{pc}(Z_0)$ and $\mathrm{pc}(Z_{\mathrm{ne}})$ are the geometric genera of X and Y respectively; cf Némethi [39].

6.4 Seiberg–Witten theoretical interpretations

Fix $h\in H$. Then, for any lifting $\sum_i c_i g_i$ of h, Corollary 5.2.1 and (6.2.6) give

$$\text{(6.4.1)}\quad \mathrm{pc}(Z_h)=\sum_{\ell\geq -\tilde{c}}\max\{0,-N_{\boldsymbol{c}}(\ell)\}=-\mathfrak{sw}_{-h*\sigma_{\mathrm{can}}}(M)-\frac{(K+2r_h)^2+|\mathcal{V}|}{8}.$$

Recall that $\sum_h \mathfrak{sw}_{-h*\sigma_{\mathrm{can}}}(M)$ is the *Casson–Walker invariant* $\lambda(M)$. Hence, for the nonequivariant version we get

$$\text{(6.4.2)}\quad \mathrm{pc}(Z_{\mathrm{ne}})=\frac{1}{2}\cdot\sum_{\substack{0\leq x_i<\alpha_i\\ 0\leq i\leq d}}\lfloor S(x)\rfloor\cdot\lfloor S(x)-1\rfloor=-\lambda(M)-\mathfrak{d}\cdot\frac{K^2+|\mathcal{V}|}{8}+\sum_h\chi(r_h).$$

For explicit formulae of $\lambda(M)$ and $K^2+|\mathcal{V}|$ in terms of Seifert invariants see eg Némethi and Nicolaescu [41, 2.6].

6.4.3 Remark Equation (6.4.1) can be compared with a known formulae of the Seiberg–Witten invariants involving the representative s_h. This will also lead us to an expression for $\chi(r_h)-\chi(r_s)$ in terms of $N_{\boldsymbol{c}}(\ell)$. Let $\boldsymbol{c}(s_h)=(c_0,\dots,c_d)$ be the coefficients of $s_{h,\mathrm{red}}$, cf Section 6.1. The set of all reduced coefficients $\boldsymbol{c}(s_h)$, when h runs in H, is characterized in Némethi [33, 11.5] by the inequalities

$$\text{(6.4.4)}\qquad \begin{cases} c_0\geq 0, & \alpha_i>c_i\geq 0\quad (\text{where } 1\leq i\leq d),\\ N_{\boldsymbol{c}}(\ell)\leq 0 & \text{for any } \ell<0.\end{cases}$$

Moreover, for this special lifting $\boldsymbol{c}(s_h)$ of h, in [33, Section 11] is proved

$$\text{(6.4.5)}\qquad \sum_{\ell\geq 0}\max\{0,-N_{\boldsymbol{c}(s_h)}(\ell)\}=-\mathfrak{sw}_{-h*\sigma_{\mathrm{can}}}(M)-\frac{(K+2s_h)^2+|\mathcal{V}|}{8}.$$

Using the discussion from the end of Section 6.1, this can be rewritten for *any* lifting $\boldsymbol{c}$ of h as

$$\text{(6.4.6)}\qquad \sum_{\ell\geq -\tilde{c}+\lfloor\tilde{c}(s_h)\rfloor}\max\{0,-N_{\boldsymbol{c}}(\ell)\}=-\mathfrak{sw}_{-h*\sigma_{\mathrm{can}}}(M)-\frac{(K+2s_h)^2+|\mathcal{V}|}{8}.$$

This compared with (6.4.1) gives for any lifting $\boldsymbol{c}$ of h

(6.4.7) $$\sum_{-\widetilde{c}+\lfloor \widetilde{c}(s_h)\rfloor > \ell \geq -\widetilde{c}} \max\{0, -N_{\boldsymbol{c}}(\ell)\} = \chi(r_h) - \chi(s_h).$$

6.4.8 Example The sum in (6.4.7), in general, can be nonzero. This happens, for example, in the case of the link of a rational singularity whose universal abelian cover is not rational. Here is a concrete example (cf Némethi [34, 4.5.4]): take the Seifert manifold with $b=-2$ and three legs, all of them with Seifert invariants $(\alpha_i,\omega_i)=(3,1)$. For $h=\sum_{i=1}^{3} g_i$ one has $s_h=\sum_{i=1}^{3} E_i^*$, the E_0–coefficient of s_h is 1, $r_h=s_h-E_0$ and $\chi(s_h)=0$, $\chi(r_h)=1$.

6.5 Ehrhart-theoretical interpretations

We fix $h\in H$ as above and *we assume that* $\widetilde{c}\in[0,1)$. Note that $Z_h(t)$ has the form $t^{\widetilde{c}}\sum_{\ell\geq 0} p_\ell t^\ell$; here the exponents $\{\widetilde{c}+\ell\}_{\ell\geq 0}$ are the possible E_0–coordinates of the elements $(r_h+L)\cap\mathcal{S}'$.

Let us compute the counting function for Z_h. If $S(t)=\sum_r p_r t^r$ is a series, we write $Q(S)(r')=\sum_{r<r'} p_r$, for $r'\in\mathbb{Q}_{\geq 0}$.

6.5.1 Lemma *For any $n\in\mathbb{N}_{\geq 0}$ one has the following facts.*

(a) $Q(Z_h)(n)=Q(Z_h)(n+\widetilde{c})$.

(b) $Q(Z_h^+)(n)$ *is a step function (hence piecewise polynomial) with*

$$Q(Z_h^+)(n)=\mathrm{pc}(Z_h) \quad \text{for } n>\deg(Z_h^+).$$

(c) $Q(Z_h^-)(n)$ *is a quasipolynomial:*

(6.5.2) $$\begin{aligned} Q(Z_h^-)(n) &= (1-e\widetilde{c})n - e\cdot\frac{n(n-1)}{2} - \sum_{i=1}^{d}\sum_{r_i=0}^{\alpha_i-1}\left\{\frac{c_i-\omega_i r_i}{\alpha_i}\right\}\left\lceil\frac{n-r_i}{\alpha_i}\right\rceil \\ &= -\frac{en^2}{2}+\frac{en}{2}(\gamma+1-2\widetilde{c}) - \sum_{i=1}^{d}\sum_{r_i=0}^{\alpha_i-1}\left\{\frac{c_i-\omega_i r_i}{\alpha_i}\right\}\left(\left\{\frac{r_i-n}{\alpha_i}\right\}-\frac{r_i}{\alpha_i}\right) \end{aligned}$$

In particular, if $n=m\alpha$ for $m\in\mathbb{Z}$, and $n>\deg(Z_h^+)$, then the double sum is zero, hence

(6.5.3) $$Q(Z_h)(n) = -\frac{en^2}{2}+\frac{en}{2}(\gamma+1-2\widetilde{c})+\mathrm{pc}(Z_h).$$

This is compatible with the expression provided by Theorem 2.2.2 and Reduction Theorem 5.4.2. Indeed, let us fix any chamber $\mathcal{C}$ such that $\mathrm{int}(\mathcal{C}\cap\mathcal{S}')\neq\varnothing$, and $\mathcal{C}$ contains the ray $\mathcal{R}=\mathbb{R}_{\geq 0}\cdot E_0^*$. Since the numerator of $f(\boldsymbol{t})$ is $(1-\boldsymbol{t}^{E_0^*})^{d-2}$, all the b_k–vectors belong to $\mathcal{R}$. In particular, $\bigcap_k(b_k+\mathcal{C})$ intersects $\mathcal{R}$ along a semiline $\mathcal{R}^{\geq c}=\mathbb{R}_{\geq\mathrm{const}}\cdot E_0^*$ of $\mathcal{R}$. Since $Q_h(l')$ is quasipolynomial on $\bigcap_k(b_k+\mathcal{C})$, cf (4.3.14), and a restriction of it is determined by (2.2.3) whose right-hand side is a quasipolynomial too, we obtain that the identity (2.2.3) is valid on $\mathcal{R}^{\geq c}$ as well.

Recall that for any $h\in H$ and $l'\in L'$ we have a unique $q_{l',h}\in\square$ with $l'+q_{l',h}\in r_h+L$. Hence we get

$$Q_h(l')=-\mathfrak{sw}_{-h*\sigma_{\mathrm{can}}}(M)-\frac{(K+2l'+2q_{l',h})^2+|\mathcal{V}|}{8}\quad (l'\in\mathcal{R}^{\geq c}). \tag{6.5.4}$$

The term $q_{l',h}$ is responsible for the nonpolynomial behavior. Nevertheless, if we assume that $l'=m\mathfrak{o}E_0^*\in\mathcal{R}^{\geq c}\cap L$, $m\in\mathbb{Z}$, then $q_{l',h}=r_h$, hence by (6.4.1)

$$Q_h(l')=-\frac{(l',l'+K+2r_h)}{2}+\mathrm{pc}(Z_h). \tag{6.5.5}$$

By Reduction Theorem 5.4.2 $Q_h(l')$ from (6.5.5) depends only on the E_0–coefficient of $l'=m\mathfrak{o}E_0^*$, which is exactly $m\alpha$. One sees that in fact (6.5.5) agrees with (6.5.3) if we set $n=m\alpha$.

The nonequivariant version can be obtained by summation of (6.5.3). For this we need $\sum_h\tilde{c}(r_h)$. Consider the group homomorphism $\varphi\colon H\to\mathbb{Q}/\mathbb{Z}$ given by $h\mapsto[\tilde{c}(r_h)]$, or $[E_v^*]\mapsto[-(E_0^*,E_v^*)]$. Its image is generated by the classes of $1/(\alpha_i|e|)$, hence its order is $\mathfrak{o}$. Hence, $\tilde{c}(r_h)$ vanishes exactly $\mathfrak{d}/\mathfrak{o}$ times (whenever $h\in\ker(\varphi)$). In all other cases $\tilde{c}(r_h)\neq 0$, and $\tilde{c}(r_h)+\tilde{c}(r_{-h})=1$. In particular, $2\sum_h\tilde{c}(r_h)=\mathfrak{d}-\mathfrak{d}/\mathfrak{o}$. Therefore, the summation of (6.5.3) provides

$$Q(Z_{\mathrm{ne}})(n)=-\frac{\mathfrak{d}en^2}{2}+\frac{\mathfrak{d}en}{2}\Big(\gamma+\frac{1}{\mathfrak{o}}\Big)+\mathrm{pc}(Z_{\mathrm{ne}})\quad(\text{for } n\in\alpha\mathbb{Z}). \tag{6.5.6}$$

Next, we will identify the coefficients of (6.5.3) and (6.5.6) with the first three coefficients of the Ehrhart quasipolynomial $\mathcal{L}_h^{\mathcal{C}}(\mathcal{T})$ via the identity (4.3.14).

For simplicity we will assume that $\mathfrak{o}=1$, in particular all the b_k–vectors belong to L.

If $l'\in\mathcal{R}$, then by Reduction Theorem 5.4.2 the counting function $\mathcal{L}_h^{\mathcal{C}}(\mathcal{T},l')$ of the polytope $P^{(l')}$ depends only on the E_0–coefficient of l'; let us denote this coefficient by l'_0.

Hence, this $\mathcal{L}_h^{\mathcal{C}}(\mathcal{T},l'_0)$ is the Ehrhart quasipolynomial of the d–dimensional simplicial polytope, being its h–class counting function. Via (6.1.3) the definition (4.3.2) of this polytope becomes

$$P_0=\Big\{(x_1,\dots,x_d)\in(\mathbb{R}_{\geq 0})^d\ \Big|\ \sum_i\frac{x_i}{|e|\alpha_i}<l'_0\Big\}. \tag{6.5.7}$$

Let

$$\mathcal{L}_h^{\mathcal{C}}(\mathcal{T},l'_0)=\sum_{j=0}^{d}\mathfrak{a}_{h,j}(l'_0)\cdot\frac{(l'_0)^j}{j!} \tag{6.5.8}$$

be the coefficients of the Ehrhart quasipolynomial: each $\mathfrak{a}_{h,j}(l'_0)$ is a periodic function in l'_0 and is normalized by $1/j!$. Since the numerator of f is $(1-t^{1/|e|})^{d-2}$, by (4.3.14) we obtain for $l'\in\mathcal{R}$

$$Q_h(l')=\sum_{j=0}^{d}\mathfrak{a}_{h,j}(l'_0)\cdot\frac{1}{j!}\sum_{k=0}^{d-2}(-1)^k\binom{d-2}{k}\Big(l'_0-\frac{k}{|e|}\Big)^j. \tag{6.5.9}$$

This equals the expression (6.5.4) above. The nonpolynomial behavior of these two expressions indicate that $\mathfrak{a}_j(l'_0)$ is indeed nonconstant periodic, and can be determined explicitly.

Since we are interested primarily in the Seiberg–Witten invariant, namely in $\mathrm{pc}(Z_h)$, we perform this explicit identification only via the expressions (6.5.3) and (6.5.5). Hence, similarly as in these cases, we take $l'=m\mathfrak{o}E_0^*\in\mathcal{R}^{\geq c}\cap L$, and we identify (6.5.3) with (6.5.9) evaluated for l', whose E_0–coefficient is $l'_0=m\alpha=n$. In this case $\mathfrak{a}_{h,j}(n)$ is a *constant*, denoted by $\mathfrak{a}_{h,j}$, and we have the following:

$$-\frac{en^2}{2}+\frac{ne}{2}(\gamma+1-2\tilde{c})+\mathrm{pc}(Z_h)=\sum_{j=0}^{d}\mathfrak{a}_{h,j}\cdot\frac{1}{j!}\sum_{k=0}^{d-2}(-1)^k\binom{d-2}{k}\Big(n-\frac{k}{|e|}\Big)^j \tag{6.5.10}$$

Here the following combinatorial expression is helpful (see for example Pólya and Szegő [55, pages 7–8]):

$$\frac{(-1)^d}{(d-2)!}\cdot\sum_{k=0}^{d-2}(-1)^k\binom{d-2}{k}k^j=\begin{cases}0 & \text{if } j<d-2,\\ 1 & \text{if } j=d-2,\\ (d-2)(d-1)/2 & \text{if } j=d-1,\\ (d-2)(d-1)d(3d-5)/24 & \text{if } j=d.\end{cases} \tag{6.5.11}$$

We obtain

$$(6.5.12)\qquad \begin{aligned} \frac{\mathfrak{a}_{h,d}}{|e|^d} &= \frac{1}{|e|},\\ \frac{\mathfrak{a}_{h,d-1}}{|e|^{d-1}} &= \frac{d-2}{2|e|} - \frac{1}{2}(\gamma+1-2\tilde{c}),\\ \frac{\mathfrak{a}_{h,d-2}}{|e|^{d-2}} &= \mathrm{pc}(Z_h) + \frac{(d-2)(3d-7)}{24|e|} - \frac{d-2}{4}(\gamma+1-2\tilde{c}). \end{aligned}$$

In particular, the $\mathfrak{a}_{h,d-2}$ can be identified (up to 'easy' extra terms) with $\mathrm{pc}(Z_h)$ (with analytical interpretation $\dim(H^1(\widetilde{Y},\mathcal{O}_{\widetilde{Y}})_{\theta(h)})$ and Seiberg–Witten theoretical interpretation (6.4.1)). The first coefficients can also be identified with the equivariant volume of P_0, (a fact already known in the nonequivariant cases). Usually (in the nonequivariant case, and when we count the points of all the facets) the second coefficient can be related with the volumes of the facets. Here we eliminate from this count some of the facets, and we are in the equivariant situation as well. The expression of the second coefficient is also a novelty of the present article (to the best of authors' knowledge).

In the nonequivariant case, if $\sum_{j=0}^{d} \mathfrak{a}_j \frac{n^j}{j!}$ is the classical Ehrhart polynomial of P_0, then

$$(6.5.13)\qquad \begin{aligned} \frac{\mathfrak{a}_d}{|e|^d} &= \prod_i \alpha_i,\\ \frac{\mathfrak{a}_{d-1}}{|e|^{d-1}} &= \prod_i \alpha_i \cdot \Big(-\frac{1}{\alpha} + \sum_i \frac{1}{\alpha_i}\Big)/2,\\ \frac{\mathfrak{a}_{d-2}}{|e|^{d-2}} &= \prod_i \alpha_i \Big(\frac{\mathrm{pc}(Z_{\mathrm{ne}})}{\prod_i \alpha_i} - \frac{(d-2)(3d-5)}{24} + \frac{d-2}{4}\Big(-\frac{1}{\alpha} + \sum_i \frac{1}{\alpha_i}\Big)\Big). \end{aligned}$$

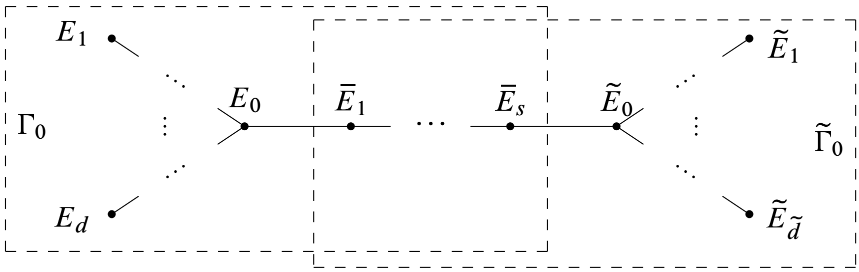
In this nonequivariant case the identities (6.5.13) are valid even without the assumption $\mathfrak{o}=1$ by the same proof.

The formulae in (6.5.12) and (6.5.13) can be further simplified if we replace P_0 by $|e|P_0$, or if we substitute in the Ehrhart polynomial the new variable $\lambda := |e|l'_0$ instead of l'_0; cf Section 8.

7 The two-node case

7.1 Notation and the group H

We consider the following graph Γ:



The nodes E_0 and $\widetilde{E}_0$ have decorations b_0 and $\widetilde{b}_0$ respectively. Similarly as in the one-node case, we codify the decorations of maximal chains by continued fraction expansions. In fact, it is convenient to consider the two maximal star-shaped graphs Γ_0 and $\widetilde{\Gamma}_0$, and the corresponding normalized Seifert invariants of their legs. Hence, let the normalized Seifert invariants of the legs with ends E_i $(1 \leq i \leq d)$ be (α_i, ω_i), while of the legs with ends $\widetilde{E}_j$ $(1 \leq j \leq \widetilde{d})$ be $(\widetilde{\alpha}_j, \widetilde{\omega}_j)$.

The chain connecting the nodes, viewed in Γ_0 has normalized Seifert invariants (α_0, ω_0), while viewed as a leg in $\widetilde{\Gamma}_0$, it has Seifert invariants $(\alpha_0, \widetilde{\omega}_0)$. One has $\omega_0\widetilde{\omega}_0 = \alpha_0\tau + 1$. Clearly, α_0 is the determinant of the chain, and

$$\omega_0 := \det\Big(\overset{\bar{E}_2}{\bullet}\!\!- \ \cdots \ -\!\!\overset{\bar{E}_s}{\bullet} \Big) \quad \widetilde{\omega}_0 := \det\Big(\overset{\bar{E}_1}{\bullet}\!\!- \ \cdots \ -\!\!\overset{\bar{E}_{s-1}}{\bullet} \Big) \quad \tau := \det\Big(\overset{\bar{E}_2}{\bullet}\!\!- \ \cdots \ -\!\!\overset{\bar{E}_{s-1}}{\bullet} \Big).$$

We denote the orbifold Euler numbers of the star-shaped subgraphs Γ_0 and $\widetilde{\Gamma}_0$ by

$$e = b_0 + \frac{\omega_0}{\alpha_0} + \sum_i \frac{\omega_i}{\alpha_i} \quad \text{and} \quad \widetilde{e} = \widetilde{b}_0 + \frac{\widetilde{\omega}_0}{\alpha_0} + \sum_j \frac{\widetilde{\omega}_j}{\widetilde{\alpha}_j}.$$

Consider the *orbifold intersection matrix* (cf Braun and Némethi [10, 4.1.4]):

$$I^{\text{orb}} = \begin{pmatrix} e & 1/\alpha_0 \\ 1/\alpha_0 & \widetilde{e} \end{pmatrix}$$

Then, the negative-definiteness of I (or Γ) implies that I^{orb} is negative-definite too, hence

$$\varepsilon := \det I^{\text{orb}} = e\widetilde{e} - \frac{1}{\alpha_0^2} > 0.$$

Then the determinant of the graph is $\det(\Gamma) = \det(-I) = \varepsilon \cdot \alpha_0 \prod_i \alpha_i \prod_j \widetilde{\alpha}_j$; cf [10].

Using (2.1.1) we have the following intersection number of the dual base elements:

$$\begin{aligned}(7.1.1)\qquad &(E_0^*)^2=\frac{\widetilde{e}}{\varepsilon},\quad (\widetilde{E}_0^*)^2=\frac{e}{\varepsilon},\quad (E_0^*,\widetilde{E}_0^*)=-\frac{1}{\alpha_0\varepsilon},\quad (E_0^*,E_i^*)=\frac{\widetilde{e}}{\alpha_i\varepsilon},\\ &(E_0^*,\widetilde{E}_j^*)=-\frac{1}{\alpha_0\widetilde{\alpha}_j\varepsilon},\quad (\widetilde{E}_0^*,E_i^*)=-\frac{1}{\alpha_0\alpha_i\varepsilon},\quad (\widetilde{E}_0^*,\widetilde{E}_j^*)=\frac{e}{\widetilde{\alpha}_j\varepsilon}.\end{aligned}$$

As in Section 5.3 or Section 6.1, we write $n^i_{k_1,k_2}$, $\widetilde{n}^j_{k_1,k_2}$ respectively $\overline{n}_{k_1,k_2}$ for the determinant of the subchains of the 'left' i^{th} leg, 'right' j^{th} leg and connecting chain connecting the vertices v_{k_1} and v_{k_2}. Let ν_i and $\widetilde{\nu}_j$ be the number of vertices in the legs; cf Section 6.1. Then (with the standard notation, where $E_{i\ell}$ and $\widetilde{E}_{j\ell}$ are the vertices of the legs) one has the following slightly technical lemma, but whose proof is standard based on the arithmetical properties of continued fractions.

7.1.2 Lemma (a) *For any* $1\le \ell<\nu_i$,

$$E^*_{i\ell}=n^i_{\ell+1,\nu_i}E_i^*+\sum_{\ell<r\le\nu_i}\frac{n^i_{1,\ell-1}n^i_{r+1,\nu_i}-n^i_{1,r-1}n^i_{\ell+1,\nu_i}}{\alpha_i}E_{ir}.$$

(There is a similar formula for $\widetilde{E}^*_{j\ell}$*.)*

(b) *For* $1<k\le s$,

$$\overline{E}_k^*=\overline{n}_{1,k-1}\overline{E}_1^*-\overline{n}_{2,k-1}E_0^*+\sum_{1\le r<k}\frac{\overline{n}_{1,r-1}\overline{n}_{k+1,s}-\overline{n}_{1,k-1}\overline{n}_{r+1,s}}{\alpha_0}\overline{E}_r.$$

(This is true even for $k=s+1$ *with the identification* $\overline{E}^*_{k+1}=\widetilde{E}^*_0$*.)*

Next, we give a presentation of $H=L'/L$. Set $g_i:=[E_i^*]$ $(1\le i\le d)$, $\widetilde{g}_j:=[\widetilde{E}_j^*]$ $(1\le j\le \widetilde{d})$, $g_0:=[E_0^*]$ and $\widetilde{g}_0:=[\widetilde{E}_0^*]$. Moreover we need to choose an additional generator corresponding to a vertex sitting on the connecting chain: we choose $\overline{g}:=[\overline{E}_1^*]$ (this motivates the choice in (b) too). The above lemma implies

$$(7.1.3)\qquad [E^*_{i\ell}]=n^i_{\ell+1,\nu_i}g_i,\quad [\widetilde{E}^*_{j\ell}]=\widetilde{n}^j_{\ell+1,\widetilde{\nu}_j}\widetilde{g}_j\quad\text{and}\quad [\overline{E}_k^*]=\overline{n}_{1,k-1}\overline{g}-\overline{n}_{2,k-1}g_0;$$

and similar arguments as in the star-shaped case provides the following presentation for H:

$$(7.1.4)\qquad H=\left\langle g_0,\widetilde{g}_0,g_i,\widetilde{g}_j,\overline{g}\;\middle|\; \begin{aligned}&g_0=\alpha_i\cdot g_i;\ \ \widetilde{g}_0=\widetilde{\alpha}_j\cdot\widetilde{g}_j\\ &\alpha_0\cdot\overline{g}=\omega_0\cdot g_0+\widetilde{g}_0;\ \ -\overline{g}-b_0\cdot g_0=\sum_i\omega_i\cdot g_i\\ &-\widetilde{\omega}_0\cdot\overline{g}+\tau\cdot g_0-\widetilde{b}_0\cdot\widetilde{g}_0=\sum_j\widetilde{\omega}_j\cdot\widetilde{g}_j\end{aligned}\right\rangle$$

Moreover, for any $l'\in L'$,

$$l' = c_0 E_0^* + \widetilde{c}_0 \widetilde{E}_0^* + \sum_k \overline{c}_k \overline{E}_k^* + \sum_{i,\ell} c_{i\ell} E_{i\ell}^* + \sum_{j\ell} \widetilde{c}_{j\ell} \widetilde{E}_{j\ell}^*,$$

if we define its *reduced transform* l'_{red} by

$$(c_0 - \sum_{k>1} \overline{n}_{2,k-1}\overline{c}_k) E_0^* + \widetilde{c}_0 \widetilde{E}_0^* + (\overline{c}_1 + \sum_{k>1} \overline{n}_{1,k-1}\overline{c}_k)\overline{E}_1^* + \sum_{i,\ell} c_{i\ell} n^i_{\ell+1,\nu_i} E_i^* + \sum_{j,\ell} \widetilde{c}_{j\ell} \widetilde{n}^j_{\ell+1,\widetilde{\nu}_j} \widetilde{E}_j^*,$$

then, by Lemma 7.1.2, $[l'] = [l'_{\mathrm{red}}]$ in H. Moreover, if for any $l'\in L'$ we distinguish the E_0 and $\widetilde{E}_0$ coefficients, that is, we set $c(l') := -(E_0^*, l')$ and $\widetilde{c}(l') := -(\widetilde{E}_0^*, l')$, then $c(l') = c(l'_{\mathrm{red}})$ and $\widetilde{c}(l') = \widetilde{c}(l'_{\mathrm{red}})$ as well. Lemma 7.1.2(b) (applied for $k = s+1$) provide these coefficients for $\overline{E}_1$:

$$(\overline{E}_1^*, E_0^*) = \frac{1}{\varepsilon\alpha_0}\Big(\omega_0\widetilde{e} - \frac{1}{\alpha_0}\Big), \quad (\overline{E}_1^*, \widetilde{E}_0^*) = \frac{1}{\varepsilon\alpha_0}\Big(e - \frac{\omega_0}{\alpha_0}\Big) \tag{7.1.5}$$

We will use the coefficients $\boldsymbol{c} = (c_0, \widetilde{c}_0, \overline{c}, c_i, \widetilde{c}_j)$ to write an element $l'_{\mathrm{red}} = c_0 E_0^* + \widetilde{c}_0 \widetilde{E}_0^* + \overline{c}\overline{E}_1^* + \sum_i c_i E_i^* + \sum_j \widetilde{c}_j \widetilde{E}_j^*$. Then (7.1.1) and (7.1.5) imply that

$$\begin{pmatrix} c \\ \widetilde{c} \end{pmatrix} = \begin{pmatrix} c(l'_{\mathrm{red}}) \\ \widetilde{c}(l'_{\mathrm{red}}) \end{pmatrix} = (-I^{\mathrm{orb}})^{-1} \cdot \begin{pmatrix} A \\ \widetilde{A} \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} -\widetilde{e} & 1/\alpha_0 \\ 1/\alpha_0 & -e \end{pmatrix} \cdot \begin{pmatrix} A \\ \widetilde{A} \end{pmatrix}, \tag{7.1.6}$$

where

$$A := c_0 + \sum_i \frac{c_i}{\alpha_i} + \frac{\omega_0}{\alpha_0}\overline{c}, \quad \widetilde{A} := \widetilde{c}_0 + \sum_j \frac{\widetilde{c}_j}{\widetilde{\alpha}_j} + \frac{1}{\alpha_0}\overline{c}.$$

Therefore, any $h\in H$ has a lift of type $l'_{h,\mathrm{red}}$. Although the corresponding coefficients c and $\widetilde{c}$ depend on the lift, by adding $\pm E_0$ and $\pm\widetilde{E}_0$ to $l'_{h,\mathrm{red}}$ we can achieve $c, \widetilde{c} \in [0,1)$, and these values are uniquely determined by h. For example, the reduced transform $(r_h)_{\mathrm{red}}$ of r_h satisfies $c((r_h)_{\mathrm{red}}) = c(r_h) \in [0,1)$ and $\widetilde{c}((r_h)_{\mathrm{red}}) = \widetilde{c}(r_h) \in [0,1)$ since $r_h \in \square$.

As we will see, for different elements of $h\in H$, we have to shift the rank two lattices by vectors of type $(c, \widetilde{c})$, hence the vectors $(c, \widetilde{c})$ will play a crucial role later.

7.2 The function Z

If we wish to compute the periodic constant of $Z^e(\boldsymbol{t})$, by Reduction Theorem 5.4.2 we can eliminate all the variables of $Z^e(\boldsymbol{t})$ except the variables of the nodes; these remaining two variables are denoted by $(t,\tilde{t})$. Therefore the equivariant form of reciprocal of the denominator is

$$Z^{/H}(t,\tilde{t}) = \prod_i (1-t^{-(E_i^*,E_0^*)}\tilde{t}^{-(E_i^*,\tilde{E}_0^*)}[g_i])^{-1}\cdot\prod_j (1-t^{-(\tilde{E}_j^*,E_0^*)}\tilde{t}^{-(\tilde{E}_j^*,\tilde{E}_0^*)}[\tilde{g}_j])^{-1}$$
$$= \sum_{x_i,\tilde{x}_j\geq 0} t^{\frac{-\tilde{\varepsilon}}{\varepsilon}\sum_i \frac{x_i}{\alpha_i}+\frac{1}{\alpha_0\varepsilon}\sum_j\frac{\tilde{x}_j}{\tilde{\alpha}_j}}\,\tilde{t}^{\frac{1}{\alpha_0\varepsilon}\sum_i\frac{x_i}{\alpha_i}+\frac{-e}{\varepsilon}\sum_j\frac{\tilde{x}_j}{\tilde{\alpha}_j}}\Big[\sum_i x_i g_i+\sum_j \tilde{x}_j\tilde{g}_j\Big].$$

We fix a lift $c_0E_0^*+\tilde{c}_0\tilde{E}_0^*+\overline{c}\,\overline{E}_1^*+\sum_i c_iE_i^*+\sum_j\tilde{c}_j\tilde{E}_j^*$ of h. Then the class of $\sum_i x_iE_i^*+\sum_j\tilde{x}_j\tilde{E}_j^*$ equals h if and only if its difference with the lift is a linear combination of the relation in (7.1.4). In other words, if there exist $\ell_0,\tilde{\ell}_0,\overline{\ell},\ell_i,\tilde{\ell}_j\in\mathbb{Z}$ such that:

(a) $-c_0=\sum_i\ell_i-b_0\ell_0+\tau\tilde{\ell}_0+\omega_0\overline{\ell}$

(b) $-\tilde{c}_0=\sum_j\tilde{\ell}_j-\tilde{b}_0\tilde{\ell}_0+\overline{\ell}$

(c) $x_i-c_i=-\omega_i\ell_0-\alpha_i\ell_i\quad(i=1,\dots,d)$

(d) $\tilde{x}_j-\tilde{c}_j=-\tilde{\omega}_j\tilde{\ell}_0-\tilde{\alpha}_j\tilde{\ell}_j\quad(j=1,\dots,\tilde{d})$

(e) $-\overline{c}=-\ell_0-\tilde{\omega}_0\tilde{\ell}_0-\alpha_0\overline{\ell}$

From (e) we deduce that

$$\ell_0+\tilde{\omega}_0\tilde{\ell}_0\equiv\overline{c}\ (\mathrm{mod}\,\alpha_0). \tag{7.2.1}$$

Since $x_i,\tilde{x}_j\geq 0$, (c) and (d) imply $(c_i-\omega_i\ell_0)/\alpha_i\geq\ell_i$, $(\tilde{c}_j-\tilde{\omega}_j\tilde{\ell}_0)/\tilde{\alpha}_j\geq\tilde{\ell}_j$. Recall also that $\omega_0\tilde{\omega}_0=\alpha_0\tau+1$. Therefore if we set $m_i:=\lfloor(c_i-\omega_i\ell_0)/\alpha_i\rfloor-\ell_i$ and $\tilde{m}_j:=\lfloor(\tilde{c}_j-\tilde{\omega}_j\tilde{\ell}_0)/\tilde{\alpha}_j\rfloor-\tilde{\ell}_j$ nonnegative integers then the number of the realization of h in the form $\sum_i x_ig_i+\sum_j\tilde{x}_j\tilde{g}_j$ is determined by the number of nonnegative integral $(d+\tilde{d})$–tuples $(m_i,\tilde{m}_j)$ satisfying

$$N_{\boldsymbol{c}}(\ell_0,\tilde{\ell}_0):=c_0+\frac{\omega_0}{\alpha_0}\overline{c}-\Big(b_0+\frac{\omega_0}{\alpha_0}\Big)\ell_0-\frac{1}{\alpha_0}\tilde{\ell}_0+\sum_i\Big\lfloor\frac{c_i-\omega_i\ell_0}{\alpha_i}\Big\rfloor=\sum_i m_i,$$

$$\tilde{N}_{\boldsymbol{c}}(\ell_0,\tilde{\ell}_0):=\tilde{c}_0+\frac{1}{\alpha_0}\overline{c}-\Big(\tilde{b}_0+\frac{\tilde{\omega}_0}{\alpha_0}\Big)\tilde{\ell}_0-\frac{1}{\alpha_0}\ell_0+\sum_j\Big\lfloor\frac{\tilde{c}_j-\tilde{\omega}_j\tilde{\ell}_0}{\tilde{\alpha}_i}\Big\rfloor=\sum_j\tilde{m}_j.$$

This number is $\binom{N_c(\ell_0,\tilde{\ell}_0)+d-1}{d-1}\binom{\tilde{N}_c(\ell_0,\tilde{\ell}_0)+\tilde{d}-1}{\tilde{d}-1}$ if $N_{\boldsymbol{c}}$ and $\tilde{N}_{\boldsymbol{c}}$ are nonnegative, otherwise it is 0. Note that (7.2.1) guarantees that both $N_{\boldsymbol{c}}$ and $\tilde{N}_{\boldsymbol{c}}$ are integers. Furthermore, (c) and (d) and (7.1.6) show that the exponent of t and $\tilde{t}$ in the formula of $Z_h^{/H}(t,\tilde{t})$ are equal to ℓ_0+c and $\tilde{\ell}_0+\tilde{c}$ respectively. Hence

$$Z_h^{/H}(t,\tilde{t})=\sum\binom{N_{\boldsymbol{c}}(\ell,\tilde{\ell})+d-1}{d-1}\binom{\tilde{N}_{\boldsymbol{c}}(\ell,\tilde{\ell})+\tilde{d}-1}{\tilde{d}-1}t^{\ell+c}\tilde{t}^{\tilde{\ell}+\tilde{c}},$$

where the sum runs over $(\ell,\widetilde{\ell})\in\mathbb{Z}^2$ with $\ell+\widetilde{\omega}_0\widetilde{\ell}\equiv\overline{c}\pmod{\alpha_0}$.

We have that $\bigl(1-t^{-(E_0^*,E_0^*)}\widetilde{t}^{-(E_0^*,\widetilde{E}_0^*)}[g_0]\bigr)^{d-1}\cdot\bigl(1-t^{-(\widetilde{E}_0^*,E_0^*)}\widetilde{t}^{-(\widetilde{E}_0^*,\widetilde{E}_0^*)}[\widetilde{g}_0]\bigr)^{\widetilde{d}-1}$ is the numerator of $Z(t,\widetilde{t})$. Hence we get Z^e by multiplying this expression by $\sum_h Z_h^{/H}[h]$. Recall that $h=c_0g_0+\widetilde{c}_0\widetilde{g}_0+\overline{cg}+\sum_i c_ig_i+\sum_j\widetilde{c}_j\widetilde{g}_j$ is paired with $\boldsymbol{c}$. Set $h':=h+kg_0+\widetilde{k}\widetilde{g}_0$ which corresponds to $\boldsymbol{c}'=\boldsymbol{c}+(k,\widetilde{k},0,0,0)$. Hence $Z_{h'}[h']$ is the next sum according to the decompositions $h'=h+kg_0+\widetilde{k}\widetilde{g}_0$:

$$\begin{aligned}
&\sum_{k=0}^{d-1}(-1)^k\binom{d-1}{k}\sum_{\widetilde{k}=0}^{\widetilde{d}-1}(-1)^{\widetilde{k}}\binom{\widetilde{d}-1}{\widetilde{k}}\\
&\qquad\cdot\sum_h\Biggl(\sum_{\ell+\widetilde{\omega}_0\widetilde{\ell}\equiv\overline{c}(\alpha_0)}\binom{N_{\boldsymbol{c}}(\ell,\widetilde{\ell})+d-1}{d-1}\binom{\widetilde{N}_{\boldsymbol{c}}(\ell,\widetilde{\ell})+\widetilde{d}-1}{\widetilde{d}-1}\\
&\qquad\qquad\cdot t^{\ell+c+(-\widetilde{e}k+\widetilde{k}/\alpha_0)/\varepsilon}\cdot\widetilde{t}^{\widetilde{\ell}+\widetilde{c}+(-e\widetilde{k}+k/\alpha_0)/\varepsilon}\Biggr)[h']\\
&=\sum_{k=0}^{d-1}(-1)^k\binom{d-1}{k}\sum_{\widetilde{k}=0}^{\widetilde{d}-1}(-1)^{\widetilde{k}}\binom{\widetilde{d}-1}{\widetilde{k}}\\
&\qquad\cdot\sum_h\Biggl(\sum_{\ell+\widetilde{\omega}_0\widetilde{\ell}\equiv\overline{c}(\alpha_0)}\binom{N_{\boldsymbol{c}'}(\ell,\widetilde{\ell})-k+d-1}{d-1}\binom{\widetilde{N}_{\boldsymbol{c}'}(\ell,\widetilde{\ell})-\widetilde{k}+\widetilde{d}-1}{\widetilde{d}-1}\\
&\qquad\qquad\cdot t^{\ell+c'}\cdot\widetilde{t}^{\widetilde{\ell}+\widetilde{c}'}\Biggr)[h']
\end{aligned}$$

Changing the orders of the sums and using the combinatorial formula

$$\sum_{k=0}^{d-1}(-1)^k\binom{N-k+d-1}{d-1}\binom{d-1}{k}=1$$

for $N\geq 0$ and $=0$ otherwise, we get the following.

7.2.2 Theorem *For any $h\in H$ one has*

$$Z_h(t,\widetilde{t})=\sum_{(\ell,\widetilde{\ell})\in\mathcal{S}_{\boldsymbol{c}}}t^{\ell+c}\widetilde{t}^{\widetilde{\ell}+\widetilde{c}},\tag{7.2.3}$$

where $\mathcal{S}_{\boldsymbol{c}}:=\{(\ell,\widetilde{\ell})\in\mathbb{Z}^2\mid N_{\boldsymbol{c}}(\ell,\widetilde{\ell})\geq 0,\ \widetilde{N}_{\boldsymbol{c}}(\ell,\widetilde{\ell})\geq 0$ *and* $\ell+\widetilde{\omega}_0\widetilde{\ell}\equiv\overline{c}\pmod{\alpha_0}\}$.

It is straightforward to verify that the right-hand side of (7.2.3) does not depend on the choice of $\boldsymbol{c}$, it depends only on h. The identity (7.2.3) is remarkable: since the coefficient of any monomial in the sum is one, it realizes the bridge between the series Z^e and the equivariant Hilbert series of *affine monoids and their modules*.

7.3 The structure of $\mathcal{S}_{\boldsymbol{c}}$

Recall that for any $h \in H$ we consider a lift of h identified by a certain $\boldsymbol{c}$ which determines the pair $(c, \widetilde{c})$ (cf (7.1.6)), and the integers $N_{\boldsymbol{c}}(\mathfrak{l})$ and $\widetilde{N}_{\boldsymbol{c}}(\mathfrak{l})$, where $\mathfrak{l} = (\ell, \widetilde{\ell}) \in \mathbb{Z}^2$. We define

$$\mathbb{Z}^2(\boldsymbol{c}) := \{(\ell, \widetilde{\ell}) \in \mathbb{Z}^2 \mid \ell + \widetilde{\omega}_0 \widetilde{\ell} \equiv \overline{c} \ (\mathrm{mod}\, \alpha_0)\}.$$

If $h = 0$ then we always choose the zero lift with $\boldsymbol{c} = \boldsymbol{0}$.

If in the definition of $N_{\boldsymbol{c}}(\mathfrak{l})$ and $\widetilde{N}_{\boldsymbol{c}}(\mathfrak{l})$ we replace each $[y]$ by y, we get the entries of

$$\begin{pmatrix} A - e\ell_0 - \widetilde{\ell}/\alpha_0 \\ \widetilde{A} - \ell_0/\alpha_0 - \widetilde{e}\widetilde{\ell}_0 \end{pmatrix} = -I^{\mathrm{orb}} \begin{pmatrix} \ell + c \\ \widetilde{\ell} + \widetilde{c} \end{pmatrix}.$$

This motivates us to define

$$\overline{\mathcal{S}}_{\boldsymbol{c}} := \left\{ \mathfrak{l} \in \mathbb{Z}^2(\boldsymbol{c}) \;\middle|\; -I^{\mathrm{orb}} \begin{pmatrix} \ell + c \\ \widetilde{\ell} + \widetilde{c} \end{pmatrix} \geq 0 \right\}. \tag{7.3.1}$$

Clearly $\mathcal{S}_{\boldsymbol{c}} \subset \overline{\mathcal{S}}_{\boldsymbol{c}}$. We also consider $\mathcal{C}^{\mathrm{orb}}$, the real cone $\{\mathfrak{l} \in \mathbb{R}^2 \mid -I^{\mathrm{orb}} \cdot \mathfrak{l} \geq 0\}$. Then $\overline{\mathcal{S}}_{\boldsymbol{c}} = (\mathcal{C}^{\mathrm{orb}} - (c, \widetilde{c})) \cap \mathbb{Z}^2(\boldsymbol{c})$.

7.3.2 Lemma (1) *$\mathcal{S}_{\boldsymbol{0}}$ and $\overline{\mathcal{S}}_{\boldsymbol{0}}$ are affine monoids. $\overline{\mathcal{S}}_{\boldsymbol{0}}$ is the normalization of $\mathcal{S}_{\boldsymbol{0}}$.*

(2) *$\mathcal{S}_{\boldsymbol{c}}$ and $\overline{\mathcal{S}}_{\boldsymbol{c}}$ are finitely generated $\mathcal{S}_{\boldsymbol{0}}$–modules, $\mathcal{S}_{\boldsymbol{c}}$ is a submodule of $\overline{\mathcal{S}}_{\boldsymbol{c}}$.*

Proof (1) is elementary. By Bruns and Gubeladze [13, 2.12] $\overline{\mathcal{S}}_{\boldsymbol{c}}$ is finitely generated over $\overline{\mathcal{S}}_{\boldsymbol{0}}$, but $\overline{\mathcal{S}}_{\boldsymbol{0}}$ itself is finitely generated as an $\mathcal{S}_{\boldsymbol{0}}$ module. □

7.3.3 Lemma *There exists $\mathfrak{v}_1$ and $\mathfrak{v}_2$ elements of $\mathbb{Z}^2$ with the following properties:*

(a) *$\mathfrak{v}_1$ and $\mathfrak{v}_2$ belong to $\mathcal{S}_{\boldsymbol{0}}$ and $\mathbb{R}_{\geq 0}\mathfrak{v}_1 + \mathbb{R}_{\geq 0}\mathfrak{v}_2 = \mathcal{C}^{\mathrm{orb}}$.*

(b) *For any $\mathfrak{l} \in \overline{\mathcal{S}}_{\boldsymbol{c}}$ one has:*
   (i) $N_{\boldsymbol{c}}(\mathfrak{l} + \mathfrak{v}_1) = N_{\boldsymbol{c}}(\mathfrak{l})$
   (ii) $\widetilde{N}_{\boldsymbol{c}}(\mathfrak{l} + \mathfrak{v}_2) = \widetilde{N}_{\boldsymbol{c}}(\mathfrak{l})$
   (iii) $N_{\boldsymbol{c}}(\mathfrak{l} + \mathfrak{v}_2) \geq 0$
   (iv) $\widetilde{N}_{\boldsymbol{c}}(\mathfrak{l} + \mathfrak{v}_1) \geq 0$

Proof We choose $\mathfrak{v}_1$ and $\mathfrak{v}_2$ such that $\widetilde{N}_{\mathbf{0}}(\mathfrak{v}_1) \geq \widetilde{d}-1$ and $N_{\mathbf{0}}(\mathfrak{v}_2) \geq d-1$, and with:

(A) $\mathfrak{v}_1 = (\ell_1, \widetilde{\ell}_1) \in \mathbb{Z}^2(\boldsymbol{c})$ such that $\{-\omega_i \ell_1/\alpha_i\} = 0$ for all i, and $N_{\mathbf{0}}(\mathfrak{v}_1) = 0$

(B) $\mathfrak{v}_2 = (\ell_2, \widetilde{\ell}_2) \in \mathbb{Z}^2(\boldsymbol{c})$ such that $\{-\widetilde{\omega}_j \widetilde{\ell}_2/\widetilde{\alpha}_j\} = 0$ for all j, and $\widetilde{N}_{\mathbf{0}}(\mathfrak{v}_2) = 0$

Then $\mathfrak{v}_1$ and $\mathfrak{v}_2$ satisfy (a), (b)(i) and (b)(ii). Furthermore, note that $N_{\boldsymbol{c}}(\mathfrak{l}+\mathfrak{v}_2) \geq N_{\boldsymbol{c}}(\mathfrak{l}) + N_{\mathbf{0}}(\mathfrak{v}_2)$ and for any $\mathfrak{l} \in \bar{\mathcal{S}}_{\boldsymbol{c}}$ one has $N_{\boldsymbol{c}}(\mathfrak{l}) \geq -(d-1)$, hence all the conditions will be satisfied. □

7.3.4 Remark Usually, the 'universal restrictions' $\widetilde{N}_{\mathbf{0}}(\mathfrak{v}_1) \geq \widetilde{d}-1$ and $N_{\mathbf{0}}(\mathfrak{v}_2) \geq d-1$ in the proof of Lemma 7.3.3 provide rather 'large' vectors $\mathfrak{v}_1$ and $\mathfrak{v}_2$. Nevertheless, usually much smaller vectors also satisfy (a) and (b). Here is another choice. Besides (A) and (B) we impose the following.

(C) Let $\square = \square(\mathfrak{v}_1, \mathfrak{v}_2) = \{\mathfrak{l} = q_1\mathfrak{v}_1 + q_2\mathfrak{v}_2 \mid 0 \leq q_1, q_2 < 1\}$ be the semiopen cube in $\mathcal{C}^{\mathrm{orb}}$. Then we require $N_{\mathbf{0}}(\mathfrak{v}_2) \geq 0$ and $N_{\boldsymbol{c}}(\mathfrak{l}_\square + \mathfrak{v}_2) \geq 0$ for any $\mathfrak{l}_\square \in (\square - (c, \widetilde{c})) \cap \mathbb{Z}^2(\boldsymbol{c})$; and symmetrically: $\widetilde{N}_{\mathbf{0}}(\mathfrak{v}_1) \geq 0$ and $\widetilde{N}_{\boldsymbol{c}}(\mathfrak{l}_\square + \mathfrak{v}_1) \geq 0$ for any $\mathfrak{l}_\square \in (\square - (c, \widetilde{c})) \cap \mathbb{Z}^2(\boldsymbol{c})$.

The desired inequality for any $\mathfrak{l} \in \bar{\mathcal{S}}_{\boldsymbol{c}}$ then follows from $N_{\boldsymbol{c}}(\mathfrak{l}_\square + k_1\mathfrak{v}_1 + k_2\mathfrak{v}_2 + \mathfrak{v}_2) = N_{\boldsymbol{c}}(\mathfrak{l}_\square + k_2\mathfrak{v}_2 + \mathfrak{v}_2) \geq N_{\boldsymbol{c}}(\mathfrak{l}_\square + \mathfrak{v}_2) + k_2 N_{\mathbf{0}}(\mathfrak{v}_2)$ (and its symmetric version).

In the sequel the next two subsets of $\bar{\mathcal{S}}_{\boldsymbol{c}}$ will be crucial:

$$\mathcal{S}^-_{\boldsymbol{c},1} := \{\mathfrak{l} \in (\square - (c, \widetilde{c})) \cap \mathbb{Z}^2(\boldsymbol{c}) \mid N_{\boldsymbol{c}}(\mathfrak{l}) < 0\}$$
$$\mathcal{S}^-_{\boldsymbol{c},2} := \{\mathfrak{l} \in (\square - (c, \widetilde{c})) \cap \mathbb{Z}^2(\boldsymbol{c}) \mid \widetilde{N}_{\boldsymbol{c}}(\mathfrak{l}) < 0\}$$

Again, both sets $\mathcal{S}^-_{\boldsymbol{c},1}$ and $\mathcal{S}^-_{\boldsymbol{c},2}$ are independent of the choice of $\boldsymbol{c}$, they depend only on h.

7.3.5 Proposition *Let $\mathfrak{v}_1$ and $\mathfrak{v}_2$ be as in Lemma 7.3.3.*

(1) *$\bar{\mathcal{S}}_{\boldsymbol{c}}$ is given by*

$$\bar{\mathcal{S}}_{\boldsymbol{c}} = \bigsqcup_{\mathfrak{l} \in (\square - (c, \widetilde{c})) \cap \mathbb{Z}^2(\boldsymbol{c})} \mathfrak{l} + \mathbb{Z}_{\geq 0}\mathfrak{v}_1 + \mathbb{Z}_{\geq 0}\mathfrak{v}_2.$$

(2) *$\bar{\mathcal{S}}_{\boldsymbol{c}} \setminus \mathcal{S}_{\boldsymbol{c}}$ is given by*

$$\bar{\mathcal{S}}_{\boldsymbol{c}} \setminus \mathcal{S}_{\boldsymbol{c}} = \Big(\bigsqcup_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},1}} \mathfrak{l} + \mathbb{Z}_{\geq 0}\mathfrak{v}_1\Big) \cup \Big(\bigsqcup_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},2}} \mathfrak{l} + \mathbb{Z}_{\geq 0}\mathfrak{v}_2\Big),$$

but

$$\Big(\bigsqcup_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},1}} \mathfrak{l} + \mathbb{Z}_{\geq 0}\mathfrak{v}_1\Big) \cap \Big(\bigsqcup_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},2}} \mathfrak{l} + \mathbb{Z}_{\geq 0}\mathfrak{v}_2\Big) = \bigsqcup_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},1} \cap \mathcal{S}^-_{\boldsymbol{c},2}} \mathfrak{l}.$$

Proof The statements follow from the choice of $\mathfrak{v}_1$ and $\mathfrak{v}_2$ and properties (a) and (b) of Lemma 7.3.3. Compare also with the structure theorem Bruns and Gubeladze [13, 4.36] of $\mathcal{S}_{\mathbf{0}}$ modules. □

7.4 The periodic constant and the SW invariant in the equivariant case

Set $\boldsymbol{t} = (t, \widetilde{t})$. Using (7.2.3) and Proposition 7.3.5 one can write $Z_h(\boldsymbol{t})/\boldsymbol{t}^{(c,\widetilde{c})}$ in the next form:

$$\sum_{\mathfrak{l} \in (\square - (c,\widetilde{c})) \cap \mathbb{Z}^2(\boldsymbol{c})} \frac{\boldsymbol{t}^{\mathfrak{l}}}{(1-\boldsymbol{t}^{\mathfrak{v}_1})(1-\boldsymbol{t}^{\mathfrak{v}_2})} - \sum_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},1}} \frac{\boldsymbol{t}^{\mathfrak{l}}}{1-\boldsymbol{t}^{\mathfrak{v}_1}} - \sum_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},2}} \frac{\boldsymbol{t}^{\mathfrak{l}}}{1-\boldsymbol{t}^{\mathfrak{v}_2}} + \sum_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},1} \cap \mathcal{S}^-_{\boldsymbol{c},2}} \boldsymbol{t}^{\mathfrak{l}}$$

Next, we apply the decomposition established in Section 4.5. Here it is important to *choose* $\boldsymbol{c}$ *in such a way that* $c \in [0,1)$ *and* $\widetilde{c} \in [0,1)$.

Note that $\mathfrak{v}_1 \in \mathbb{R}_{>0}(1/\alpha_0, -e)$ and $\mathfrak{v}_2 \in \mathbb{R}_{>0}(-\widetilde{e}, 1/\alpha_0)$, hence $\mathfrak{v}_2$ sits in the cone determined by $\mathfrak{v}_1$ and $(1,0)$. Then, as in Section 4.5, we set $\Xi_1 := \{(\ell, \widetilde{\ell}) \mid 0 \le \ell <$ first coordinate of $\mathfrak{v}_1\}$ and $\Xi_2 := \{(\ell, \widetilde{\ell}) \mid 0 \le \widetilde{\ell} <$ second coordinate of $\mathfrak{v}_2\}$, and for any $\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},i}$ the unique $n_{\mathfrak{l},i}$ such that $\mathfrak{l} - n_{\mathfrak{l},i}\mathfrak{v}_i \in \Xi_i$, $i = 1, 2$. Then Section 4.5 provides the following decomposition:

$$Z_h^+(\boldsymbol{t}) = \boldsymbol{t}^{(c,\widetilde{c})} \Big(\sum_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},1}} \sum_{j=1}^{n_{\mathfrak{l},1}} \boldsymbol{t}^{\mathfrak{l} - j\mathfrak{v}_1} + \sum_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},2}} \sum_{j=1}^{n_{\mathfrak{l},2}} \boldsymbol{t}^{\mathfrak{l} - j\mathfrak{v}_2} + \sum_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},1} \cap \mathcal{S}^-_{\boldsymbol{c},2}} \boldsymbol{t}^{\mathfrak{l}} \Big)$$

$$Z_h^-(\boldsymbol{t}) = \boldsymbol{t}^{(c,\widetilde{c})} \Big(\sum_{\mathfrak{l} \in (\square - (c,\widetilde{c})) \cap \mathbb{Z}^2(\boldsymbol{c})} \frac{\boldsymbol{t}^{\mathfrak{l}}}{(1-\boldsymbol{t}^{\mathfrak{v}_1})(1-\boldsymbol{t}^{\mathfrak{v}_2})} - \sum_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},1}} \frac{\boldsymbol{t}^{\mathfrak{l} - n_{\mathfrak{l},1}\mathfrak{v}_1}}{1-\boldsymbol{t}^{\mathfrak{v}_1}} - \sum_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},2}} \frac{\boldsymbol{t}^{\mathfrak{l} - n_{\mathfrak{l},2}\mathfrak{v}_2}}{1-\boldsymbol{t}^{\mathfrak{v}_2}} \Big) \tag{7.4.1}$$

Therefore, by Remark 4.4.7 and Theorem 4.5.1 we get

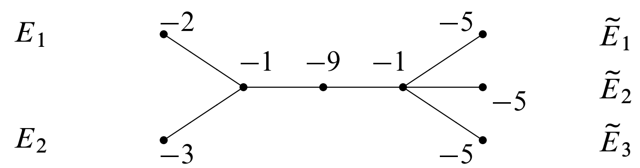
$$\mathrm{pc}_h^{\mathcal{C}^{\mathrm{orb}}}(Z) = \mathrm{pc}^{\mathcal{C}^{\mathrm{orb}}}(Z_h(\boldsymbol{t})/\boldsymbol{t}^{(c,\widetilde{c})}) = Z_h^+(1,1) = \sum_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},1}} n_{\mathfrak{l},1} + \sum_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},2}} n_{\mathfrak{l},2} + |\mathcal{S}^-_{\boldsymbol{c},1} \cap \mathcal{S}^-_{\boldsymbol{c},2}|.$$

7.4.2 Corollary *Choose* $\boldsymbol{c}$ *in such a way that* $c \in [0,1)$ *and* $\widetilde{c} \in [0,1)$. *Then one has the following combinatorial formula for the normalized Seiberg–Witten invariant of* M:

$$-\frac{(K + 2r_h)^2 + |\mathcal{V}|}{8} - \mathfrak{sw}_{-h*\sigma_{\mathrm{can}}}(M) = \sum_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},1}} n_{\mathfrak{l},1} + \sum_{\mathfrak{l} \in \mathcal{S}^-_{\boldsymbol{c},2}} n_{\mathfrak{l},2} + |\mathcal{S}^-_{\boldsymbol{c},1} \cap \mathcal{S}^-_{\boldsymbol{c},2}|$$

Proof Use Corollary 5.2.1, Reduction Theorem 5.4.2 and the above computation. □

7.4.3 Example Consider the following plumbing graph:



The corresponding Seifert invariants are $\alpha_1 = 2$, $\alpha_2 = 3$, $\widetilde{\alpha}_j = 5$, $\alpha_0 = 9$ and $\omega_i = \widetilde{\omega}_j = \omega_0 = \widetilde{\omega}_0 = 1$ for all i and j. Hence $e = -1/18$, $\widetilde{e} = -13/45$ and $\varepsilon = 1/(3^3 \cdot 10)$. For $h = 0$ we choose $\boldsymbol{c} = \mathbf{0}$. Then

$$\mathcal{S}_{\mathbf{0}} = \Big\{(\ell,\widetilde{\ell}) \in \mathbb{Z}^2 \;\Big|\; 8\ell - \widetilde{\ell} + 9\cdot\Big(\Big[\frac{-\ell}{2}\Big] + \Big[\frac{-\ell}{3}\Big]\Big) \geq 0,\ 8\widetilde{\ell} - \ell + 27\cdot\Big[\frac{-\widetilde{\ell}}{5}\Big] \geq 0,\ \ell + \widetilde{\ell} \equiv 0 \ (\mathrm{mod}\ 9)\Big\},$$

$$\bar{\mathcal{S}}_{\mathbf{0}} = \{(\ell,\widetilde{\ell}) \in \mathbb{Z}^2 \mid \ell - 2\widetilde{\ell} \geq 0,\ -5\ell + 13\widetilde{\ell} \geq 0,\ \ell + \widetilde{\ell} \equiv 0 \ (\mathrm{mod}\ 9)\}.$$

If we take the generators $\mathfrak{v}_1 = (60,30)$ and $\mathfrak{v}_2 = (26,10)$ (via conditions (A), (B), (C) following Lemma 7.3.3), one can calculate explicitly the sets

$$\mathcal{S}^{-}_{\mathbf{0},1} = \{(13,5), (19,8), (25,11), (31,14), (37,17), (43,20), (49,23), (55,26), (61,29), (67,32)\},$$

$$\mathcal{S}^{-}_{\mathbf{0},2} = \{(6,3), (19,8), (12,6), (25,11), (24,12), (37,17), (42,21), (55,26)\}.$$

This generates the next counting function of $\bar{\mathcal{S}}_{\mathbf{0}} \setminus \mathcal{S}_{\mathbf{0}}$, namely

$$\sum_{(\ell,\widetilde{\ell})\in\bar{\mathcal{S}}_{\mathbf{0}}\setminus\mathcal{S}_{\mathbf{0}}} t^{\ell}\widetilde{t}^{\widetilde{\ell}} = (t^{13}\widetilde{t}^{5} + t^{19}\widetilde{t}^{8} + t^{25}\widetilde{t}^{11} + t^{31}\widetilde{t}^{14} + t^{37}\widetilde{t}^{17} + t^{43}\widetilde{t}^{20} + t^{49}\widetilde{t}^{23} + t^{55}\widetilde{t}^{26} + t^{61}\widetilde{t}^{29} + t^{67}\widetilde{t}^{32})/(1 - t^{60}\widetilde{t}^{30}) + (t^{6}\widetilde{t}^{3} + t^{12}\widetilde{t}^{6} + t^{19}\widetilde{t}^{8} + t^{24}\widetilde{t}^{12} + t^{25}\widetilde{t}^{11} + t^{37}\widetilde{t}^{17} + t^{42}\widetilde{t}^{21} + t^{55}\widetilde{t}^{26})/(1 - t^{26}\widetilde{t}^{10}) - t^{19}\widetilde{t}^{8} - t^{25}\widetilde{t}^{11} - t^{37}\widetilde{t}^{17} - t^{55}\widetilde{t}^{26},$$

which by (7.4.1) provides $Z_0^{+}(t,\widetilde{t}) = t\widetilde{t}^{-1} + t^{3}\widetilde{t}^{2} + t^{-2}\widetilde{t}^{2} + t^{-1}\widetilde{t} + t^{11}\widetilde{t}^{7} + t^{16}\widetilde{t}^{11} + t^{-10}\widetilde{t} + t^{29}\widetilde{t}^{16} + t^{3}\widetilde{t}^{6} + t^{19}\widetilde{t}^{8} + t^{25}\widetilde{t}^{11} + t^{37}\widetilde{t}^{17} + t^{55}\widetilde{t}^{26}$. Hence $\mathrm{pc}_0^{\mathcal{C}^{\mathrm{orb}}}(Z) = Z_0^{+}(1,1) = 13$.

(It can be verified that there exists a splice quotient type normal surface singularity whose link is given by the above graph. It is a complete intersection in $(\mathbb{C}^4, 0)$ with equations $z^3 + (y_2 + 2y_3)^2 - y_1y_2(2y_2 + 3y_3) = y_1^5 + (2y_2 + 3y_3)y_2y_3 = 0$.)

7.5 The periodic constant in the nonequivariant case and $\lambda(M)$

Though the nonequivariant Z_{ne} can be obtained by the sum $\sum_h Z_h$ treated in the previous subsection, here we provide a more direct procedure, which leads to a new formula. Write $J := (-I^{\mathrm{orb}})^{-1}$ and $\boldsymbol{t}^{\binom{a}{b}}$ for $t^a\tilde{t}^b$. Applying Reduction Theorem 5.4.2 to the definition (2.2.1) of Z, we get

$$Z_{\mathrm{ne}}(\boldsymbol{t}) = \frac{\big(1-\boldsymbol{t}^{J\binom{1}{0}}\big)^{d-1}\big(1-\boldsymbol{t}^{J\binom{0}{1}}\big)^{\tilde{d}-1}}{\prod_i \big(1-\boldsymbol{t}^{J\binom{1/\alpha_i}{0}}\big)\prod_j \big(1-\boldsymbol{t}^{J\binom{0}{1/\tilde{\alpha}_j}}\big)}.$$

Set $S(x) := \sum_i x_i/\alpha_i$ and $\widetilde{S}(\tilde{x}) := \sum_j \tilde{x}_j/\tilde{\alpha}_j$. Similarly as in (6.2.7), $Z_{\mathrm{ne}}(\boldsymbol{t})$ can be written as

$$\sum_{\substack{0\le x_i<\alpha_i,\,0\le i\le d\\ 0\le \tilde{x}_j<\tilde{\alpha}_j,\,0\le j\le \tilde{d}}} f(x,\tilde{x}), \quad \text{where } f(x,\tilde{x}) = \frac{\boldsymbol{t}^{J\binom{S(x)}{\widetilde{S}(\tilde{x})}}}{\big(1-\boldsymbol{t}^{J\binom{1}{0}}\big)\big(1-\boldsymbol{t}^{J\binom{0}{1}}\big)}.$$

Substituting $u_1=\boldsymbol{t}^{J\binom{1}{0}}$ and $u_2=\boldsymbol{t}^{J\binom{0}{1}}$, $f(x,\tilde{x})$ becomes $u_1^{S(x)}u_2^{\widetilde{S}(\tilde{x})}/(1-u_1)(1-u_2)$. The division of this fraction (with remainder) is elementary, hence $f(x,\tilde{x})$ equals

$$\boldsymbol{t}^{J\binom{S_{\mathrm{rat}}}{\widetilde{S}_{\mathrm{rat}}}}\Bigg(\sum_{n=0}^{S_{\mathrm{int}}-1}\sum_{k=0}^{\widetilde{S}_{\mathrm{int}}-1}\boldsymbol{t}^{J\binom{n}{k}} - \sum_{k=0}^{S_{\mathrm{int}}-1}\frac{\boldsymbol{t}^{J\binom{k}{0}}}{1-\boldsymbol{t}^{J\binom{0}{1}}} - \sum_{\tilde{k}=0}^{\widetilde{S}_{\mathrm{int}}-1}\frac{\boldsymbol{t}^{J\binom{0}{\tilde{k}}}}{1-\boldsymbol{t}^{J\binom{1}{0}}} + \frac{1}{\big(1-\boldsymbol{t}^{J\binom{1}{0}}\big)\big(1-\boldsymbol{t}^{J\binom{0}{1}}\big)}\Bigg),$$

where $S_{\mathrm{int}} := \lfloor S(x)\rfloor$, $\widetilde{S}_{\mathrm{int}} := \lfloor \widetilde{S}(\tilde{x})\rfloor$, $S_{\mathrm{rat}} := \{S(x)\}$ and $\widetilde{S}_{\mathrm{rat}} := \{\widetilde{S}(\tilde{x})\}$.

Then, by Lemma 4.4.12, $\mathrm{pc}^{\mathcal{C}^{\mathrm{orb}}}\big(\boldsymbol{t}^{J\binom{S_{\mathrm{rat}}}{\widetilde{S}_{\mathrm{rat}}}}/\big(1-\boldsymbol{t}^{J\binom{1}{0}}\big)\big(1-\boldsymbol{t}^{J\binom{0}{1}}\big)\big) = 0$. Furthermore, Section 4.5 gives a unique integer $s(k)\ge 0$ for $k\in\{0,\dots,S_{\mathrm{int}}-1\}$ such that

$$\frac{\boldsymbol{t}^{J\binom{k+S_{\mathrm{rat}}}{-s(k)+\widetilde{S}_{\mathrm{rat}}}}}{1-\boldsymbol{t}^{J\binom{0}{1}}}$$

has vanishing periodic constant with respect to $\mathcal{C}^{\mathrm{orb}}$. It turns out we have that $s(k) = \lfloor -\tilde{e}\alpha_0(k+S_{\mathrm{rat}})+\widetilde{S}_{\mathrm{rat}}\rfloor$. Similarly $s(\tilde{k}) = \lfloor -e\alpha_0(\tilde{k}+\widetilde{S}_{\mathrm{rat}})+S_{\mathrm{rat}}\rfloor$ in the case of

$$\frac{\boldsymbol{t}^{J\binom{-s(\tilde{k})+S_{\mathrm{rat}}}{\tilde{k}+\widetilde{S}_{\mathrm{rat}}}}}{1-\boldsymbol{t}^{J\binom{1}{0}}}.$$

Therefore, by Theorem 4.5.1, for

$$\mathrm{pc}(Z_{\mathrm{ne}}) = -\lambda(M) - \mathfrak{d}\cdot\frac{K^2+|\mathcal{V}|}{8} + \sum_h \chi(r_h)$$

we get

$$\sum_{\substack{0\le x_i<\alpha_i,0\le i\le d\\ 0\le \widetilde{x}_j<\widetilde{\alpha}_j,0\le j\le \widetilde{d}}} \Big(S_{\mathrm{int}}\widetilde{S}_{\mathrm{int}}+\sum_{k=0}^{S_{\mathrm{int}}-1}\lfloor -\widetilde{e}\alpha_0(k+S_{\mathrm{rat}})+\widetilde{S}_{\mathrm{rat}}\rfloor+\sum_{\widetilde{k}=0}^{\widetilde{S}_{\mathrm{int}}-1}\lfloor -e\alpha_0(\widetilde{k}+\widetilde{S}_{\mathrm{rat}})+S_{\mathrm{rat}}\rfloor\Big).$$

7.6 Ehrhart-theoretical interpretation

In general, in contrast with the one-node case Section 6.5, the direct determination of the counting function of $Z_h(\boldsymbol{t})$, or equivalently, of the complete equivariant Ehrhart quasipolynomial associated with the corresponding polytope, is rather hard. Nevertheless, those coefficients which are relevant to us (eg those ones which contain the information about the Seiberg–Witten invariants of the 3–manifold) can be identified using the right-hand side of (2.2.3). The computation is more transparent when $L'=L$. In that case, the two-variable Ehrhart polynomial has degree $d+\widetilde{d}$, and a specific $d+\widetilde{d}-2$ degree coefficient is exactly the normalized Seiberg–Witten invariant of the 3–manifold. We will not provide here the formulae, since this identification will be established for any negative-definite plumbing graph with arbitrary number of nodes; see Section 8 where several other coefficients will be computed as well.

8 Ehrhart-theoretical interpretation of the SW invariant (the general case)

8.1

Let Γ be a negative-definite plumbing graph, a connected tree as in Section 2.1. Let $\mathcal{N}$ and $\mathcal{E}$ be the set of nodes and end vertices as above. We assume that $\mathcal{N}\neq\varnothing$. If δ_n denotes the valency of a node n, then $|\mathcal{E}|=2+\sum_{n\in\mathcal{N}}(\delta_n-2)$.

We consider the matrix J with entries $J_{nm}:=-(E^*_n,E^*_m)$ for $n,m\in\mathcal{N}$. By Section 2.1 it is a principal minor of $-I^{-1}$ (with rows and columns corresponding to the nodes).

Another incarnation of the matrix J already appeared in Section 7.5, as the negative of the inverse of the orbifold intersection matrix. Indeed, let for any $n\in\mathcal{N}$ take that component of $\Gamma\setminus\bigcup_{m\in\mathcal{N}\setminus n}\{m\}$ which contains n. It is a star-shaped graph, let e_n be its orbifold Euler number. Furthermore, for any two nodes n and m which are connected by a chain, let α_{nm} be the determinant of that chain (not including the nodes). Then define the orbifold intersection matrix (of size $|\mathcal{N}|$) as $I^{\mathrm{orb}}_{nn}=e_n$, $I^{\mathrm{orb}}_{nm}=1/\alpha_{nm}$ if the two nodes $n\neq m$ are connected by a chain, and $I^{\mathrm{orb}}_{nm}=0$ otherwise; cf Braun and Némethi [10, 4.1.4] or Section 7.1. One can show (see [10, 4.1.4]) that I^{orb} is invertible, negative-definite and $\det(-I^{\mathrm{orb}})$ is the product of $\det(-I)$ with the determinants of all (maximal) chains and legs of Γ. This fact and (2.1.1) imply that $J=(-I^{\mathrm{orb}})^{-1}$.

8.2 The Ehrhart polynomial

In the sequel we assume that $L = L'$, that is $H = 0$.

By Section 5.2, $P^{(l)}$ sits in $\mathbb{R}^{|\mathcal{E}|}$. Moreover, by Reduction Theorem 5.4.2, we can take l of the form $l = \sum_{n\in\mathcal{N}} \lambda_n E_n^*$, from the subcone of the Lipman cone generated by $\{E_n^*\}_{n\in\mathcal{N}}$.

Then Section 5.4 guarantees that the associated polytope is $P^{(l)} = \bigcup_{n\in\mathcal{N}} P_n^{(l_n)}$, $P_n^{(l_n)}$ depending only on the component $l_n = -(l, E_n^*)$. Note that the coefficients $\{\lambda_n\}_n$ and the entries $\{l_n\}_n$ are connected exactly by the transformation law $(l_n)_n = J\,(\lambda_n)_n$.

Take any chamber $\mathcal{C}$ such that $\mathrm{int}(\mathcal{C}\cap\mathcal{S}) \neq \varnothing$, as in Corollary 5.2.1. Let $\widehat{\mathcal{L}}^{\mathcal{C}}(P, \mathcal{T}, (\lambda_n)_n)$ be the Ehrhart quasipolynomial $\mathcal{L}^{\mathcal{C}}(P, \mathcal{T}, (l_n)_n)$, associated with the denominator of Z, after changing the variables to $(\lambda_n)_n$ via $(l_n)_n = J\,(\lambda_n)_n$. It is convenient to normalize the coefficient of $\prod_n \lambda_n^{m_n}$ by a factor $\prod_n m_n!$, hence we write

$$\widehat{\mathcal{L}}^{\mathcal{C}}(P, \mathcal{T}, (\lambda_n)_n) = \sum_{\substack{\sum_n m_n \le |\mathcal{E}| \\ m_n \ge 0;\, n\in\mathcal{N}}} \widehat{\mathfrak{a}}^{\mathcal{C}}_{(m_n)_n} \prod_n \frac{\lambda_n^{m_n}}{m_n!}$$

for certain periodic functions $\widehat{\mathfrak{a}}^{\mathcal{C}}_{(m_n)_n}$ in variables $(\lambda_n)_n$. By (2.2.3), Corollary 4.3.11 and Reduction Theorem 5.4.2

$$\chi\Big(\sum_{n\in\mathcal{N}} \lambda_n E_n^*\Big) + \mathrm{pc}^{\mathcal{S}}(Z) = \Delta((\lambda_n)_n), \tag{8.2.1}$$

where

$$\begin{aligned}\Delta((\lambda_n)_n) &= \sum_{\substack{0\le k_n\le \delta_n-2 \\ \forall n\in\mathcal{N}}} (-1)^{\sum_n k_n} \prod_n \binom{\delta_n-2}{k_n} \widehat{\mathcal{L}}^{\mathcal{C}}(P, \mathcal{T}, (\lambda_n - k_n)_n) \\ &= \sum_{\substack{\sum_n m_n \le |\mathcal{E}| \\ m_n \ge 0;\, n\in\mathcal{N}}} \Bigg(\sum_{\substack{0\le p_n\le m_n \\ n\in\mathcal{N}}} (-1)^{\sum_n p_n} \cdot \prod_n \binom{m_n}{p_n} \Big(\sum_{k_n=0}^{\delta_n-2} (-1)^{k_n} \binom{\delta_n-2}{k_n} k_n^{p_n} \Big)\Bigg) \\ &\qquad\qquad \cdot \widehat{\mathfrak{a}}^{\mathcal{C}}_{(m_n)_n} \prod_n \frac{\lambda_n^{m_n-p_n}}{m_n!}.\end{aligned}$$

On the other hand, since $\chi(l) = -(K+l, l)/2$, the left-hand side of (8.2.1) is the quadratic function

$$\sum_{n,m\in\mathcal{N}} (J_{nm}/2)\lambda_n\lambda_m + \sum_{n\in\mathcal{N}} (-(K, E_n^*)/2)\lambda_n + \mathrm{pc}^{\mathcal{S}}(Z).$$

Now we identify these coefficients with those of $\Delta((\lambda_n)_n)$ above. The additional ingredient is the combinatorial formula (6.5.11), which also shows that for the nonzero summands one necessarily has $p_n \geq \delta_n - 2$ for any n. One gets the following result.

8.2.2 Theorem *We have:*

$$\begin{aligned}
\widehat{\mathfrak{a}}^{\mathcal{C}}_{(\delta_n,(\delta_m-2)_{m\neq n})} &= J_{nn}\\
\widehat{\mathfrak{a}}^{\mathcal{C}}_{(\delta_n-1,\delta_m-1,(\delta_q-2)_{q\neq n,m})} &= J_{nm} \quad \text{for } n \neq m\\
\widehat{\mathfrak{a}}^{\mathcal{C}}_{(\delta_n-1,(\delta_m-2)_{m\neq n})} &= -\frac{1}{2}(K, E_n^*) + \frac{1}{2}\sum_{m\in\mathcal{N}}(\delta_m-2)J_{nm}\\
\widehat{\mathfrak{a}}^{\mathcal{C}}_{(\delta_n-2)_n} &= \mathrm{pc}^{\mathcal{S}}(Z) - \sum_{n\in\mathcal{N}}\frac{(\delta_n-2)(K,E_n^*)}{4}\\
&\quad + \sum_{n\in\mathcal{N}}\frac{(\delta_n-2)(3\delta_n-7)J_{nn}}{24} + \sum_{\substack{n,m\in\mathcal{N}\\ m\neq n}}\frac{(\delta_n-2)(\delta_m-2)J_{nm}}{8}
\end{aligned}$$

Recall that $\mathrm{pc}^{\mathcal{S}}(Z) = -(K^2+|\mathcal{V}|)/8 - \lambda(M)$, where $\lambda(M)$ is the Casson invariant of M. Hence $\widehat{\mathfrak{a}}^{\mathcal{C}}_{(\delta_n-2)_n}$ equals the normalized Casson invariant modulo some 'easy terms.'

We emphasize that these formulae also show that the above coefficients are constants (as periodic functions in $(\lambda_n)_n$) and independent of the chosen chamber $\mathcal{C}$ in the Lipman cone.

References

[1] **V I Arnold**, **S M Gusein-Zade**, **A N Varchenko**, *Singularities of differentiable maps, Vol. 2: Monodromy and asymptotics of integrals*, Springer, NY (2012) MR2919697

[2] **A I Barvinok**, *Computing the Ehrhart polynomial of a convex lattice polytope*, Discrete Comput. Geom. 12 (1994) 35–48 MR1280575

[3] **A Barvinok**, **J E Pommersheim**, *An algorithmic theory of lattice points in polyhedra*, from: "New perspectives in algebraic combinatorics", (L J Billera, A Björner, C Greene, R E Simion, R P Stanley, editors), MSRI Publ. 38, Cambridge Univ. Press (1999) 91–147 MR1731815

[4] **M Beck**, *A closer look at lattice points in rational simplices*, Electron. J. Combin. 6 (1999) 9 MR1725701

[5] **M Beck**, *Multidimensional Ehrhart reciprocity*, J. Combin. Theory Ser. A 97 (2002) 187–194 MR1879135

[6] **M Beck**, *The partial-fractions method for counting solutions to integral linear systems*, Discrete Comput. Geom. 32 (2004) 437–446 MR2096740

[7] **M Beck**, **R Diaz**, **S Robins**, *The Frobenius problem, rational polytopes, and Fourier–Dedekind sums*, J. Number Theory 96 (2002) 1–21 MR1931190

[8] **M Beck**, **S Robins**, *Explicit and efficient formulas for the lattice point count in rational polygons using Dedekind–Rademacher sums*, Discrete Comput. Geom. 27 (2002) 443–459 MR1902672

[9] **M Beck**, **S Robins**, *Computing the continuous discretely: Integer–point enumeration in polyhedra*, Undergrad. Texts Math. 128, Springer, New York (2007) MR2271992

[10] **G Braun**, **A Némethi**, *Invariants of Newton non-degenerate surface singularities*, Compos. Math. 143 (2007) 1003–1036 MR2339837

[11] **G Braun**, **A Némethi**, *Surgery formula for Seiberg–Witten invariants of negative definite plumbed 3–manifolds*, J. Reine Angew. Math. 638 (2010) 189–208 MR2595340

[12] **M Brion**, **M Vergne**, *Residue formulae, vector partition functions and lattice points in rational polytopes*, J. Amer. Math. Soc. 10 (1997) 797–833 MR1446364

[13] **W Bruns**, **J Gubeladze**, *Polytopes, rings, and K–theory*, Springer Monographs in Mathematics, Springer, Dordrecht (2009) MR2508056

[14] **W Bruns**, **J Herzog**, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics 39, Cambridge Univ. Press (1993) MR1251956

[15] **A Campillo**, **F Delgado**, **S M Gusein-Zade**, *Poincaré series of a rational surface singularity*, Invent. Math. 155 (2004) 41–53 MR2025300

[16] **S E Cappell**, **J L Shaneson**, *Genera of algebraic varieties and counting of lattice points*, Bull. Amer. Math. Soc. 30 (1994) 62–69 MR1217352

[17] **P Clauss**, **V Loechner**, *Parametric analysis of polyhedral iteration spaces*, Journal of VLSI signal processing systems 19 (1998) 179–194

[18] **S D Cutkosky**, **J Herzog**, **A Reguera**, *Poincaré series of resolutions of surface singularities*, Trans. Amer. Math. Soc. 356 (2004) 1833–1874 MR2031043

[19] **R Diaz**, **S Robins**, *The Ehrhart polynomial of a lattice polytope*, Ann. of Math. 145 (1997) 503–518 MR1454701

[20] **I V Dolgachev**, *On the link space of a Gorenstein quasihomogeneous surface singularity*, Math. Ann. 265 (1983) 529–540 MR721886

[21] **A H Durfee**, *The signature of smoothings of complex surface singularities*, Math. Ann. 232 (1978) 85–98 MR0466620

[22] **D Eisenbud**, **W Neumann**, *Three-dimensional link theory and invariants of plane curve singularities*, Annals Math. Studies 110, Princeton Univ. Press (1985) MR817982

[23] **R E Gompf**, **A I Stipsicz**, *4–manifolds and Kirby calculus*, Graduate Studies in Mathematics 20, Amer. Math. Soc. (1999) MR1707327

[24] **S Goto**, **K Watanabe**, *On graded rings. I*, J. Math. Soc. Japan 30 (1978) 179–213 MR494707

[25] **S M Gusein-Zade**, **F Delgado**, **A Campillo**, *Universal abelian covers of rational surface singularities, and multi-index filtrations*, Funktsional. Anal. i Prilozhen. 42 (2008) 3–10 MR2438013

[26] **J-M Kantor**, **A Khovanskii**, *Une application du théorème de Riemann–Roch combinatoire au polynôme d'Ehrhart des polytopes entiers de* $\mathbf{R}^d$, C. R. Acad. Sci. Paris Sér. I Math. 317 (1993) 501–507 MR1239038

[27] **C Lescop**, *Global surgery formula for the Casson–Walker invariant*, Annals Math. Studies 140, Princeton Univ. Press (1996) MR1372947

[28] **M Marcolli**, **B-L Wang**, *Seiberg–Witten and Casson-Walker invariants for rational homology* 3*–spheres*, from: "Proceedings of the Euroconference on Partial Differential Equations and their Applications to Geometry and Physics", Geom. Dedicata 91 (2002) 45–58 MR1919892

[29] **P McMullen**, *Lattice invariant valuations on rational polytopes*, Arch. Math. (Basel) 31 (1978/79) 509–516 MR526617

[30] **M Merle**, **B Teissier**, *Conditions d'adjonction, d'après du val*, from: "Séminaire sur les Singularitiés des Surfaces", (M Demazure, H C Pinkham, B Teissier, editors), Lecture Notes in Mathematics 777, Springer Berlin Heidelberg (1980) 229–245

[31] **A Némethi**, *Line bundles associated with normal surface singularities* `arXiv:math/0310084`

[32] **A Némethi**, *Five lectures on normal surface singularities*, from: "Low-dimensional topology", (K Böröczky, Jr, W Neumann, A Stipsicz, editors), Bolyai Soc. Math. Stud. 8, János Bolyai Math. Soc., Budapest (1999) 269–351 MR1747271

[33] **A Némethi**, *On the Ozsváth–Szabó invariant of negative definite plumbed* 3*–manifolds*, Geom. Topol. 9 (2005) 991–1042 MR2140997

[34] **A Némethi**, *Graded roots and singularities*, from: "Singularities in geometry and topology", (J-P Brasselet, J Damon, M Oka, editors), World Sci. Publ., Hackensack, NJ (2007) 394–463 MR2311495

[35] **A Némethi**, *Lattice cohomology of normal surface singularities*, Publ. Res. Inst. Math. Sci. 44 (2008) 507–543 MR2426357

[36] **A Némethi**, *Poincaré series associated with surface singularities*, from: "Singularities I", (J-P Brasselet, J L Cisneros-Molina, D Massey, J Seade, B Teissier, editors), Contemp. Math. 474, Amer. Math. Soc. (2008) 271–297 MR2454352

[37] **A Némethi**, *The Seiberg–Witten invariants of negative definite plumbed* 3*–manifolds*, J. Eur. Math. Soc. (JEMS) 13 (2011) 959–974 MR2800481

[38] **A Némethi**, *Two exact sequences for lattice cohomology*, from: "Noncommutative geometry and global analysis", Contemp. Math. 546, Amer. Math. Soc. (2011) 249–269 MR2815139

[39] **A Némethi**, *The cohomology of line bundles of splice-quotient singularities*, Adv. Math. 229 (2012) 2503–2524 MR2880230

[40] **A Némethi**, **L I Nicolaescu**, *Seiberg–Witten invariants and surface singularities*, Geom. Topol. 6 (2002) 269–328 MR1914570

[41] **A Némethi**, **L I Nicolaescu**, *Seiberg–Witten invariants and surface singularities, II: Singularities with good $\mathbb{C}^*$–action*, J. London Math. Soc. 69 (2004) 593–607 MR2050035

[42] **A Némethi**, **T Okuma**, *The Seiberg–Witten invariant conjecture for splice-quotients*, J. Lond. Math. Soc. 78 (2008) 143–154 MR2427056

[43] **A Némethi**, **T Okuma**, *On the Casson invariant conjecture of Neumann–Wahl*, J. Algebraic Geom. 18 (2009) 135–149 MR2448281

[44] **A Némethi**, **T Okuma**, *The embedding dimension of weighted homogeneous surface singularities*, J. Topol. 3 (2010) 643–667 MR2684515

[45] **A Némethi**, **F Román**, *The lattice cohomology of $S^3_{-d}(K)$*, from: "Zeta functions in algebra and geometry", (A Campillo, G Cardona, A Melle-Hernández, W Veys, editors), Contemp. Math. 566, Amer. Math. Soc. (2012) 261–292 MR2858927

[46] **W D Neumann**, *Abelian covers of quasihomogeneous surface singularities*, from: "Singularities, Part 2", (P Orlik, editor), Proc. Sympos. Pure Math. 40, Amer. Math. Soc. (1983) 233–243 MR713252

[47] **W Neumann**, **J Wahl**, *Casson invariant of links of singularities*, Comment. Math. Helv. 65 (1990) 58–78 MR1036128

[48] **W D Neumann**, **J Wahl**, *Complete intersection singularities of splice type as universal abelian covers*, Geom. Topol. 9 (2005) 699–755 MR2140991

[49] **L I Nicolaescu**, *Seiberg–Witten invariants of rational homology 3–spheres*, Commun. Contemp. Math. 6 (2004) 833–866 MR2111431

[50] **T Okuma**, *The geometric genus of splice-quotient singularities*, Trans. Amer. Math. Soc. 360 (2008) 6643–6659 MR2434304

[51] **P Ozsváth**, **Z Szabó**, *On the Floer homology of plumbed three-manifolds*, Geom. Topol. 7 (2003) 185–224 MR1988284

[52] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and three-manifold invariants: Properties and applications*, Ann. of Math. 159 (2004) 1159–1245 MR2113020

[53] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. 159 (2004) 1027–1158 MR2113019

[54] **H Pinkham**, *Normal surface singularities with C^* action*, Math. Ann. 227 (1977) 183–193 MR0432636

[55] **G Pólya**, **G Szegő**, *Problems and theorems in analysis, I: Series, integral calculus, theory of functions*, Springer, Berlin (1998) MR1492447

[56] **J E Pommersheim**, *Toric varieties, lattice points and Dedekind sums*, Math. Ann. 295 (1993) 1–24 MR1198839

[57] **R Rustamov**, *A surgery formula for renormalized Euler characteristic of Heegaard Floer homology* arXiv:math/0409294

[58] **R P Stanley**, *Combinatorial reciprocity theorems*, Advances in Math. 14 (1974) 194–253 MR0411982

[59] **B Sturmfels**, *On vector partition functions*, J. Combin. Theory Ser. A 72 (1995) 302–309 MR1357776

[60] **A Szenes**, **M Vergne**, *Residue formulae for vector partitions and Euler–MacLaurin sums*, Adv. in Appl. Math. 30 (2003) 295–342 MR1979797

[61] **P Wagreich**, *The structure of quasihomogeneous singularities*, from: "Singularities, Part 2", (P Orlik, editor), Proc. Sympos. Pure Math. 40, Amer. Math. Soc. (1983) 593–611 MR713284

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