Tyre induced vibration of the car-trailer system

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Abstract

It is well known, that since tyres make contact between the vehicle and the road surface the behaviour of the tyres has a great impact on the dynamics of road vehicles, as tyre deformations strongly influence the manoeuvrability and the stability [1]*. Thus, choosing the tyre model is one of the most essential part of the development of mechanical models for vehicles. In our study, we analyse the stability of the car-trailer combination. Although this vehicle system was thoroughly analysed in former studies, the reanalysis of the bicycle model of the four-wheeled car showed* [2]*, that new unstable parameter domains can be found by using a time delayed tyre model. These results motivated us to focus on the accurate modelling of the tyre deformation in the contact patch and to implement the time delayed tyre model in a car-trailer system, which is capable to describe the regenerative vibrations of the tyres.*

1. Mechanical model

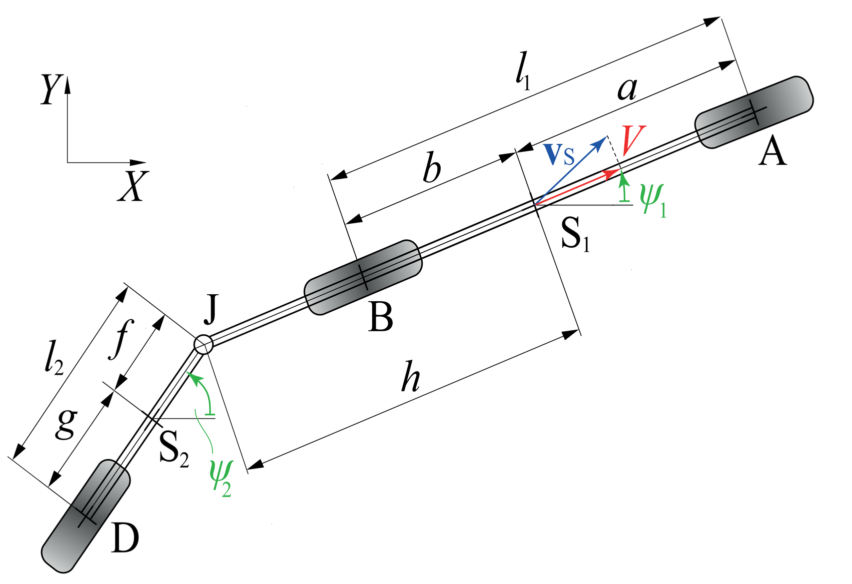

Figure 1: The single track model of the car-trailer combination

The car-trailer combination is represented by the so-called single track model (in plane bicycle model with a trailer, see Fig. 1). The vertical motion as well as the lateral extension of the vehicle is neglected, which is acceptable in the case of the analysis of the steady rectilinear motion.

The model has four degrees of freedom (DoF), namely, the motion of the vehicle can be described by four generalized coordinates: the X and Y coordinates of the centre of gravity S_1 of the car (X_1 and Y_1), the deflection angles ψ_1 and ψ_2 of the longitudinal axes of the car and the trailer, respectively. A kinematic constraint is applied to the system, i.e. the longitudinal velocity V of the car is kept constant in time:

$$\dot{X}_1\cos\psi_1 + \dot{Y}_1\sin\psi_1 = V, \qquad (1)$$

where dots refer to the derivatives with respect to time.

The equations of motion of the non-holonomic mechanical system can be determined by means of the Appell-Gibbs equation, which requires the introduction of the so-called pseudo velocities. A possible choice for the pseudo velocities is

$$\beta_1 = -\dot{X}_1\sin\psi_1 + \dot{Y}_1\cos\psi_1\,,\quad \beta_2 = \dot{\psi}_1\,,\quad \beta_3 = \dot{\psi}_2\,, \qquad (2)$$

where the β_1 is the lateral velocity of the car, while β_2 and β_3 are the angular velocities of the car and the trailer, respectively. Thus, the governing equations of the nonlinear system can be expressed as

$$\begin{aligned}
&m_1\left(\dot{\beta}_1 + V\beta_2\right) + m_2\left(\dot{\beta}_1 + V\beta_2 - h\dot{\beta}_2 - f\dot{\beta}_3\cos(\psi_2-\psi_1) + f\beta_3^2\sin(\psi_2-\psi_1)\right)\\
&\quad = F_1 + F_2 + F_3\cos(\psi_2-\psi_1)\,,\\
&\theta_{S_1}\dot{\beta}_2 + m_2\left(h^2\dot{\beta}_2 - hV\beta_2 - h\dot{\beta}_1 + hf\dot{\beta}_3\cos(\psi_2-\psi_1) - hf\beta_3^2\sin(\psi_2-\psi_1)\right)\\
&\quad = aF_1 - bF_2 - hF_3\cos(\psi_2-\psi_1) + M_1 + M_2\,,\\
&\theta_{S_2}\dot{\beta}_3 + m\left(f^2\dot{\beta}_3 - fV\beta_2\cos(\psi_2-\psi_1) - f\dot{\beta}_1\cos(\psi_2-\psi_1) - f\beta_1\beta_2\sin(\psi_2-\psi_1)\right.\\
&\quad \left.+hf\dot{\beta}_2\cos(\psi_2-\psi_1) + hf\beta_2^2\sin(\psi_2-\psi_1)\right) = M_3 - l_2F_3\,.
\end{aligned} \qquad (3)$$

where F_i and M_i, $i = 1,2,3$ are the forces and aligning torques originated in the tyre deformation. The parameters m_1 and m_2 are the masses, whereas θ_{S_1} and θ_{S_2} are the mass moments of inertia about the Z axis at the centres of gravity of the car and the trailer, respectively.

2. Tyre model with memory effect

In order to calculate the forces and aligning torques of the tyres, a time delayed tyre model (see Fig. 2) was used that is based on the so-called "brush-model". In the tyre-ground contact patch of length $2a_0$, the tyre particles are modelled as massless elements with distributed stiffness k and damping d. In case of rolling, the tyre particles in contact have zero velocity relative to the ground. Although the lateral tyre deformation $q(x,t)$ is governed by nonlinear partial differential equation, the travelling wave solution can be composed:

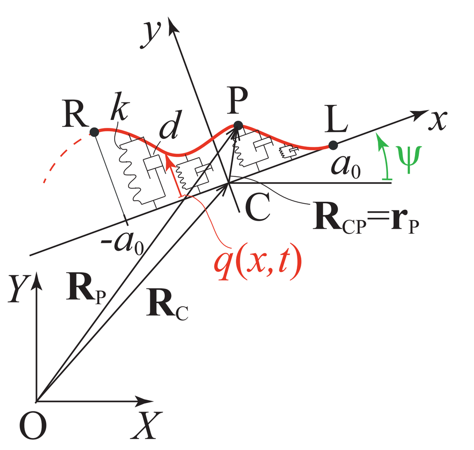

Figure 2: Tyre deformation in the contact patch

$$\begin{aligned}
q(x,t) &= \left(X_C(t) - X_C(t-\tau)\right)\sin\psi(t) - \left(Y_C(t) - Y_C(t-\tau)\right)\cos\psi(t)\\
&\quad - a_0\sin\left(\psi(t) - \psi(t-\tau)\right)\,,
\end{aligned} \qquad (4)$$

where the time delay τ for small oscillations around the rectilinear motion reads $\tau \approx (a_0 - x)/V$. Thus, the forces and aligning torques of the tyre can be calculated by integration of the deformation function along the contact patches.

3. Linear stability and self-excited vibrations

The linearised form of the equation of motion Eqn. (3) can be expressed as a system of delayed differential equations:

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{D}\dot{\mathbf{y}}(t) + \mathbf{S}\mathbf{y}(t) = k\int_0^{\frac{2a_0}{V}}\left(\mathbf{B}_0 - V\tau\mathbf{B}_1\right)\mathbf{y}(t-\tau)V\,\mathrm{d}\tau\,, \qquad (5)$$

where $\mathbf{y}(t)$ is the vector of the state coordinates of the linearised system: $\mathbf{y}(t) = [Y_1(t)\ \psi_1(t)\ \psi_2(t)]^T$. $\mathbf{M}$ is the mass matrix

$$\mathbf{M} = \begin{bmatrix} m_1+m_2 & -hm_2 & -fm_2\\ -hm_2 & \theta_{S_1}+m_2h^2 & m_2hf\\ -fm_2 & m_2hf & \theta_{S_2}+m_2f^2\end{bmatrix}. \qquad (6)$$

$\mathbf{S} = \mathbf{S}_k + \mathbf{S}_d$ is the stiffness matrix, which can be separated into a symmetrical and a non-symmetrical parts that are related to the tyre stiffness and damping, respectively:

$$\mathbf{S}_k = 2a_0k\begin{bmatrix} 3 & a-b-h & -l_2\\ a-b-h & a^2+b^2+h^2+\frac{2}{3}a_0^2 & l_2h\\ -l_2 & l_2h & \frac{1}{3}a_0^2+l_2^2\end{bmatrix},\quad \mathbf{S}_d = 2a_0dV\begin{bmatrix}0 & -2 & -1\\ 0 & -a+b & h\\ 0 & 0 & l_2\end{bmatrix}. \qquad (7)$$

The damping matrix $\mathbf{D}$ is proportional to the symmetrical part $\mathbf{S}_k$ of the stiffness matrix, i.e. $\mathbf{D} = (d/k)\mathbf{S}_k$. Finally, $\mathbf{B}_0$ and $\mathbf{B}_1$ are coefficient matrices containing vehicle and tyre geometry parameters:

$$\mathbf{B}_0 = \begin{bmatrix} 3 & a-b-h & a_0-l_2\\ 2a_0+a-b-h & a^2+b^2-h^2+2a_0(a_0+a-b) & -h(a_0-l_2)\\ a_0-l_2 & -h(a_0-l_2) & (a_0-l_2)^2\end{bmatrix}, \qquad (8)$$

$$\mathbf{B}_1 = \begin{bmatrix}0 & 0 & 0\\ 2 & 2a_0+a-b & 0\\ 1 & -h & a_0-l_2\end{bmatrix}. \qquad (9)$$

With the help of these, the characteristic equation can be composed:

$$\det\left(\lambda^2\mathbf{M} + \lambda\mathbf{D} + \mathbf{S} - k\frac{V}{\lambda}\left(\mathbf{B}_0\left(1-\mathrm{e}^{-\frac{2a_0}{V}\lambda}\right)\right) - \mathbf{B}_1\left(\frac{V}{\lambda}\left(1-\mathrm{e}^{-\frac{2a_0}{V}\lambda}\right) - 2a_0\mathrm{e}^{-\frac{2a_0}{V}\lambda}\right)\right) = 0\,. \qquad (10)$$

4. Results and conclusions

The stability chart was drawn (see Fig. 3) in the plane of the forward speed V and the trailer-load location parameter $p = f/l_2$ (i.e., the distance between the king pin and the centre of gravity of the trailer divided by the distance between the king pin and the trailer axle). In the figure, the shaded domains are linearly unstable, in which self-excited lateral vibrations of the vehicle can emerge. The different colours in the figure correspond to the three different vibration modes of the vehicle system.

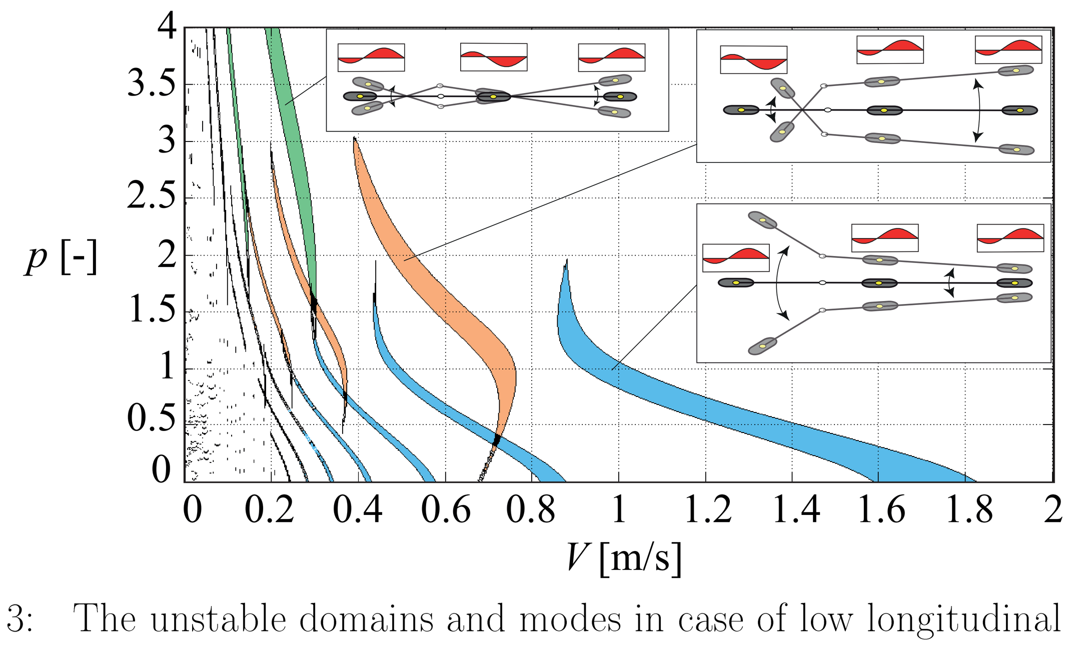

Figure 3: The unstable domains and modes in case of low longitudinal velocity

Consequently, new unstable parameter domains are captured in a simple car-trailer mechanical model. Although, these critical domains relate to low forward speeds of the vehicle, they can be relevant at certain vehicle manoeuvres (e.g. parking). The detected self-excited vibration can influence the vehicle dynamics and/or can lead to unwanted noise and tire wear.

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