Hypergraph Turán numbers of linear cycles

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Abstract

A *k-uniform linear cycle* of length ℓ, denoted by $\mathbb{C}_\ell^{(k)}$, is a cyclic list of k-sets $A_1, \ldots, A_\ell$ such that consecutive sets intersect in exactly one element and nonconsecutive sets are disjoint. For all $k \geq 5$ and $\ell \geq 3$ and sufficiently large n we detemine the largest size of a k-uniform set family on $[n]$ not containing a linear cycle of length ℓ. For odd $\ell = 2t+1$ the unique extremal family $\mathcal{F}_S$ consists of all k-sets in $[n]$ intersecting a fixed t-set S in $[n]$. For even $\ell = 2t+2$, the unique extremal family consists of $\mathcal{F}_S$ plus all the k-sets outside S containing some fixed two elements. For $k \geq 4$ and large n we also establish an exact result for so-called *minimal cycles*. For all $k \geq 4$ our results substantially extend Erdős' result on largest k-uniform families without $t+1$ pairwise disjoint members and confirm, in a stronger form, a conjecture of Mubayi and Verstraëte [23]. Our main method is the delta system method.

1 Introduction

The delta system method is a very useful tool for set system problems. It was fully developed in a series of papers including [11] and [8]. It was successfully used for starlike configurations in [8] and [14] and recently also for larger configurations (as paths and trees) in [12] and [13]. In this paper we apply the delta system method, particularly tools from [11] and [8], to determine, for all $k \geq 5$ and large n, the Turán numbers of certain hypergraphs called *k-uniform linear cycles*. This confirms, in a stronger form, a conjecture of Mubayi and Verstraëte [23] for $k \geq 5$ and adds to the limited list of hypergraphs whose Turán numbers have been known either exactly or asymptotically.

We organize the paper as follows. Section 2 and 3 contain definitions concerning hypergraphs. Section 4 gives a rough upper bound establishing the correct order of the magnitude. Section 6 contains the statements of the main results. Section 7 introduces the delta system method and lemmas needed for the linear cycle problem and Sections 8–10 contain proofs.

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2 Definitions: shadows, degrees, delta systems

A *hypergraph* $\mathcal{F} = (V, \mathcal{E})$ consists of a set V of vertices and a set $\mathcal{E}$ of edges, where each edge is a subset of V. If V has n vertices, then it is often convenient to just assume that $V = [n] = \{1, 2, \dots, n\}$. Let $\binom{V}{k}$ denote the collection of all the k-subsets of V. If all the edges of $\mathcal{F}$ are k-subsets of V, then we write $\mathcal{F} \subseteq \binom{V}{k}$ and say that $\mathcal{F}$ is a *k-uniform* hypergraph, or a *k-graph* for brevity, on V. Note that the usual graphs are precisely 2-graphs on respective vertex sets. A hypergraph $\mathcal{F} = (V, \mathcal{E})$ is also often times called a *set system* or *set family* on V with its edges referred to as the *members* of the set system/family. A k-graph $\mathcal{F}$ is *k-partite* if its vertex set V can be partitioned into k subsets $V_1, \dots, V_k$ such that each edge of $\mathcal{F}$ contains precisely one vertex from each V_i.

The *shadow* of $\mathcal{F}$, denoted by $\partial(\mathcal{F})$, is defined as

$$\partial(\mathcal{F}) = \{D : \exists F \in \mathcal{F}, D \subsetneq F\}.$$

Here, we treat $\emptyset$ as a member of $\partial(\mathcal{F})$. We define the *p-shadow* of $\mathcal{F}$ to be

$$\partial_p(\mathcal{F}) = \{D : D \in \partial(\mathcal{F}), |D| = p\}.$$

The Lovász' [20] version of the Kruskal-Katona theorem states that if $\mathcal{F}$ is a k-graph of size $|\mathcal{F}| = \binom{x}{k}$ where $x \geq k-1$ is a real number, then for $k \geq p \geq 1$

$$|\partial_p(\mathcal{F})| \geq \binom{x}{p}. \tag{1}$$

Let $\mathcal{F}$ be a hypergraph on $[n]$ and $D \subseteq V(\mathcal{F})$. The *degree* $\deg_{\mathcal{F}}(D)$ of D in $\mathcal{F}$, is defined as

$$\deg_{\mathcal{F}}(D) = |\{F : F \in \mathcal{F}, D \subseteq F\}|.$$

A family of sets $F_1, \dots, F_s$ is said to form an *s-star* or *Δ-system* of size s with *kernel* D if $F_i \cap F_j = D$ for all $1 \leq i < j \leq s$ and $\forall i \in [s], F_i \setminus D \neq \emptyset$. The sets $F_1, \dots, F_s$ are called the *petals* of this s-star. Note that we allow $D = \emptyset$. Let $\mathcal{F}$ be a hypergraph and $D \subseteq V(\mathcal{F})$. The kernel degree $\deg^*_{\mathcal{F}}(D)$ of D in $\mathcal{F}$ is defined as

$$\deg^*_{\mathcal{F}}(D) = \max\{s : \mathcal{F} \text{ contains an } s\text{-star with kernel } D\}.$$

Given a k-graph $\mathcal{F}$ on a set V and a positive integer s, the *kernel graph* of $\mathcal{F}$ with threshold s, denoted by $\mathrm{Ker}_s(\mathcal{F})$, is defined as

$$\mathrm{Ker}_s(\mathcal{F}) = \{D \subseteq V : \deg^*_{\mathcal{F}}(D) \geq s\}.$$

For convenience, if $D \in \mathrm{Ker}_s(\mathcal{F})$, we will just say that D *is a kernel.* For each $1 \leq r \leq k-1$, the r-kernel graph of $\mathcal{F}$ with threshold s, denoted by $\mathrm{Ker}^{(r)}_s(\mathcal{F})$, is defined as

$$\mathrm{Ker}^{(r)}_s(\mathcal{F}) = \{D \subseteq V : |D| = r, \deg^*_{\mathcal{F}}(D) \geq s\}.$$

If $D \in \mathrm{Ker}^{(r)}_s(\mathcal{F})$ we will just say that D is an *r-kernel.* Throughout the paper, we will frequently use the following fact which follows easily from the definition of $\deg^*_{\mathcal{F}}(D)$.

$$\text{Given sets } D, Y, \text{ if } \deg^*_{\mathcal{F}}(D) > |Y| \text{ then } \exists F \in \mathcal{F} \text{ such that } D \subseteq F \text{ and } (F \setminus D) \cap Y = \emptyset. \tag{2}$$

3 Matchings, intersecting hypergraphs, paths and cycles

Given n, k, t, let $\mathcal{F}_S$ be the k-graph on $[n]$ formed by taking a t-set S in $[n]$ and taking as edges all the k-sets in $[n]$ that intersect S. Clearly, $\mathcal{F}_S$ contains no $t+1$ pairwise disjoint members, i.e., its matching number is t. Erdős [1] showed that there is a smallest positive integer $n_0(k,t)$ such that, for all $n > n_0(k,t)$, $\mathcal{F}_S$ is the largest k-uniform set system on $[n]$ not containing $t+1$ pairwise disjoint members. The function $n_0(k,t)$ has not been completely determined. The value of $n_0(k,2)$ was determined in the classical Erdős-Ko-Rado Theorem [3] about intersecting families. For $k=2$ (graphs) the value of $n_0(2,t)$ was determined by Erdős and Gallai [2]. The case $k=3$ was recently investigated by Frankl, Rödl, and Ruciński [10] and $n_0(3,t)$ was finally determined by Łuczak and Mieczkowska [21] for large t, and by Frankl [6] for all t. In general, Huang, Loh, and Sudakov [16] showed $n_0(k,t) < 3tk^2$, which was slightly improved in [9] and greatly improved to $n_0(k,t) \le (2t+1)k - t$ by Frankl [7].

Frankl [5] showed that for every n, k, t if a k-graph $\mathcal{F}$ on $[n]$ has no $t+1$ pairwise disjoint edges then $|\mathcal{F}| \le t\binom{n-1}{k-1}$. This implies

$$\forall D \subseteq [n] \text{ if } \deg^*_{\mathcal{F}}(D) \le s, \text{ then } \deg_{\mathcal{F}}(D) \le s\binom{n-|D|-1}{k-|D|-1}. \tag{3}$$

Frankl [4] considered set systems that do not contain two members intersecting in exactly one element. This condition is equivalent to forbidding a linear path of length 2 (see definition below). He showed that for all $k \ge 4$ there exists a bound $m(k)$ such that

$$\text{if } \mathcal{H} \subseteq \binom{[m]}{k} \text{ satisfies } \forall A, B \in \mathcal{H}, |A \cap B| \ne 1 \text{ and } m > m(k), \text{ then } |\mathcal{H}| \le \binom{m-2}{k-2}. \tag{4}$$

The unique extremal family is obtained by taking as members all the k-sets in $[m]$ containing a fixed set of two elements.

We now introduce some notions of hypergraph paths and cycles. While the notion of a hypergraph matching is a straightforward extension of that of a graph matching, there are different possibilities for paths and cycles. We discuss three versions, Berge path, minimal path and linear (or loose) path. A *Berge path* of length ℓ in the hypergraph $\mathcal{F}$ is a list of distinct hyperedges $F_1, \ldots, F_\ell \in \mathcal{F}$ and $\ell+1$ distinct vertices $P := \{v_1, \ldots, v_{\ell+1}\}$ such that for each $1 \le i \le \ell$, F_i contains v_i and v_{i+1}.

If we allow only consecutive F_i's to intersect, i.e., $F_i \cap F_j = \emptyset$ when $|i-j| \ge 2$, then the resulting Berge path is called a *minimal* path. We denote the family of all k-uniform minimal paths of length ℓ by $\mathcal{P}^{(k)}_\ell$. If we require all the F_i's to be pairwise disjoint outside P and $F_i \cap P = \{v_i, v_{i+1}\}$, then the path is unique. We call it the k-uniform *linear path* of length ℓ and denote it by $\mathbb{P}^{(k)}_\ell$. Note that $\mathbb{P}^{(k)}_\ell$ is a member of $\mathcal{P}^{(k)}_\ell$.

Likewise, a k-uniform *Berge cycle* of length ℓ is a cyclic list of distinct k-sets $F_1, \ldots, F_\ell$ and ℓ distinct vertices $C = \{v_1, \ldots, v_\ell\}$ such that for each $1 \le i \le \ell$, F_i contains v_i and v_{i+1} (where $v_{\ell+1} = v_1$). If we allow only consecutive F_i's in the cyclic list to intersect then the resulting cycle is called a *minimal* cycle. We denote the family of all k-uniform minimal cycles of length ℓ by $\mathcal{C}_\ell^{(k)}$. If we require all the F_i's to be pairwise disjoint outside C and $F_i \cap C = \{v_i, v_{i+1}\}$, then the cycle is unique and we call it the k-uniform *linear cycle* of length ℓ and denote it by $\mathbb{C}_\ell^{(k)}$. Note that $\mathbb{C}_\ell^{(k)}$ is a member of $\mathcal{C}_\ell^{(k)}$.

The *triangulated cycle* $\mathbb{T}_\ell^{(3)}$, is a triple system on 2ℓ vertices $\{v_1, \dots, v_\ell, u_1, \dots, u_\ell\}$ with $2\ell - 2$ edges $A_i := \{v_1, v_i, v_{i+1}\}$ $(1 < i < \ell)$ and $B_j := \{v_j, v_{j+1}, u_j\}$ $(\ell \geq 3)$. Note that the B_j's form $\mathbb{C}_\ell^{(3)}$.

4 Hypergraph extensions and an estimate of the Turán number

Given a hypergraph $\mathcal{H}$ whose edges have size at most k, the k-*expansion* of $\mathcal{H}$, denoted by $\mathcal{H}^{(k)}$, is the k-graph obtained by enlarging each edge of $\mathcal{H}$ into a k-set by using new vertices (called *expansion vertices*) such that different edges are enlarged using disjoint sets of expansion vertices. For instance, if $\mathcal{H} = \{1, 12, 123\}$, then $\{1ab, 12c, 123\}$ is the 3-expansion of $\mathcal{H}$ and $\{1abc, 12de, 123f\}$ is the 4-expansion of $\mathcal{H}$. Note that for any k, b where $b \geq 2$ and $k \geq b+1$, the k-expansion of a b-uniform linear (or minimal) ℓ-cycle is a k-uniform linear (or minimal) ℓ-cycle.

Proposition 4.1 *Let k be a positive integer. Let $\mathcal{H} := \{E_1, \dots, E_t\}$ be a hypergraph whose edges are sets of size at most k. Let $\mathcal{F}$ be a k-graph, $s = tk$. If $\mathcal{H} \subseteq \mathrm{Ker}_s(\mathcal{F})$, then $\mathcal{H}^{(k)} \subseteq \mathcal{F}$.*

Proof. We want to expand the edges of $\mathcal{H}$ into edges $F_1, \dots, F_t$ of $\mathcal{F}$ such that different edges of $\mathcal{H}$ are enlarged through disjoint sets of expansion vertices. We find F_i's one by one by using (2). Suppose $F_1, \dots, F_{i-1}$ have been defined. Let $Y = (\cup_{j<i} F_j) \cup (\cup_j E_j)$. Since $\deg^*_{\mathcal{F}}(E_i) \geq s > |Y|$, by (2) one can find an $F_i \in \mathcal{F}$ such that $F_i \supseteq E_i$ and $F_i \cap Y = E_i$. We do this for $i = 1, \dots, t$. The F_i's form a copy of $\mathcal{H}^{(k)}$. ■

Proposition 4.2 *Suppose that $\mathcal{F}$ is a triple system not containing $\mathbb{C}_\ell^{(3)}$. Then $|\mathcal{F}| \leq (2\ell - 3)\binom{n}{2}$.*

Proof. Starting with $\mathcal{F}$, whenever we can find a pair $\{x, y\}$ such that the number of triples containing the pair is at least one and at most $2\ell - 3$, we remove all triples containing the pair from the system. Repeat this process until no more triple can be removed. Let $\mathcal{H}$ be the remaining triple system. If $\mathcal{H} \neq \emptyset$, then we must have $\deg_{\mathcal{H}}(\{x, y\}) \geq 2\ell - 2$ for all $\{x, y\} \in \partial_2(\mathcal{H})$. Clearly $|\mathcal{F} \setminus \mathcal{H}| \leq (2\ell - 3)\binom{n}{2}$. We claim that $\mathcal{H} = \emptyset$. Otherwise, starting with any triple $A_1 = \{v_1, v_2, v_3\} \in \mathcal{H}$ we can embed one by one $A_1, \dots, A_{\ell-2}$ then $B_1, \dots, B_\ell$, the edges of a triangulated cycle $\mathbb{T}_\ell^{(3)}$ in $\mathcal{H}$ in the same way as we did in the proof of Proposition 4.1. But $\mathbb{T}_\ell^{(3)}$ contains $\mathbb{C}_\ell^{(3)}$. This contradicts $\mathbb{C}_\ell^{(3)} \not\subseteq \mathcal{F}$. ■

Given a family $\mathcal{H}$ of k-graphs, the *Turán number* of $\mathcal{H}$, for fixed n, denoted by $\mathrm{ex}_k(n, \mathcal{H})$, is the maximum number edges in a k-graph on $[n]$ that does not contain any member of $\mathcal{H}$ as a subgraph. If $\mathcal{H}$ consists of a single k-graph H, we will write $\mathrm{ex}_k(n, H)$ for $\mathrm{ex}_k(n, \{H\})$.

Corollary 4.3 *For all n and $k, \ell \geq 3$ we have*

$$\mathrm{ex}_k(n, \mathbb{C}_\ell^{(k)}) \leq (k\ell - 1)\binom{n}{k-1}.$$

Proof. Consider an $\mathcal{F} \subseteq \binom{[n]}{k}$ avoiding $\mathbb{C}_\ell^{(k)}$. Let $s = k\ell$. By Proposition 4.1 the triple-system $\mathrm{Ker}_s^{(3)}(\mathcal{F})$ does not contain $\mathbb{C}_\ell^{(3)}$ so we can apply Proposition 4.2 for its size. Use the upper bound (3) for the degrees of the other triples of $[n]$. We obtain

$$|\mathcal{F}|\binom{k}{3} = \sum_{|T|=3,\, T\subseteq[n]} \deg_{\mathcal{F}}(T) \le |\mathrm{Ker}_s^{(3)}(\mathcal{F})|\binom{n-3}{k-3} + \binom{n}{3}(s-1)\binom{n-4}{k-4}.$$

An easy calculation completes the proof. ■

5 Some previous results and a conjecture

For the class of k-uniform Berge paths of length ℓ, Győri et al. [15] determined $\mathrm{ex}_k(n, \mathcal{B}_\ell^{(k)})$ exactly for infinitely many n. For the Turán problem for k-uniform minimal paths of length ℓ, observe that to forbid such a path it suffices to forbid a matching of size $t+1$, $\mathcal{M}_{t+1}^{(k)}$, where $t = \lfloor(\ell-1)/2\rfloor$. So $\mathrm{ex}_k(n, \mathcal{P}_\ell^{(k)}) \ge \mathrm{ex}_k(n, \mathcal{M}_{t+1}^{(k)}) \ge \binom{n}{k} - \binom{n-t}{k}$, where the last lower bound is attained by taking all the k-sets in $[n]$ intersecting some fixed t-set S. Mubayi and Verstraete [23] showed that this lower bound is tight up to a factor of 2. Note that $\binom{n}{k} - \binom{n-t}{k} = t\binom{n-1}{k-1} + O(n^{k-2})$. They proved that if $k, \ell \ge 3$, $t = \lfloor(\ell-1)/2\rfloor$ and $n \ge (\ell+1)k/2$, then $\mathrm{ex}_k(n, \mathcal{P}_3^{(k)}) = \binom{n-1}{k-1}$ and for $\ell, k > 3$

$$t\binom{n-1}{k-1} + O(n^{k-2}) \le \mathrm{ex}_k(n, \mathcal{P}_\ell^{(k)}) \le 2t\binom{n-1}{k-1} + O(n^{k-2}). \tag{5}$$

Using the delta system method, Füredi, Jiang, and Seiver [13] were able to sharpen (5) to determine the exact value of $\mathrm{ex}_k(n, \mathcal{P}_\ell^{(k)})$ for all $k \ge 3, t \ge 1$ and sufficiently large n

$$\mathrm{ex}_k(n, \mathcal{P}_{2t+1}^{(k)}) = \binom{n}{k} - \binom{n-t}{k}, \quad \text{and} \quad \mathrm{ex}(n, \mathcal{P}_{2t+2}^{(k)}) = \binom{n}{k} - \binom{n-t}{k} + 1. \tag{6}$$

For $\mathcal{P}_{2t+1}^{(k)}$, the only extremal family consists of all the k-sets in $[n]$ that intersect some fixed t-set S. For $\mathcal{P}_{2t+2}^{(k)}$, the only extremal family consists of all the k-sets in $[n]$ that intersect some fixed set S of t vertices plus one additional k-set that is disjoint from S.

The Turán problem for a linear path $\mathbb{P}_\ell^{(k)}$ was also solved in [13] for all $k \ge 4$ and sufficiently large n

$$\mathrm{ex}_k(n, \mathbb{P}_{2t+1}^{(k)}) = \binom{n}{k} - \binom{n-t}{k}, \quad \text{and} \quad \mathrm{ex}_k(n, \mathbb{P}_{2t+2}^{(k)}) = \binom{n}{k} - \binom{n-t}{k} + \binom{n-t-2}{k-2}. \tag{7}$$

For $\mathbb{P}_{2t+1}^{(k)}$, the only extremal family consists of all the k-sets in $[n]$ that meet some fixed k-set S. For $\mathbb{P}_{2t+2}^{(k)}$, the only extremal family consists of all the k-sets in $[n]$ that intersect some fixed t-set S plus all the k-sets in $[n] \setminus S$ that contain some two fixed elements.

For minimal cycles of length ℓ, the same lower bound of $\binom{n}{k} - \binom{n-t}{k}$ for $\mathcal{M}_{t+1}^{(k)}$ applies, where $t = \lfloor(\ell-1)/2\rfloor$. Answering a conjecture of Erdős, Mubayi and Verstraëte [22] showed that for all $k \ge 3$ and $n \ge 3k/2$, we have $\mathrm{ex}_k(n, \mathcal{C}_3^{(k)}) = \binom{n-1}{k-1}$. Later for general minimal cycles they [23] showed that the lower bound for $\mathcal{M}_{t+1}^{(k)}$ is tight up to a factor of 3. For $k \ge 3$, $\ell \ge 4$, $t = \lfloor(\ell-1)/2\rfloor$ they have $\mathrm{ex}_3(n, \mathcal{C}_\ell^{(3)}) \le \frac{5\ell-1}{6}\binom{n}{2}$, $\mathrm{ex}_4(n, \mathcal{C}_\ell^{(4)}) \le \frac{5\ell}{4}\binom{n}{3}$ and $\mathrm{ex}_k(n, \mathcal{C}_4^{(k)}) = \binom{n-1}{k-1} + O(n^{k-2})$. For $k, \ell \ge 5$, they obtained

$$t\binom{n-1}{k-1} + O(n^{k-2}) \le \mathrm{ex}_k(n, \mathcal{C}_\ell^{(k)}) \le 3t\binom{n-1}{k-1} + O(n^{k-2}). \tag{8}$$

For $k, \ell \ge 3$, Mubayi and Verstraëte [23] conjectured their lower bound to be asymptotically tight.

Conjecture 5.1 [23] *Let $n, k, \ell \ge 3$ be integers and $t = \lfloor \frac{\ell-1}{2} \rfloor$. Then as $n \to \infty$*

$$ex_k(n, \mathcal{C}_\ell^{(k)}) = t\binom{n-1}{k-1} + O(n^{k-2}).$$

6 Main results: Turán numbers of cycles

As our main result, in Theorem 6.1 we determine for all $k \ge 5$ and sufficiently large n the exact value of the Turán number of the linear cycle $\mathbb{C}_\ell^{(k)}$. In Theorem 6.2, we determine the exact Turán numbers of minimal cycles $\mathcal{C}_\ell^{(k)}$ for all $k \ge 4$ and large n. Theorem 6.2 confirms the truth of Conjecture 5.1 for all $k \ge 4$ in a stronger sense. For $k \ge 5$ and odd ℓ, Theorem 6.1 is even stronger than Theorem 6.2.

Theorem 6.1 (Main result) *Let k, t be positive integers, $k \ge 5$. For sufficiently large n, we have*

$$\mathrm{ex}_k(n, \mathbb{C}_{2t+1}^{(k)}) = \binom{n}{k} - \binom{n-t}{k}, \quad \textit{and} \quad \mathrm{ex}_k(n, \mathbb{C}_{2t+2}^{(k)}) = \binom{n}{k} - \binom{n-t}{k} + \binom{n-t-2}{k-2}.$$

For $\mathbb{C}_{2t+1}^{(k)}$, the only extremal family consists of all the k-sets in $[n]$ that meet some fixed k-set S. For $\mathbb{C}_{2t+2}^{(k)}$, the only extremal family consists of all the k-sets in $[n]$ that intersect some fixed t-set S plus all the k-sets in $[n] \setminus S$ that contain some two fixed elements.

Note that the case $\ell = 3$ was already proved in [8].

Theorem 6.2 *Let t be a positive integer, $k \ge 4$. For sufficiently large n, we have*

$$\mathrm{ex}_k(n, \mathcal{C}_{2t+1}^{(k)}) = \binom{n}{k} - \binom{n-t}{k}, \quad \textit{and} \quad \mathrm{ex}_k(n, \mathcal{C}_{2t+2}^{(k)}) = \binom{n}{k} - \binom{n-t}{k} + 1.$$

For $\mathcal{C}_{2t+1}^{(k)}$, the only extremal family consists of all the k-sets in $[n]$ that meet some fixed k-set S. For $\mathcal{C}_{2t+2}^{(k)}$, the only extremal family consists of all the k-sets in $[n]$ that intersect some fixed t-set S plus one additional k-set outside S.

Our method does not work for $k = 3$, however, we were informed that Kostochka, Mubayi, and Verstraëte [19] have some new results on this case.

Note that the answers in the main theorem are exactly the same as for $\mathbb{P}_{2t+1}^{(k)}$ and $\mathbb{P}_{2t+2}^{(k)}$ with the same extremal constructions as well. However, neither the path result nor the cycle result imply each other and the proofs for cycles are more involved and require additional ideas.

7 The delta-system method

In this section, we introduce our main tools we need from the delta-system method. Given a hypergraph $\mathcal{F}$ and an edge F of $\mathcal{F}$, we define the *intersection structure* of F relative to $\mathcal{F}$ to be

$$\mathcal{I}(F,\mathcal{F}) = \{F \cap F' : F' \in \mathcal{F}, F' \neq F\}.$$

Let $\mathcal{F}$ be a k-partite k-graph with a k-partition $(X_1,\dots,X_k)$. Hence, each edge of $\mathcal{F}$ contains exactly one element of each X_i. Given any subset S of $[n]$, let

$$\Pi(S) = \{i : S \cap X_i \neq \emptyset\} \subseteq [k].$$

So $\Pi(S)$ records which parts in the given k-partition that S meets. If $\mathcal{L}$ is a collection of subsets of $[n]$, then we define

$$\Pi(\mathcal{L}) = \{\Pi(S) : S \in \mathcal{L}\} \subseteq 2^{[k]}.$$

We will call $\Pi(\mathcal{I}(F,\mathcal{F}))$ the *intersection pattern* of F relative to $\mathcal{F}$. Given $F \in \mathcal{F}$ and $I \subseteq [k]$, let $F[I] = F \cap (\bigcup_{i\in I} X_i)$. So $F[I]$ is the restriction of F onto those parts indexed by I.

Lemma 7.1 (The intersection semilattice lemma [11]) *For any positive integers s and k, there exists a positive constant $c(k,s)$ such that* **every** *family $\mathcal{F} \subseteq \binom{[n]}{k}$ contains a subfamily $\mathcal{F}^* \subseteq \mathcal{F}$ satisfying*

1. *$|\mathcal{F}^*| \geq c(k,s)|\mathcal{F}|$.*
2. *$\mathcal{F}^*$ is k-partite, together with a k-partition $(X_1,\dots,X_k)$.*
3. *There exists a family $\mathcal{J}$ of proper subsets of $[k]$ such that $\Pi(\mathcal{I}(F,\mathcal{F}^*)) = \mathcal{J}$ holds for all $F \in \mathcal{F}^*$.*
4. *$\mathcal{J}$ is closed under intersection, i.e., for all $I, I' \in \mathcal{J}$ we have $I \cap I' \in \mathcal{J}$ as well.*
5. *For every $F \in \mathcal{F}^*$, and every $I \in \mathcal{J}$, $F[I] \in \mathrm{Ker}_s(\mathcal{F})$.*

Definition 7.2 We call a family $\mathcal{F}^*$ that satisfies items (2)-(5) of Lemma 7.1 (k,s)-*homogeneous* with intersection pattern $\mathcal{J}$.

Given a family $\mathcal{L}$ of subsets of $[k]$, the *rank* of $\mathcal{L}$ is the minimum size of a set in $[k]$ that is not contained in any member of $\mathcal{L}$. Formally

$$r(\mathcal{L}) = \min\{|D| : D \subseteq [k], \nexists B \in \mathcal{L}, D \subseteq B\}.$$

The next three lemmas were used in many earlier papers, e.g., we can refer to [8, 13].

Lemma 7.3 (The rank bound) *Let k,s be positive integers. Let $\mathcal{F}^*$ be a (k,s)-homogeneous family on n vertices with intersection pattern $\mathcal{J}$. If $r(\mathcal{J}) = p$, then $|\mathcal{F}^*| \leq \binom{n}{p}$.*

Lemma 7.4 *Let $k \geq 3$ be a positive integer. Let $\mathcal{L}$ be a family of proper subsets of $[k]$ that is closed under intersection.*

1. *If $\mathcal{L}$ has rank k, then it contains all the proper subsets of $[k]$.*

2. *If $\mathcal{L}$ has rank $k-1$, then the elements of $[k]$ can be listed as $x_1, x_2, \dots, x_t, x_{t+1}, \dots, x_k$ such that for every $t+1 \le i \le k$, $[k] \setminus \{x_i\} \in \mathcal{L}$ and for all $1 \le i < j \le t$, $[k] \setminus \{x_i, x_j\} \in \mathcal{L}$. If $t = 1$, then we say that $\mathcal{L}$ is of* **type 1**. *If $t \ge 2$, then we say that $\mathcal{L}$ is of* **type 2**. *If $\mathcal{L}$ is of type 1, then there exists an element $x \in [k]$ such that $\mathcal{L}$ contains all the proper subsets of $[k]$ that contains x; we call x the* **central element** *of $\mathcal{L}$. If $\mathcal{L}$ is of type 2, then $\forall C \subseteq \{x_1, \dots, x_t\}, \forall D \subseteq \{x_{t+1}, \dots, x_k\}$, where $|C| \le t-2$, we have $C \cup D \in \mathcal{L}$.*

Lemma 7.5 (The partition lemma) *Let n, k, s be positive integers, let $\mathcal{F} \subseteq \binom{[n]}{k}$. Then $\mathcal{F}$ can be partitioned into subfamilies $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_m$ and $\mathcal{F}_0$ such that $|\mathcal{F}_0| \le \frac{1}{c(k,s)}\binom{n}{k-2}$ and for $1 \le i \le m$ each $\mathcal{G}_i$ is (k,s)-homogeneous with intersection pattern $\mathcal{J}_i$ of rank at least $k-1$.*

Proof. Apply Lemma 7.1 to $\mathcal{F}$ to get a (k,s)-homogeneous subfamily $\mathcal{G}_1$ with intersection pattern $\mathcal{J}_1$ such that $|\mathcal{G}_1| \ge c(k,s)|\mathcal{F}|$. Then apply Lemma 7.1 again to $\mathcal{F} \setminus \mathcal{G}_1$ to get a (k,s)-homogeneous subfamily $\mathcal{G}_2$ with intersection pattern $\mathcal{J}_2$ such that $|\mathcal{G}_2| \ge c(k,s)|\mathcal{F} \setminus \mathcal{G}_1|$. We continue like this. Let m be the smallest nonnegative integer such that $\mathcal{J}_{m+1}$ has rank $k-2$ or less and let $\mathcal{F}_0 = \mathcal{F} \setminus (\cup_{i \le m} \mathcal{G}_i)$. By our procedure, $|\mathcal{G}_{m+1}| \ge c(k,s)|\mathcal{F}_0|$ and Lemma 7.3 gives the upper bound. ■

8 Homogeneous families without cycles are not of type 2

The aim of this section is to describe the typical intersection structures of the members of a k-uniform hypergraph avoiding cycles.

Definition 8.1 A set-family $\mathcal{F}$ is *centralized* with threshold s if $\forall F \in \mathcal{F}$ there exists an element $c(F) \in F$ such that if D is a proper subset of F containing $c(F)$ then $D \in \mathrm{Ker}_s(\mathcal{F})$. We call $c(F)$ a *central element* of F. (The choice of $c(F)$ may not be unique, but we will fix one.)

Theorem 8.2 (The partition theorem) *Let k, ℓ, s be positive integers, where $k \ge 4$, $\ell \ge 3$, and $s \ge k\ell$. Let $\mathcal{F} \subseteq \binom{[n]}{k}$. If $k = 4$, then suppose $\mathcal{F}$ contains no member of $\mathcal{C}_\ell^{(4)}$. If $k \ge 5$, then suppose $\mathbb{C}_\ell^{(k)} \not\subseteq \mathcal{F}$. Then $\mathcal{F}$ can be partitioned into subfamilies $\mathcal{F}_1, \mathcal{F}_0$ such that $\mathcal{F}_1$ is centralized with threshold s and $|\mathcal{F}_0| \le \frac{1}{c(k,s)}\binom{n}{k-2}$.*

The proof consists of several small steps and is given at the end of this Section.

The following proposition follows immediately from Lemma 7.1 and Lemma 7.4.

Proposition 8.3 *If $\mathcal{F} = \bigcup_{i=1}^m \mathcal{G}_i$, where $\forall i \in [m]$, $\mathcal{G}_i$ is a (k,s)-homogeneous family whose intersection pattern $\mathcal{J}_i$ has rank $k-1$ and is of type 1, then $\mathcal{F}$ is centralized with threshold s.* ■

Recall that given a set S, 2^S denotes the collection of all subsets of S.

Lemma 8.4 *Let $k \ge 5$ be an integer and let $\mathcal{L}$ be a family of subsets of $[k]$ that is closed under intersection and has rank $k-1$ and is of type 2. Then there exists $S \subseteq [n]$ such that $|S| = 3$ and $2^S \subseteq \mathcal{L}$.*

Proof. Let S be a 3-subset of $[k] \setminus \{x_1, x_2\}$. Any subset A of S can be written as $C \cup D$, where $C \subseteq \{x_1, \dots, x_t\}$, $|C| \leq t-2$, and $D \subseteq \{x_{t+1}, \dots, x_k\}$. By Lemma 7.4, $A \in \mathcal{L}$. ∎

For $k = 4$, we prove something a bit weaker.

Proposition 8.5 *Let $\mathcal{L}$ be a family of subsets of $[4]$ that is closed under intersection and has rank 3 and is of type 2. Then $\mathcal{L}$ contains a minimal 3-cycle where each edge has size 2 or 3.*

Proof. By Lemma 7.4, $\forall C \subseteq \{x_1, \dots, x_t\}$, where $|C| \leq t-2$, and $\forall D \subseteq \{x_{t+1}, \dots x_k\}$, we have $C \cup D \in \mathcal{L}$. If $t = 4$, then we have $\{x_1, x_2\}$, $\{x_1, x_3\}$, $\{x_2, x_3\} \in \mathcal{L}$. If $t = 3$, then we have $\{x_1, x_2, x_3\}$, $\{x_1, x_4\}$, $\{x_3, x_4\} \in \mathcal{L}$. If $t = 2$, then we have $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$, $\{x_3, x_4\} \in \mathcal{L}$. ∎

Lemma 8.6 *Let k, ℓ, s be positive integers, where $k \geq 5, \ell \geq 3$, and $s \geq k\ell$. Let $\mathcal{G}$ be a (k, s)-homogeneous family with a k-partition $(X_1, \dots, X_k)$ and intersection pattern $\mathcal{J}$ such that either $\mathcal{J}$ has rank k or has rank $k-1$ and is of type 2. Then $\mathbb{C}_\ell^{(k)} \subseteq \mathcal{G}$.*

Proof. By Lemma 7.4 and Lemma 8.4, there exists a 3-set $S \subseteq [k]$ such that $2^S \subseteq \mathcal{J}$. By definition, this means that $\forall F \in \mathcal{G}, \forall A \subseteq S$, $F \cap (\cup_{i \in A} X_i)$ is a member of $\mathrm{Ker}_s(\mathcal{G})$. Let $\mathcal{H} = \{F \cap (\cup_{i \in A} X_i) : F \in \mathcal{G}, A \subseteq S\}$. Then $\mathcal{H} \subseteq \mathrm{Ker}_s(\mathcal{G})$. Note that $\mathcal{H}$ is down-closed. Let $D \in \partial_2(\mathcal{H})$. Then $D \in \mathrm{Ker}_s(\mathcal{F})$. So $\mathcal{F}$ contains an s-star $\mathcal{L}$ with kernel D. The restriction of the s petals of $\mathcal{L}$ on $\bigcup_{i \in S} X_i$ are s distinct triples in $\mathcal{H}$ containing D. So $\deg^*_{\mathcal{H}}(D) = \deg_{\mathcal{H}}(D) \geq s$ for all $D \in \partial_2(\mathcal{H})$. This allows us to embed the triangulated cycle $\mathbb{T}_\ell^{(3)}$ into $\mathcal{H}$ as we did in the proof of Proposition 4.2. Since $\mathbb{C}_\ell^{(3)} \subseteq \mathbb{T}_\ell^{(3)}$ and $\mathcal{H} \subseteq \mathrm{Ker}_s(\mathcal{G})$ Proposition 4.1 implies that $\mathcal{G}$ contains a k-expansion of $\mathbb{C}_\ell^{(3)}$, which is $\mathbb{C}_\ell^{(k)}$. ∎

For $k = 4$, Lemma 8.5 and induction yield

Proposition 8.7 *Let ℓ, s be positive integers, where $\ell \geq 3$ and $s \geq 4\ell$. Let $\mathcal{G}$ be a $(4, s)$-homogeneous family with a 4-partition $(X_1, \dots, X_4)$ intersection pattern $\mathcal{J}$ such that either $\mathcal{J}$ has rank 4 or has rank 3 and is of type 2. Then $\mathcal{G}$ contains a member of $\mathcal{C}_\ell^{(4)}$.*

Proof. By Lemma 8.5 $\mathcal{J}$ contains a minimal 3-cycle L. Consider first the case where the edges of L are $I_1 = \{1,2,3\}, I_2 = \{1,2,4\}$ and $I_3 = \{3,4\}$. Then $\forall F \in \mathcal{G}, F[I_1], F[I_2], F[I_3] \in \mathrm{Ker}_s(\mathcal{G})$. We use induction on ℓ to show that $\mathcal{G}$ contains a member of $\mathcal{C}_\ell^{(4)}$ such that for any two consecutive edges E, E' on the cycle, either they intersect in exactly one vertex and that vertex lies in X_3 or X_4 or they intersect in two vertices and those two vertices lie in X_1 and X_2, respectively; we call such a member of $\mathcal{C}_\ell^{(4)}$ a *good* member.

For the basis step let $\ell = 3$. Let $E_0 = \{a_1, a_2, a_3, a_4\}$ be any edge in $\mathcal{G}$, where $\forall i \in [4], a_i \in X_i$. By our assumption, $\{a_1, a_2, a_3\}, \{a_1, a_2, a_4\}, \{a_3, a_4\}$ all have kernel degree at least $s \geq 4\ell$. So we can find $E_1 = \{a_1, a_2, a_3, a'_4\}, E_2 = \{a_1, a_2, a'_3, a_4\}, E_3 = \{a'_1, a'_2, a_3, a_4\} \in \mathcal{G}$, where $\forall i \in [4], a'_i \in X_i$ and $a_1, \dots, a_4, a'_1, \dots, a'_4$ are all distinct. Now, E_1, E_2, E_3 form a minimal 3-cycle that satisfies the claim. For the induction step, suppose $\ell \geq 4$ and that the claim holds for $\ell - 1$ and that $\mathcal{G}$ contains a good member L of $\mathcal{C}_{\ell-1}^{(4)}$. Let E be an edge of L. Let E', E'' be the edge preceding E and succeeding E, respectively on L. Then $\{|E \cap E'|, |E \cap E''|\} = \{1,1\}$ or $\{1,2\}$. In the former case, we may assume $E \cap E' = \{b_3\} \subseteq X_3$ and $E \cap E'' = \{b_4\} \subseteq X_4$. By our assumption $E \setminus \{b_4\} = E[I_1] \in \mathrm{Ker}_s(\mathcal{G})$ and $E \setminus \{b_3\} = E[I_2] \in \mathrm{Ker}_s(\mathcal{G})$. Since $s \geq 4\ell$ and $n(L) \leq 4\ell - 1$, we can find $b'_3 \in X_3 \setminus V(L), b'_4 \in X_4 \setminus V(L)$

such that $(E \setminus \{b_3\}) \cup \{b_3'\} \in \mathcal{G}$ and $(E \setminus \{b_4\}) \cup \{b_4'\} \in \mathcal{F}$. Replacing E with these two members of $\mathcal{G}$ in L yields a good member of $\mathcal{C}_\ell^{(4)}$. The case where $\{|E \cap E'|, |E \cap E''|\} = \{1, 2\}$ can be handled similarly. This completes the induction.

Similar arguments apply if $\mathcal{J}$ contains other kinds of minimal 3-cycles. ■

Proof of Theorem 8.2. Consider the partition $\mathcal{F} = \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m \cup \mathcal{F}_0$ given by the Partition Lemma 7.5. For each $i \in [m]$, $\mathcal{J}_i$ has rank at least $k-1$. If some $\mathcal{J}_i$ has rank k or has rank $k-1$ and is of type 2, then $\mathbb{C}_\ell^{(k)} \subseteq \mathcal{G}_i \subseteq \mathcal{F}$ by Lemma 8.6, a contradiction in the case of $k \geq 5$. So each $\mathcal{J}_i$ has rank $k-1$ and is of type 1. By Proposition 8.3, $\mathcal{F}_1 := \bigcup_{i=1}^m \mathcal{G}_i$ is centralized with threshold s.

For $k = 4$ we use Proposition 8.7 in place of Lemma 8.6. ■

9 The kernel structure of centralized families

Theorem 9.1 *Let k, ℓ be integers, where $k \geq 4$ and $\ell \geq 3$. Let $t = \lfloor \frac{\ell-1}{2} \rfloor$. Let $s = k\ell$. For all $n \geq n_1(k, \ell)$ the following holds: If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a centralized family with threshold s, $\mathbb{C}_\ell^{(k)} \not\subseteq \mathcal{F}$, and $|\mathcal{F}| \geq t\binom{n}{k-1} - o(n^{k-1})$, then there exist $S, T \subseteq [n]$, where $S \cap T = \emptyset$, $|S| = t$ and $|T| \geq n - o(n)$, such that $\mathrm{Ker}_s^{(2)}(\mathcal{F})$ contains all the edges between S and T.*

Lemma 9.2 *Let n, p, q be positive integers and $x_1, \ldots, x_n$ reals such that $q \geq p$ and $x_1 \geq \ldots \geq x_n \geq q-1$. Let $M = \sum_{i=1}^n \binom{x}{p}$. Then $\forall h \in [n]$, we have*

$$\sum_{i=h}^{n} \binom{x_i}{q} \leq p^{q-p} \frac{M^{\frac{q}{p}}}{h^{\frac{q}{p}-1}}.$$

Proof. Since $\sum_{i=1}^n \binom{x_i}{p} = M$ and $\binom{x_1}{p} \geq \binom{x_2}{p} \geq \ldots \geq \binom{x_n}{p}$, we have $\binom{x_h}{p} \leq \frac{M}{h}$. Since $\binom{x_h}{p} \geq (\frac{x_h}{p})^p$, this yields $x_h < p(\frac{M}{h})^{\frac{1}{p}}$. Also, trivially for $i \geq h$, $\binom{x_i}{q} \leq x_i^{q-p}\binom{x_i}{p} \leq x_h^{q-p}\binom{x_i}{p}$. Hence,

$$\sum_{i=h}^{n} \binom{x_i}{q} \leq (x_h)^{q-p} \sum_{i=h}^{n} \binom{x_i}{p} = x_h^{q-p} M < M \left[p \left(\frac{M}{h}\right)^{\frac{1}{p}} \right]^{q-p} = p^{q-p} \frac{M^{\frac{q}{p}}}{h^{\frac{q}{p}-1}}.$$

■

Proof of Theorem 9.1. Let us partition $\mathcal{F}$ according to $c(F)$. For each $i \in [n]$, let

$$\mathcal{A}_i = \{F \in \mathcal{F} : c(F) = i\}, \quad \text{and} \quad \mathcal{A}_i' = \mathcal{F}_i - \{i\}.$$

Let $D \in \partial_2(\mathcal{A}_i')$. Then $D \cup \{i\}$ is a proper 3-subset of F containing $i = c(F)$. Since $\mathcal{F}$ is centralized with threshold s, $D \cup \{i\} \in \mathrm{Ker}_s^{(3)}(\mathcal{F})$. Thus $\forall D \in \partial_2(\mathcal{A}_i')$, we have $D \cup \{i\} \in \mathrm{Ker}_s^{(3)}(\mathcal{F})$. This yields

$$|\mathrm{Ker}_s^{(3)}(\mathcal{F})| \geq \frac{1}{3} \sum_{i=1}^{n} |\partial_2(\mathcal{A}_i')|. \tag{9}$$

Since $\mathbb{C}_\ell^{(k)} \not\subseteq \mathcal{F}$ and $s = k\ell$, by Proposition 4.1, $\mathbb{C}_\ell^{(3)} \not\subseteq \mathrm{Ker}_s^{(3)}(\mathcal{F})$. By Proposition 4.2, we have

$$|\mathrm{Ker}_s^{(3)}(\mathcal{F})| \le \mathrm{ex}_3(n, \mathbb{C}_\ell^{(3)}) < 2\ell\binom{n}{2}. \tag{10}$$

By (9) and (10), we have

$$\sum_{i=1}^{n} |\partial_2(\mathcal{A}'_i)| \le 6\ell\binom{n}{2} < 3\ell n^2. \tag{11}$$

For each $i \in [n]$, let $x_i \ge 1$ be the real such that $|\partial_2(\mathcal{A}'_i)| = \binom{x_i}{2}$, where without loss of generality we may assume that $x_1 \ge \ldots \ge x_n$. Let $M = \sum_{i=1}^n \binom{x_i}{2}$. Then $M < 3\ell n^2$. By Kruskal-Katona's theorem (1), $\forall i \in [n]$, $|\mathcal{A}'_i| \le \binom{x_i}{k-1}$. Now, set $\varepsilon = \frac{1}{2(t+1)}$ and $h = \lceil n^\varepsilon \rceil$. Applying Lemma 9.2 with $p = 2, q = k-1$, we have

$$\sum_{i \ge h} |\mathcal{A}_i| = \sum_{i=h}^{n} |\mathcal{A}'_i| = \sum_{i=h}^{n} \binom{x_i}{k-1} \le 2^{k-3}\frac{M^{\frac{k-1}{2}}}{h^{\frac{k-3}{2}}} = O(n^{k-1-\frac{k-3}{2}\varepsilon}) = O(n^{k-1-\frac{k-3}{4(t+1)}}). \tag{12}$$

Let $L = [h]$. Let $\mathcal{F}_1 = \{F \in \mathcal{F} : c(F) \notin L\}$. Then $\mathcal{F}_1 \subseteq \bigcup_{i>h} \mathcal{A}_i$. By (12), we have

$$|\mathcal{F}_1| = O(n^{k-1-\frac{k-3}{4(t+1)}}). \tag{13}$$

By our definition, $\forall F \in \mathcal{F} \setminus \mathcal{F}_1$ we have $c(F) \in L$. Let $\mathcal{F}_2 = \{F : F \in \mathcal{F} \setminus \mathcal{F}_1, |F \cap L| \ge 2\}$. Then

$$|\mathcal{F}_2| \le \binom{|L|}{2}\binom{n-|L|}{k-2} \le n^{k-2+2\varepsilon} < n^{k-\frac{3}{2}}. \tag{14}$$

Let $\mathcal{F}' = \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)$. Then $\forall F \in \mathcal{F}'$, we have $F \cap L = \{c(F)\}$. For each $A \subseteq L$, let

$$\mathcal{F}_A = \{F \in \mathcal{F}' : \forall a \in A, (F \setminus L) \cup \{a\} \in \mathcal{F}'\}, \quad \text{and} \quad \mathcal{F}'_A = \{F \setminus L : F \in \mathcal{F}_A\}.$$

Then $\mathcal{F}' = \bigcup_{A \subseteq L} \mathcal{F}_A$.

A set W of vertices in a hypergraph $\mathcal{G}$ is *strongly independent* if no two vertices of W lie in the same edge of $\mathcal{G}$. The cycle $\mathbb{C}_\ell^{(k)}$ has a strongly independent set W of $t+1 = 1 + \lfloor(\ell-1)/2\rfloor = \lceil \ell/2 \rceil$ vertices whose removal leaves a $(k-1)$-uniform hypergraph $\mathcal{T}$ with ℓ edges. It is easy to see that $\mathcal{T} \subseteq \mathbb{C}_{\lceil 3\ell/2 \rceil}^{(k-1)}$. By Corollary 4.3 we obtain

$$\mathrm{ex}_{k-1}(n, \mathcal{T}) \le \mathrm{ex}_{k-1}(n, \mathbb{C}_{\lceil 3\ell/2 \rceil}^{(k-1)}) < 2k\ell\binom{n}{k-2}. \tag{15}$$

Note that one can easily get a sharper bound on $\mathrm{ex}_{k-1}(n, \mathcal{T})$ than (15). But (15) suffices for our purposes. Suppose there exists $A \subseteq L$, where $|A| \ge t+1$, such that $\mathcal{F}'_A$ contains a copy $\mathcal{T}'$ of $\mathcal{T}$. Then since each edge of $\mathcal{T}'$ together with each $a \in A$ forms an edge of $\mathcal{F}$, we can extend $\mathcal{T}'$ to a copy of $\mathbb{C}_\ell^{(k)}$ in $\mathcal{F}$, contradicting $\mathbb{C}_\ell^{(k)} \not\subseteq \mathcal{F}$. So, $\forall A \subseteq L, |A| \ge t+1$, we have $\mathcal{T} \not\subseteq \mathcal{F}_A$ and by (15) $|\mathcal{F}_A| \le 2k\ell\binom{n-|L|}{k-2}$. Let $\mathcal{F}_3 = \bigcup_{A \subseteq L, |A| \ge t+1} \mathcal{F}_A$. By our discussion above, we have

$$|\mathcal{F}_3| \le \binom{|L|}{t+1} 2k\ell \binom{n-|L|}{k-2} < 2k\ell n^{k-2+\varepsilon(t+1)} = O(n^{k-\frac{3}{2}}). \tag{16}$$

Let $\mathcal{F}^* = \bigcup_{A\subseteq L, |A|\le t} \mathcal{F}_A$. Then $\mathcal{F}^* = \mathcal{F}' \setminus \mathcal{F}_3 = \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$. By (13), (14), and (16), we have

$$|\mathcal{F}^*| \ge t\binom{n}{k-1} - o(n^{k-1}).$$

Furthermore, by the definition of $\mathcal{F}^*$, we have $\forall F \in \mathcal{F}^*, F \cap L = \{c(F)\}$ and $\deg_{\mathcal{F}^*}(F \setminus L) \le t$.

Let

$$\mathcal{F}^*_0 = \{F \in \mathcal{F}^* : \deg_{\mathcal{F}^*}(F \setminus L) \le t-1\}.$$

Obviously

$$|\mathcal{F}^*| + \frac{1}{t-1}|\mathcal{F}^*_0| \le t\binom{n-|L|}{k-1}.$$

Since $|\mathcal{F}^*| \ge t\binom{n}{k-1} - o(n^{k-1})$, we have

$$|\mathcal{F}^*_0| = o(n^{k-1}) \quad \text{ and } |\mathcal{F}^* \setminus \mathcal{F}^*_0| \ge t\binom{n}{k-1} - o(n^{k-1}).$$

By our definition, $\mathcal{F}^* \setminus \mathcal{F}^*_0 = \bigcup_{A\subseteq L, |A|=t} \mathcal{F}_A$. Note that $|\mathcal{F}_A| = t|\mathcal{F}'_A|$ for each $A \subseteq L, |A| = t$.

Fix any $A \subseteq L$, $D \in \partial(\mathcal{F}'_A)$, and $a \in A$. By definition, $D \cup \{a\}$ is a proper subset of some $F \in \mathcal{F}_A$ where $c(F) = a$. So $D \cup \{a\} \in \mathrm{Ker}_s(\mathcal{F})$. In particular, $\forall x \in \partial_1(\mathcal{F}'_A) = V(\mathcal{F}'_A)$ and $\forall a \in A$, we have $\{a, x\} \in \mathrm{Ker}^{(2)}_s(\mathcal{F})$ and $\forall \{x, y\} \in \partial_2(\mathcal{F}'_A)$ and $\forall a \in A$, we have $\{a, x, y\} \in \mathrm{Ker}^{(3)}_s(\mathcal{F})$. This means that $\mathrm{Ker}^{(2)}_s(\mathcal{F})$ contains all the edges between A and $V(\mathcal{F}'_A)$. Let

$$\mathcal{A}_1 = \{A \subseteq L : |A| = t, |V(\mathcal{F}'_A)| \ge t+1\}, \quad \mathcal{A}_2 = \{A \subseteq L : |A| = t, |V(\mathcal{F}'_A)| \le t\}.$$

Recall that $|L| = O(n^{\varepsilon}) = O(n^{\frac{1}{2(t+1)}})$. We have

$$|\bigcup_{A\in\mathcal{A}_2} \mathcal{F}_A| \le \binom{|L|}{t}\binom{t}{k-1}t = O(n^{\frac{1}{2}}).$$

Hence,

$$|\bigcup_{A\in\mathcal{A}_1} \mathcal{F}_A| \ge t\binom{n}{k-1} - o(n^{k-1}). \tag{17}$$

Claim 1. $\forall A, B \in \mathcal{A}_1, A \ne B$, we have $\partial_2(\mathcal{F}'_A) \cap \partial_2(\mathcal{F}'_B) = \emptyset$.

Proof of Claim 1. Suppose otherwise that there are $A, B \in \mathcal{A}_1, A \ne B$ and $x, y \in [n] \setminus L$, such that $\{x, y\} \in \partial_2(\mathcal{F}'_A) \cap \partial_2(\mathcal{F}'_B)$. Since $\mathrm{Ker}^{(2)}_s(\mathcal{F})$ contains all the edges between A and $V(\mathcal{F}'_A)$ and $|V(\mathcal{F}'_A)| \ge t+1$, we can find an x, y-path P of length $2t$ in $\mathrm{Ker}^{(2)}_s(\mathcal{F})$ using the edges between A and $V(\mathcal{F}'_A)$. Let $b \in B \setminus A$. Since $\{x, y\} \in \partial_2(\mathcal{F}'_B)$, we have $\{x, y, b\} \in \mathrm{Ker}^{(3)}_s(\mathcal{F})$. Now, $P \cup \{x, y, b\}$ is a linear cycle of length $2t+1$ in $\mathrm{Ker}_s(\mathcal{F})$. Since $s \ge k\ell$, by Proposition 4.1, $\mathcal{F}$ contains a linear cycle $\mathbb{C}^{(k)}_{2t+1}$. Note that we also have $\{x, b\}, \{y, b\} \in \mathrm{Ker}^{(2)}_s(\mathcal{F})$. So $P \cup \{xb, yb\}$ is a linear cycle of length $2t+2$ in $\mathrm{Ker}_s(\mathcal{F})$ and by Proposition 4.1, $\mathcal{F}$ contains a linear cycle of length $2t+2$. Since $\ell = 2t+1$ or $2t+2$, $\mathcal{F}$ contains a copy of $\mathbb{C}^{(k)}_\ell$, contradicting our assumption about $\mathcal{F}$. ■

For convenience, suppose $\mathcal{A}_1 = \{A_1, \dots, A_p\}$. For each $i \in [p]$, let $y_i \ge k-1$ denote the positive real such that $|\mathcal{F}'_{A_i}| = \binom{y_i}{k-1}$, where without loss of generality, we may assume that $y_1 \ge y_2 \ge \ldots \ge y_p$. By the Kruskal-Katona theorem (1), $\forall i \in [p], |\partial_2(\mathcal{F}'_{A_i})| \ge \binom{y_i}{2}$. By Claim 1, $\partial_2(\mathcal{F}'_{A_1}), \dots, \partial_2(\mathcal{F}'_{A_p})$

are pairwise disjoint. So we have

$$\sum_{i=1}^{p} \binom{y_i}{2} \leq \binom{n-|L|}{2} < \binom{n}{2}.$$

For each $i = 1, \ldots, p$, observe that $\frac{\binom{y_i}{k-1}}{\binom{y_1}{k-1}} \leq \frac{\binom{y_i}{2}}{\binom{y_1}{2}}$ and hence $\binom{y_i}{k-1} \leq \binom{y_1}{k-1}\frac{\binom{y_i}{2}}{\binom{y_1}{2}}$. This yields

$$|\bigcup_{A\in\mathcal{A}_1} \mathcal{F}_A| = t\sum_{A\in\mathcal{A}_1} |\mathcal{F}'_A| = t\sum_{i=1}^{p} \binom{y_i}{k-1} \leq t\binom{y_1}{k-1}\frac{\sum_{i=1}^{p}\binom{y_i}{2}}{\binom{y_1}{2}} < t\binom{y_1}{k-1}\frac{\binom{n}{2}}{\binom{y_1}{2}}.$$

This and (17) imply $y_1 \geq n - o(n)$. Applying Kruskal-Katona theorem (1) again we get

$$|V(\mathcal{F}'_{A_1})| = |\partial_1(\mathcal{F}'_{A_1})| \geq y_1 \geq n - o(n).$$

Since $\mathrm{Ker}_s^{(2)}(\mathcal{F})$ has the all the edges between A_1 and $V(\mathcal{F}'_{A_1})$ the sets $S = A_1$ and $T = V(\mathcal{F}'_{A_1})$ satisfy the claim of Theorem 9.1. ■

10 Proofs of the main results

In this section we prove Theorem 6.1 and Theorem 6.2. The lower bound is presented in Section 6. It remains to prove the upper bounds for large n.

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, where n is sufficiently large. To prove Theorem 6.1 we assume that $k \geq 5$ and $\mathcal{F}$ contains no copy of $\mathbb{C}_\ell^{(k)}$. To prove Theorem 6.2, we assume that $k \geq 4$ and $\mathcal{F}$ contains no member of $\mathcal{C}_\ell^{(k)}$. Each upper bound in Theorem 6.1 and 6.2 is at least $t\binom{n}{k-1} - O(n^{k-2})$. So we may assume that $|\mathcal{F}| \geq t\binom{n}{k-1} - o(n^{k-1})$. By Theorem 8.2, we can partition $\mathcal{F}$ into two subfamilies $\mathcal{F}_1$ and $\mathcal{F}_0$, where $\mathcal{F}_1$ is centralized with threshold $s = k\ell$ and $|\mathcal{F}_0| = O(n^{k-2})$. In particular, $|\mathcal{F}_1| \geq t\binom{n}{k-1} - o(n^{k-1})$.

By Theorem 9.1, there exists a set $S \subseteq [n]$, where $|S| = t$ and a set $T \subseteq [n]\setminus S$ where $|T| \geq n - o(n)$ such that $\mathrm{Ker}_s^{(2)}(\mathcal{F}_1)$, as a 2-graph, contains all the edges between S and T. Let $W \subseteq [n] \setminus S$ be a set of maximum size such that $\mathrm{Ker}_s^{(2)}(\mathcal{F})$ contains all the edges between S and W. We have $|W| \geq n - o(n)$. Let $Z = [n] \setminus (S \cup W)$, $z = |Z|$. We have $z = o(n)$. Let

$$\mathcal{F}_S = \{F \in \binom{[n]}{k} : F \cap S \neq \emptyset\}.$$

Then $|\mathcal{F}_S| = \binom{n}{k} - \binom{n-t}{k}$.

We split $\mathcal{F} \setminus \mathcal{F}_S$ into three (later into four) parts and will give an estimate for their sizes one by one. We also estimate a class of missing edges, $\mathcal{D} \subseteq \mathcal{F}_S \setminus \mathcal{F}$, and finally compare $|\mathcal{D}|$ to $|\mathcal{F} \setminus \mathcal{F}_S|$.

Define $\mathcal{F} \setminus \mathcal{F}_S = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$, $\mathcal{G}_1 = \mathcal{A} \cup \mathcal{B}$ and $\mathcal{D}$ as follows.

$$\begin{aligned}
\mathcal{G}_0 &= \{F \in \mathcal{F} : F \subseteq Z\}, \quad \text{i.e., } F \cap S = \emptyset,\ |F \cap W| = 0,\\
\mathcal{G}_1 &= \{F \in \mathcal{F} : F \cap S = \emptyset,\ |F \cap W| = 1\},\\
\mathcal{A} &= \{F \in \mathcal{G}_1 : \deg_{\mathcal{G}_1}(F \setminus W) < \ell\}, \quad \mathcal{B} = \{F \in \mathcal{G}_1 : \deg_{\mathcal{G}_1}(F \setminus W) \geq \ell\},\\
\mathcal{G}_2 &= \{F \in \mathcal{F} : F \cap S = \emptyset,\ |F \cap W| \geq 2\},\\
\mathcal{D} &= \{F \in \binom{[n]}{k} : |F \cap S| = |F \cap Z| = 1,\ F \notin \mathcal{F}\}.
\end{aligned}$$

The family $\mathcal{G}_0$ does not contain a $\mathbb{C}_\ell^{(k)}$ on z vertices so Proposition 4.3 yields

$$|\mathcal{G}_0| \leq k\ell \binom{z}{k-1} = O(z^{k-1}) = o(zn^{k-2}). \tag{18}$$

Clearly

$$|\mathcal{A}| \leq \ell \binom{|Z|}{k-1} = O(z^{k-1}) = o(zn^{k-2}). \tag{19}$$

Let $\mathcal{B}' = \{F \setminus W : F \in \mathcal{B}\}$, it is a $(k-1)$-graph on Z. If $\mathcal{B}'$ contains a copy L of $\mathbb{C}_\ell^{(k-1)}$, then since $\forall F \in \mathcal{B}, \deg_{\mathcal{G}_1}(F \setminus W) \geq \ell$, L can be extended to a copy of $\mathbb{C}_\ell^{(k)}$ in $\mathcal{F}$, a contradiction. So $\mathcal{B}'$ contains no linear ℓ-cycle, and Proposition 4.3 gives $|\mathcal{B}'| \leq k\ell\binom{z}{k-2} = O(z^{k-2})$. Since $|\mathcal{B}| \leq |\mathcal{B}'| \cdot |W|$ we get

$$|\mathcal{B}| = O(z^{k-2}n) = o(zn^{k-2}). \tag{20}$$

Claim 2. For $\ell = 2t+1$ we have $\mathcal{G}_2 = \emptyset$. For $\ell = 2t+2$, if $\mathcal{F}$ has no linear ℓ-cycle then $\mathcal{G}_2$ has no two members meeting in a singleton and if $\mathcal{F}$ has no minimal ℓ-cycle then $|\mathcal{G}_2| \leq 1$.

Proof of Claim 2. Suppose first that $\ell = 2t+1$. Suppose $\mathcal{G}_2$ has a member F. By definition, $F \cap S = \emptyset$ and $|F \cap W| \geq 2$. Let x, y be two elements of $F \cap W$. Let H denote the subgraph of $\mathrm{Ker}_s^{(2)}(\mathcal{F})$ consisting of all of its edges between S and W. By our choice of W, H is a complete bipartite graph. Since $|W| \geq n - o(n) > t + k$, for large n, we can find an x, y-path P of length $2t$ in H such that $P \cap F = \{x, y\}$. Since each edge on P has kernel degree at least $s = k\ell$ in $\mathcal{F}$, we can expand $F \cup P$ into a linear ℓ-cycle in $\mathcal{F}$, a contradiction. So $\mathcal{G}_2 = \emptyset$.

Next, consider the case $\ell = 2t+2$. Suppose that $\mathcal{F}$ has no linear ℓ-cycle and $\mathcal{G}_2$ contains two members F and F' that intersect in exactly one element u. Let x be a vertex in $(F \cap W) \setminus \{u\}$ and y a vertex in $(F' \cap W) \setminus \{u\}$. Like before, since H has all the edges between S and W and $|W|$ is large, we can find an x, y-path in H of length $2t$ such that $P \cap (F \cup F') = \{x, y\}$. We can expand $F \cup F' \cup P$ into a linear cycle of length $2t + 2 = \ell$ in $\mathcal{F}$, a contradiction. Suppose $\mathcal{F}$ contains no minimal ℓ-cycle instead and $\mathcal{G}_2$ contains two different edges F and F'. Then we can get a contradiction by constructing a minimal ℓ-cycle in $\mathcal{F}$ using a procedure similar to above. We omit the details. ■

For $\ell = 2t+1$, by Claim 2 we have $|\mathcal{G}_2| = 0$. For $\ell = 2t+2$, if $\mathcal{F}$ has no linear ℓ-cycle, then Claim 2 and Frankl's theorem (4) yield $|\mathcal{G}_2| \leq \binom{n-t-2}{k-2}$ (for large enough n) and if $\mathcal{F}$ has no minimal ℓ-cycle then $|\mathcal{G}_2| \leq 1$.

Finally, consider $\mathcal{D}$. Let $u \in Z$. The maximality of W implies that there exists an $x \in S$ such that $xu \notin \mathrm{Ker}_s^{(2)}(\mathcal{F})$ and hence $\deg^*_{\mathcal{F}}(\{x, u\}) < s$. Then (3) implies that the $\deg_{\mathcal{F}}(\{x, u\}) \leq s\binom{|W|-1}{k-3}$.

Hence $\deg_{\mathcal{D}}(\{x,u\}) \ge \binom{|W|}{k-2} - s\binom{|W|-1}{k-3} \ge \Omega(n^{k-2})$. Since this holds for every $u \in Z$ we get

$$|\mathcal{D}| \ge |Z| \times \left(\binom{|W|}{k-2} - s\binom{|W|-1}{k-3}\right) \ge \Omega(zn^{k-2}). \tag{21}$$

Now, we are ready to prove the desired bound on $|\mathcal{F}|$. By our definition, $\mathcal{F} \subseteq (\mathcal{F}_S \backslash \mathcal{D}) \cup G_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$. For $\ell = 2t+1$ we have $|\mathcal{G}_2| = 0$. So, for sufficiently large n, (18)–(21) yield

$$|\mathcal{F}| \le |\mathcal{F}_S| - \Omega(zn^{k-2}) + o(zn^{k-2}) \le |\mathcal{F}_S| - \Omega(zn^{k-2}) \le |\mathcal{F}_S| = \binom{n}{k} - \binom{n-t}{k}. \tag{22}$$

For $\ell = 2t+2$, if we assume that $k \ge 5$ and $\mathcal{F}$ has no linear ℓ-cycle, then $|\mathcal{G}_2| \le \binom{n-t-2}{k-2}$ and by (18)–(21) we have

$$|\mathcal{F}| \le |\mathcal{F}_S| - \Omega(zn^{k-2}) + o(zn^{k-2}) + \binom{n-t-2}{k-2} \le \binom{n}{k} - \binom{n-t}{k} + \binom{n-t-2}{k-2}, \tag{23}$$

for large n. If we assume that $k \ge 4$ and $\mathcal{F}$ has no minimal ℓ-cycle, then $|\mathcal{G}_2| \le 1$ and we have $|\mathcal{F}| \le |\mathcal{F}_S| + 1 = \binom{n}{k} - \binom{n-t}{k} + 1$. ■

11 Stability and concluding remarks

By (22) and (23), we also have the following stability statement.

Proposition 11.1 *Let k, ℓ be positive integers, where $\ell \ge 3$ and $k \ge 4$. Let ε be any small positive real. There exists a positive real δ such that for all $n \ge n_2(k,\ell)$ the follows holds. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a family that contains no copy of $\mathbb{C}_\ell^{(k)}$ if $k \ge 5$ and no member of $\mathcal{C}_\ell^{(k)}$ if $k = 4$ and $|\mathcal{F}| \ge (1-\delta)t\binom{n}{k-1}$. Then there exists a set $S \subseteq [n]$, where $|S| = t$, such that all except at most $\varepsilon\binom{n}{k}$ of the members of $\mathcal{F}$ intersect S.*

In Section 3 we observed that the 3-uniform linear cycle $\mathbb{C}_\ell^{(3)}$ is a subgraph of the triangulated cycle $\mathbb{T}_\ell^{(3)}$ and $\mathbb{C}_\ell^{(k)}$ is a k-expansion of $\mathbb{C}_\ell^{(3)}$. The triangulated cycle $\mathbb{T}_\ell^{(k)}$ is an example of a so-called q-forest where $q = 3$. A *q-forest* is a q-graph whose edges can be linearly ordered as $E_1, \dots, E_m$ such that for all $i \ge 2$ there exists some $a(i) < i$ such that $E_i \cap ((\bigcup_{j<i} E_j) \subseteq E_{a(i)}$. A subgraph of a q-forest is called a *partial q-forest*. So $\mathbb{C}_\ell^{(3)}$ is a partial 3-forest. In a forthcoming paper, for all k, q satisfying $q \ge 3$ and $k \ge 2q-1$, we will asymptotically determine the Turán numbers for the rather wide family of hypergraphs that are k-expansions of partial q-forests.

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