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Matching Problems*

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Abstract

We give necessary and sufficient conditions for coverability of parallelepipeds by a given figure. Two types of figures are considered: 1. parallelepiped, 2. figure consisting of two relatively fixed, not necessarily connected, unit cubes, e.g. the field on the chess-board where the knight stands and a field attacked by the knight.

In 1962 *Matematikai Lapok* published the interesting problem [1] of N. G. de Bruijn: *An n-dimensional rectangular parallelepiped is to decompose to congruent rectangular parallelepipeds the edge lengths of which are the given natural numbers* $a_1, a_2, \ldots, a_n$. *Under what conditions can we say that such a decomposition exists if and only if there exists a decomposition with parallel parallelepipeds (i.e., the parallel edges of the parallelepipeds involved in the decomposition are equal)*?

The solution of the problem was given by G. Hajós and the authors [2].

(α) In solving the problem, the question arosed: What is the necessary and sufficient condition in general of the decomposibility of a parallelepiped to congruent parallelepipeds of given edge lengths?

* The results of the first part of this paper are contained in the author's work entered for the competition of the Hungarian Scientific Circle of Students in 1962.

(β) After answering this question we should like to give conditions for the decomposibility of a parallelepiped to congruent lattice figures of given type: a very simple case in which the lattice figure consists of two cubes. We solved this problem only in the two-dimensional case.

(γ) We considered another generalization of problem (α). In this case we allowed decomposition of the parallelepiped to parallelepipeds of several given types.

Definitions and Results

Let us consider the set of n-dimensional lattice points (i.e., the points with integer coordinates).

An n-dimensional *lattice figure* is an arbitrary subset of lattice points.

There exists a natural correspondence between the lattice points and lattice fields (unit cubes). Thus, sometimes we shall use the more illustrative expression "lattice field" instead of "lattice point."

We accept the usual concept of congruency, that is, we allow shift, rotation, and symmetry.

For the sake of simplicity we suppose (unless we emphasize the contrary) that the parallelepiped we want to decompose will be situated in the non-negative octant and one of its vertices is the origin (i.e., a parallelepiped B with edge lengths $b_1, b_2, \ldots, b_n$ consists of the lattice points $(x_1, x_2, \ldots, x_n)$ satisfying the conditions $0 \leqslant x_i < b_i$ $(1 \leqslant i \leqslant n)$).

Definition A. We say that a parallelepiped B can be filled up (covered) by the given lattice figures $A_1, A_2, \ldots, A_m$ if we can decompose B into disjoint subsets each of which is congruent to one of A_i's; and in this case we write

$$(A_1, A_2, \ldots, A_m) \mid B.$$

(If $m = 1$, we write simply $A_1 \mid B$.)

If we use the above natural definition of coverability, then the necessary and sufficient conditions are valid only if all the edges of parallelepiped B are large enough. However, in the case of the next definition we can omit this.

Definition B. We say that a parallelepiped B can be filled up (covered) in weak sense by the given lattice figures $A_1, A_2, \ldots, A_m$ if there exist the parallelepipeds $A_1(1),..., A_1(r_1), A_2(1),..., A_2(r_2),..., A_m(1),..., A_m(r_m)$ and the integers $\nu_{11},..., \nu_{1r_1}, \nu_{21},..., \nu_{2r_2},..., \nu_{m1},..., \nu_{mr_m}$ such that

$$\sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant r_i \\ x \in A_i(j)}} \nu_{ij} = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \notin B, \end{cases}$$

where $A_i(j)$ $(1 \leqslant j \leqslant r_i)$ is congruent to A_i $(1 \leqslant i \leqslant m)$. In this case we write

$$(A_1, A_2,..., A_m) \mid^* B.$$

The number ν_{ij} is called the multiplicity of $A_i(j)$.

It is easy to see that $(A_1, A_2,..., A_m) \mid B$ results in $(A_1, A_2,..., A_m) \mid^* B$ and we can choose the integers ν_{ij} so that $\nu_{ij} = 1$ $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant r_i)$.

DEFINITION C. We say that a parallelepiped B can be filled up (covered) in a parallel manner by a given parallelepiped A if we can decompose B into disjoint subsets each of which is congruent to A and the parallel edges are equal. In this case we write $A \mid^p B$.

DEFINITION D. If $a_1, a_2,..., a_n$ are given non-negative integers $(\sum_{i=1}^{n} a_i > 0)$, then the lattice points $(x_1, x_2,..., x_n)$ $(y_1, y_2,..., y_n)$ form a knight figure of type $a_1 \times a_2 \times \cdots \times a_n$ if $|x_1 - y_1|, |x_2 - y_2|,..., |x_n - y_n|$ is a permutation of the integers $a_1, a_2,..., a_n$. The knight figure of type $a_1 \times a_2 \times \cdots \times a_n$ will be denoted by $K(a_1, a_2,..., a_n)$.

In Part 1 we give a necessary and sufficient condition for the validity of $(A_1 A_2,..., A_m) \mid^* B$ (Theorem 2), and for the validity of $(A_1, A_2,..., A_m) \mid B$ and $A \mid B$ if B is large enough (Theorems 3 and 4, respectively). Some special cases are also explained because of the simpler form of conditions (Theorems 1, 5, 6, 7). In the course of the proofs we need a generalization (Lemma 7) of the well-known marriage principle, which may be interesting in itself.

In Part 2 we give a necessary and sufficient condition for the validity of $K(a, b) \mid B$ if B is large enough (Theorem 10). For the case $K(a, 1)$ a covering is constructed. Two simple n-dimensional generalizations (Theorems 11, 12) are also given.

1. COVERINGS WITH PARALLELEPIPEDS

The simplest but very interesting case is the case of the parallelepiped with edge lengths 1, 1,..., 1, a. The following theorem concerning this type is obviously a special case of the general Theorem 2, but the proof is very simple and interesting; we think it is worth writing [6]:

THEOREM 1. *Let A and B be n-dimensional parallelepiped and the edge lengths of A be* 1, 1,..., 1, a, *then* $A \mid B$ *if and only if at least one edge of B is divisible by* a.

Proof. The sufficiency of the divisibility condition is trivial; we have to prove only the necessity.

Denote by $S(k)$ the number of points in B for which

$$x_1 + x_2 + \cdots + x_n \equiv k \pmod{a} \quad (0 \leqslant k < a),$$

where $x_1, x_2, \ldots, x_n$ are the coordinates of the point. Consider now the points of a covering parallelepiped $\tilde{A}$ congruent to A. There is an $i\,(1 \leqslant i \leqslant n)$ for which the i-th coordinates are a consecutive integers; the other coordinates are the same for all points. Thus, the sums $x_1 + x_2 + \cdots + x_n$ are incongruent (mod a) for points of $\tilde{A}$. Obviously, if $A \mid B$, then the set of points of B is a union of $\tilde{A}$'s; thus $S(0) = S(1) = \cdots = S(a-1)$.

We will now prove that $S(0) = \cdots = S(a-1)$ if and only if at least one side of B is divisible by a. The following generator function is used:

$$F(x) = (1 + x + \cdots + x^{b_1 - 1}) \cdots (1 + x + \cdots + x^{b_n - 1}). \tag{1}$$

Denote now by $S'(k)$ the number of points in B for which

$$x_1 + x_2 + \cdots + x_n = k.$$

Obviously

$$F(x) = \sum_{k=0}^{\infty} S'(k)\, x^k.$$

Substitute the complex number $\epsilon_a = \cos(2\pi/a) + i \sin(2\pi/a)$:

$$F(\epsilon_a) = \sum_{k=0}^{\infty} S'(k)\, \epsilon_a{}^k = \sum_{k=0}^{a-1} \epsilon_a{}^k \Big[\sum_{l \equiv k \pmod{a}} S'(l) \Big].$$

However, $\sum_{l \equiv k \pmod{a}} S'(l) = S(k)$; thus

$$F(\epsilon_a) = \sum_{k=0}^{a-1} S(k)\, \epsilon_a{}^k,$$

or using $S(0) = S(1) = \cdots = S(a-1)$ we obtain

$$F(\epsilon_a) = S(0) \sum_{K=0}^{a-1} \epsilon_a{}^k = 0.$$

(In the case of $a = 1$ it is not true, but in this case the theorem is trivial.) Therefore at least one factor of (1) vanishes for ϵ_a, say the i-th, which results from

$$\sum_{k=0}^{b_i-1} \epsilon_a{}^k = \frac{\epsilon_a^{b_i} - 1}{\epsilon_a - 1} = 0,$$

the desired relation $a \mid b_i$.

Passing over to the general case, first we prove a recursive property. Let us introduce the following definition:

DEFINITION 1. Let A be an n-dimensional parallelepiped and d a natural number. We denote by $R(A, d)$ the set of those $n - 1$-dimensional parallelepipeds obtained from A by omitting any edge of A such that it is not divisible by d. If all the edges of A are divisible by d, then $R(A, d) = \varnothing$.

If $A_1, \ldots, A_n$ are n-dimensional parallelepipeds and d is a natural number, then

$$R(A_1, \ldots, A_m, d) = \bigcup_{i=1}^{m} R(A_i, d).$$

After this definition we can formulate the recursive property:

LEMMA 1. *If* $(A_1, \ldots, A_m) \mid^* B$ *and*

$$d \nmid b_n$$

then

$$R(A_1, \ldots, A_m, d) \mid^* B',$$

where B' *is an* $n - 1$ *dimensional parallelepiped with edges* $b_1, b_2, \ldots, b_{n-1}$.

Heuristic Proof. Since $d \nmid b_n$, the number of layers whose n-th coordinate $\equiv 0 \pmod m$ is greater by 1 than the number of layers whose n-th coordinate $\equiv d - 1 \pmod d$. Let us now consider a fixed filling up of B, and project the layers congruent with 0 or $d - 1$ onto B. Taking with sign $+$ the former and with sign $-$ the latter, we obtain a filling up of B'. However, we do not need the $A_i(j)$'s, whose edge lying in the n-th dimension is divisible by d, because such $A_i(j)$'s contain the same number of layers taken with $+$ and $-$. In this way, we obtained a filling up of B' with $R(A_1, \ldots, A_m, d)$.

Formal Proof. If $(A_1, \ldots, A_m) \mid^* B$ by Definition B, we know that there exist the parallelepipeds $A_1(1), \ldots, A_1(r_1)$, $A_2(1), \ldots, A_2(r_2)$, $A_m(1), \ldots, A_m(r_m)$ and the integers $\nu_{11}, \ldots, \nu_{1r_1}, \nu_{21}, \ldots, \nu_{2r_2}, \ldots, \nu_{m1}, \ldots, \nu_{mr_m}$ such that

$$\sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant r_i \\ x \in A_i(j)}} \nu_{ij} = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \notin B. \end{cases} \tag{2}$$

Denote by prime the projection on the first $n - 1$ dimension, that is, if $x \in E^n$ then $x' \in E^{n-1}$ and the coordinates of x' are equal to the first $n - 1$ ones of x, and C' is the set of the points x' for all $x \in C$. Let us consider a fixed point $y \in B'$ and the difference

$$\sum_{x: \begin{cases} x' = y \\ 0 \leqslant x_n \leqslant b_n - 1 \\ x_n \equiv 0 \pmod d \end{cases}} \sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant r_i \\ x \in A_i(j)}} \nu_{ij} - \sum_{x: \begin{cases} x' = y \\ 0 \leqslant x_n \leqslant b_n - 1 \\ x_n \equiv d-1 \pmod d \end{cases}} \sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant r_i \\ x \in A_i(j)}} \nu_{ij}. \tag{3}$$

Using (2) we trivially obtain that (3) gives

$$\sum_{x: \begin{cases} x' = y \\ 0 \leqslant x_n \leqslant b_n - 1 \\ x_n \equiv 0 \pmod d \end{cases}} 1 - \sum_{x: \begin{cases} x' = y \\ 0 \leqslant x_n \leqslant b_n - 1 \\ x_n \equiv d-1 \pmod d \end{cases}} 1 = \sum_{x_n: \begin{cases} 0 \leqslant x_n \leqslant b_n - 1 \\ x_n \equiv 0 \pmod d \end{cases}} 1 - \sum_{x_n: \begin{cases} 0 \leqslant x_n \leqslant b_n - 1 \\ x_n \equiv d-1 \pmod d \end{cases}} 1,$$

which equals 1 by assumption $d \nmid b_n$. If $y \notin B'$, then for every x satisfying $x' = y$ the relation $x \notin B$ holds; thus using (2) the sum (3) gives 0. In this way, we have seen that (3) gives 1 for $y \in B'$ and 0 for $y \notin B'$. However, we may write (3) in the following manner:

$$\sum_{i,j: \begin{cases} 1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant r_i \\ y \in A_i'(j) \end{cases}} \left(\sum_{x: \begin{cases} x \in A_i(j) \\ x' = y \\ 0 \leqslant x_n \leqslant b_n - 1 \\ x_n \equiv 0 \pmod d \end{cases}} \nu_{ij} - \sum_{x: \begin{cases} x \in A_i(j) \\ x' = y \\ 0 \leqslant x_n \leqslant b_n - 1 \\ x_n \equiv d-1 \pmod d \end{cases}} \nu_{ij} \right) \tag{4}$$

That means the parallelepipeds $A_i'(j)$ fill up B' with multiplicity (which is independent of y, if $y \in A_i'(j)$):

$$\sum_{x: \begin{cases} x \in A_i(j) \\ x' = y \\ 0 \leqslant x_n \leqslant b_n - 1 \\ x_n \equiv 0 \pmod d \end{cases}} \nu_{ij} - \sum_{x: \begin{cases} x \in A_i(j) \\ x' = y \\ 0 \leqslant x_n \leqslant b_n - 1 \\ x_n \equiv d-1 \pmod d \end{cases}} \nu_{ij}$$

$$= \nu_{ij} \left(\sum_{x: \begin{cases} x \in A_i(j) \\ x' = y \\ 0 \leqslant x_n \leqslant b_n - 1 \\ x_n \equiv 0 \pmod d \end{cases}} 1 - \sum_{x: \begin{cases} x \in A_i(j) \\ x' = y \\ 0 \leqslant x_n \leqslant b_n - 1 \\ x_n \equiv d-1 \pmod d \end{cases}} 1 \right).$$

If the n-th edge of $A_i(j)$ is divisible by d, then the latter formula vanishes, i.e., its multiplicity is 0. Hence it follows that we can replace $y \in A_i'(j)$ in (4) by $y \in A_i'(j) \in R(A_1, ..., A_m, d)$, which means $R(A_1, ..., A_m, d) |^* B'$, indeed.

From the above proof it is clear that, if d is a divisor of all edges of $A_1, A_2, ..., A_m$, then $R(A_1, ..., A_m, d) |^* B'$ is impossible, since $R(A_1, A_2, ..., A_m, d) = \varnothing$. In other words, in this case, $d \mid b_n$ must hold and in the same way $d \mid b_j$ $(1 \leqslant j \leqslant n)$. Thus, we have the following lemma:

LEMMA 2. *If* $(A_1, A_2, ..., A_m) |^* B$ *and all the edges of* A_i*'s* $(1 \leqslant i \leqslant m)$ *are divisible by* d, *then*

$$d \mid b_j \qquad (1 \leqslant i \leqslant n).$$

DEFINITION 2. If $e_1, e_2, ..., e_n$ are natural numbers and B an n-dimensional parallelepiped, then $M(B, e_1, ..., e_n)$ denotes the "divisibility matrix": the j-th element of the i-th row is 1 if $e_i \mid b_j$ (where b_j is the j-th side of B, and 0 if $e_i \nmid b_j$.

DEFINITION 3. We say that a $n \times n$ matrix M has no independent 0's (or 1's) if there are no n 0's (or 1's) in different rows and columns.

We can now formulate the following theorems:

THEOREM 2. $(A_1, A_2, ..., A_m) |^* B$ *holds if and only if*

(F_1) *choosing in arbitrary manner* k_i $(\geqslant 1)$ *edges of* A_i, *denoting by* d_i *their greatest common divisor, and making* n *sets of the numbers* d_i *in an arbitrary manner but using every* d_i *exactly in* $n - k_i + 1$ *sets, finally, denoting by* $e_1, ..., e_n$ *the greatest common divisor of the numbers in one set* ($e_j = \infty$ *if the j-th set is void*), *the matrix* $M(B, e_1, e_2, ..., e_n)$ *has no* n *independent* 0*'s.*

THEOREM 3. *In the case of*

$$b_i \geqslant 3^{nm \cdot 2^{nm}} \cdot a^{2^{nm}+2} \tag{5}$$

(*where* a *is the maximum of edges of* A_j*'s*) $(A_1, ..., A_m) \mid B$ *holds if and only if* (F_1) *holds.*

The proofs of the two theorems will be given together. More exactly, we will prove the necessity of the condition of Theorem 2; obviously, the same condition must be necessary in the case of "|" instead of "|*".

On the other hand we will prove that condition (F_1) is sufficient in Theorem 3; from the proof it will be clear that in the case of "$|*$" the condition is sufficient even without (5).

Proof of Necessity. Let $e_{n-l+1}, e_{n-l+2}, \ldots, e_n$ be the finite numbers among $e_1, e_2, \ldots, e_n$ ($e_i = \infty$, $1 \leqslant i \leqslant n - l$; $e_i < \infty$, $n - l < i \leqslant n$), where $e_1, e_2, \ldots, e_n$ are some greatest common divisors defined in the theorems. Suppose that, in contradiction with our assumption, $M(B, e_1, \ldots, e_n)$ has n independent 0's, and that the edges of B are indexed in such a way that the 0's are in the main diagonal. That is,

$$e_{n-l+1} \nmid b_{n-l+1}, \ldots, e_n \nmid b_n . \tag{6}$$

Using now $l - 1$ times Lemma 1 we obtain

$$R(R(R \cdots R(A_1, A_2, \ldots, A_m, e_n), e_{n-1}) \cdots), e_{n-l+3}), e_{n-l+2}) \,|^*\, B^{(l-1)}, \tag{7}$$

where $B^{(l-1)}$ denotes the $n - l + 1$ dimensional parallelepiped with edges $b_1, b_2, \ldots, b_{n-l}, b_{n-l+1}$. Here, obviously

$$\begin{aligned} &R(R(R \cdots R(A_1, \ldots, A_m, e_n), \ldots), e_{n-l+3}), e_{n-l+2}) \\ &\quad = \bigcup_{i=1}^{m} R(R \cdots R(A_i, e_n) \cdots)\, e_{n-l+2}). \end{aligned} \tag{8}$$

On the right-hand side all the $R(R \cdots R(A_i, e_n) \cdots)\, e_{n-l+2})$'s cannot be void. In the contrary case, there would be a maximal r ($n - l + 1 \leqslant r < n$) for which

$$R(R(R \cdots R(A_1, \ldots, A_m, e_n), e_{n-1}), \ldots, e_{r+1}) \neq \varnothing$$

and

$$R(R(R \cdots R(A_1, \ldots, A_m, e_n), e_{n-1}), \ldots, e_r) = \varnothing \tag{9}$$

should hold. However, (9) means that all the edges of all parallelepipeds in $R(R(R \cdots R(A_1, \ldots, A_m)\, e_n), \ldots, e_{r+1})$ are divisible by e_r. Taking into account that

$$R(R(R \cdots R(A_1, A_2, \ldots, A_m, e_n), e_{n-1}), \ldots, e_{r+1}) \,|^*\, B^{(n-r)}$$

and using Lemma 2, we obtain that e_r is a divisor of all the edges of $B^{(n-r)}$, that is, $e_r \mid b_r$, which contradicts (6).

By Definition 1, $R(R \cdots R(A_i, e_n), \ldots,)\, e_{n-l+2})$ consists of the parallelepipeds obtained by omitting from A_i one edge non-divisible by e_n, one edge non-divisible by $e_{n-1}, \ldots,$ and one side non-divisible by e_{n-l+2}.

If $R(R \cdots R(A_i, e_n),..., e_{n-l+2})$ is non-void, then all the remaining edges are divisible by e_{n-l+1}, because in the contrary case A_i would have

$$\begin{array}{ll} & \text{one edge non-divisible by } e_n \\ (10) & \quad\vdots \qquad\qquad\qquad\qquad \vdots \\ & \text{one edge non-divisible by } e_{n-l+2}, \\ & \text{one edge non-divisible by } e_{n-l+1}, \\ & \text{one edge non-divisible by } e_{n-l} = \infty, \\ & \quad\vdots \\ & \text{one edge non-divisible by } e_1 = \infty. \end{array}$$

However, we have k_i ($\geqslant 1$) edges of A_i with greatest common divisor d_i and $n - k_i + 1$ of e_j's are divisors of d_i. Thus, there are $n - k_i + 1$ e_j's which are divisors of at least k_i edges of A_i; but this contradicts (10) by the pigeon hole principle. The proof is finished.

LEMMAS TO THE PROOF OF SUFFICIENCY. In the proof we shall use a number of lemmas:

LEMMA 3. *If c_1, c_2, c_3 are arbitrary natural numbers, then*

$$([c_1, c_3], c_2) \mid [c_3, (c_1, c_2)],$$

where (x, y) denotes the greatest common divisor and $[x, y]$ denotes the least common multiple of x and y.

Proof. If p is a prime number and $p^i \mid ([c_1, c_3], c_2)$, then $p^i \mid c_2$, and either $p^i \mid c_1$ or $p^i \mid c_3$ holds.

In the first case, $p^i \mid (c_1, c_2)$ and, in the second case, $p^i \mid c_3$, holds, that is, $p^i \mid [c_3, (c_1, c_2)]$, which proves the lemma.

LEMMA 4. *If the natural number b is divisible by c_3 and (c_1, c_2), further, if*

$$b \geqslant 3[c_1, c_3]\, c_2, \tag{11}$$

then there are two natural numbers $b(1)$ and $b(2)$ such that

$$b = b(1) + b(2), \tag{12}$$

$$c_3, c_1 \mid b(1), \qquad c_3, c_2 \mid b(2), \tag{13}$$

$$b(1) \geqslant \frac{b}{3}, \qquad b(2) \geqslant \frac{b}{3}. \tag{14}$$

Proof. Let us consider the Diophantine equation

$$[c_1, c_3]x + c_2 y = b. \tag{15}$$

It is solvable because by Lemma 3 $([c_1, c_3], c_2) \mid [c_3, (c_1, c_2)]$ and by the condition of the lemma $[c_3, (c_1, c_2)] \mid b$; thus $([c_1, c_3], c_2) \mid b$.

For arbitrary solution x_0, y_0 of (15) the numbers $b(1) = [c_1, c_3]\, x_0$, $b(2) = c_2 y_0$ satisfy the conditions (12) and (13). We have to search for only such a solution of (15) for which (14) also holds. If x_0, y_0 is a solution, $x_0 + tc_2, y_0 - t[c_1, c_3]$ is also a solution (t integer). Thus, if the above $b(1)$ and $b(2)$ ($[c_1, c_3]\, x_0$ and $c_2 y_0$) satisfy (12) and (13), the pair $[c_1, c_3]\, x_0 + t[c_1, c_3]\, c_2$, $c_2 y_0 - t[c_1, c_3]\, c_2$ also satisfies them.

Hence it is clear that, if $b/3 \geqslant [c_1, c_3]\, c_2$, then we have a $b(1) = [c_1, c_3]\, x_0 + t[c_1, c_3]\, c_2$ lying in the interval $[b/3, 2b/3]$ and from (12) it follows that $b(2) = c_2 y_0 - t[c_1, c_3]\, c_2 \geqslant b/3$, too.

LEMMA 5. *If b is a natural number divisible by $(c_{11}, c_{12}, \ldots, c_{1s_1}), \ldots, (c_{u1}, c_{u2}, \ldots, c_{us_u})$ and*

$$b \geqslant 3c^{u+2} \tag{16}$$

(where $c = \max_{1 \leqslant i \leqslant u; 1 \leqslant j \leqslant s_i} c_{ij}$), then we can divide b into two parts

$$b = b(1) + b(2)$$

satisfying the conditions

$$(c_{21}, \ldots, c_{2s_2}), (c_{31}, \ldots, c_{3s_3}), \ldots, (c_{u1}, \ldots, c_{us_u}) \mid b(1), b(2),$$

$$c_{11} \mid b(1), \qquad (c_{12}, \ldots, c_{1s_1}) \mid b(2),$$

$$b(1) \geqslant b/3, \qquad b(2) \geqslant b/3.$$

Proof. We use Lemma 4 with c_{11}, $(c_{12}, \ldots, c_{1s_1})$ and

$$[(c_{21}, \ldots, c_{2s_2}), (c_{31}, \ldots, c_{3s_3}), \ldots, (c_{u1}, \ldots, c_{us_u})]$$

instead of c_1, c_2, and c_3, taking into account that $(c_{11}, (c_{12}, c_{13}, \ldots, c_{1s_1})) = (c_{11}, \ldots, c_{1s_1})$. The only problem is whether (16) ensures (11) or does not. But this follows from the inequality

$$\begin{aligned} 3[c_1, c_3]\, c_2 &\leqslant 3c_1c_2c_3 \\ &= 3c_{11}(c_{12}, \ldots, c_{1s_1}) \cdot [(c_{21}, \ldots, c_{2s_2}), \ldots, (c_{u1}, \ldots, c_{us_u})] \\ &\leqslant 3c \cdot c \cdot c^{u-1} \\ &= 3c^{u+1}. \end{aligned}$$

LEMMA 6. *If b is a natural number, divisible by $(c_{11},..., c_{1s_1}),...,$ $(c_{u1},..., c_{us_u})$ and*

$$b \geqslant 3^{s_1+\cdots+s_u} \cdot c^{u+1},$$

then we can divide b into $s_1 s_2 \cdots s_u$ non-negative parts

$$b = \sum_{i=1}^{s_1 s_2 \cdots s_u} b(i)$$

in such a way that for every i $(1 \leqslant i \leqslant s_1 \cdots s_u)$ there are (depending on i) $i_1, i_2,..., i_u$ $(1 \leqslant i_1 \leqslant s_1,..., 1 \leqslant i_u \leqslant s_u)$ such that

$$c_{1i_1} \mid b(i),\ c_{2i_2} \mid b(i),...,\ c_{us_u} \mid b(i).$$

(That is, every part is divisible by at least one from every group of c_{ij}'s.)

Proof. The lemma is a simple consequence of Lemma 5.

The proof will be given by induction over $v = \sum_{i=1}^{u} s_i$. $\sum_{i=1}^{u} s_i = 1$ can hold only if $u = 1$, $s_1 = 1$. For this case the lemma is trivial.

Suppose now that the lemma is proved for all numbers less than v and prove for $v = \sum_{i=1}^{u} s_i$. If $s_1 = \cdots = s_u = 1$, then the statement is trivial; thus we may assume that some s_i, e.g., s_u, >1.

We can use Lemma 5; there are $b'(1)$ and $b'(2)$ such that

$$b = b'(1) + b'(2), \tag{17}$$

$$(c_{11},..., c_{1s_1}),..., (c_{u-1,1}, c_{u-1,2},..., c_{u-1,s_{u-1}}) \mid b'(1), b'(2),$$

$$c_{us_u} \mid b'(1), \qquad (c_{u,1},..., c_{u,s_u-1}) \mid b'(2),$$

and

$$b'(1), b'(2) \geqslant b/3 \geqslant 3^{s_1+\cdots+s_u-1} \cdot c^{u+1}.$$

For $b'(2)$ we may use our inductive hypothesis with $(c_{11},..., c_{1s_1}),...,$ $(c_{u1},..., c_{u,s_u-1})$: There is a partition

$$b'(2) = \sum_{i=1}^{s_1 \cdots (s_u-1)} b(i) \tag{18}$$

such that for all $b(i)$ $(1 \leqslant i \leqslant s_1 \cdots (s_n - 1))$ there are $i_1, i_2,..., i_u$ $(1 \leqslant i_1 \leqslant s_1,..., 1 \leqslant i_u \leqslant s_u - 1)$ satisfying

$$c_{1i_1}, c_{2i_2},..., c_{u\tilde{i}_u} \mid b(i). \tag{19}$$

On the other hand, for $b'(1)$ we may also apply our induction hypothesis with $(c_{11},..., c_{1s_1}),..., (c_{u-1,1},..., c_{u-1,s_{u-1}}), c_{u,s_u}$: There is a partition

$$b'(1) = \sum_{i=s_1\cdots(s_u-1)+1}^{s_1\cdots s_u} b(i) \tag{20}$$

such that for all $b(i)$ $(s_1 \cdots (s_u - 1) < i \leqslant s_1 s_2 \cdots s_u)$ we have the indices $j_1, j_2,..., j_u = s_u$ satisfying

$$c_{1j_1}, c_{2j_2},..., c_{u-1,j_{u-1}}, c_{u,s_u} \mid b(i). \tag{21}$$

Thus (17), (18), and (20) give a partition of b whose members satisfy the desired conditions by (19) and (21); the lemma is proved.

DEFINITION 4. Let ϵ_1 and ϵ_2 be equal to 0 or 1. We call the logical sum of ϵ_1 and ϵ_2 the number

$$\epsilon_1 \vee \epsilon_2 = \begin{cases} 0, & \textit{if} \quad \epsilon_1 = \epsilon_2 = 0, \\ 1, & \textit{otherwise}. \end{cases}$$

Similarly, if t_1 and t_2 are row vectors with 0, 1 coordinates then the coordinates of the logical sum of t_1 and t_2 are the logical sums of the corresponding coordinates.

LEMMA 7. *Let $M_1, M_2,..., M_m$ be $n \times m$ matrices with elements* 0 *and* 1. *If they have the property that*

(F_2) *choosing in arbitrary manner $k_i \geqslant 1$ rows from M_i $(1 \leqslant i \leqslant m)$, denoting by w_i the logical sum of these rows, and making n sets of the row vectors w_i in arbitrary manner but using every w_i exactly in $n - k_i + 1$ sets, finally denoting by z_j $(1 \leqslant j \leqslant n)$ the logical sum of the w_i's lying in the j-th set (if the j-th set is void, then $w_i =$ (0, 0,..., 0)) the matrix formed from $z_1, z_2,..., z_n$ as rows has no n independent* 0*'s,*

then there is an index p $(1 \leqslant p \leqslant m)$ such that M_p has n independent 1*'s.*

Remark. This lemma is a generalization of the well-known marriage principle [3], which states:

MARRIAGE PRINCIPLE. *Let M be an $n \times n$ matrix with elements* 0 *and* 1. *If choosing in arbitrary manner k $(1 \leqslant k \leqslant n)$ rows, the number of columns containing* 1 *in these rows (or the numbers of* 1*'s of the logical sum of these rows) $\geqslant k$, then M has n independent* 1*'s.*

Proof of Lemma 7. We prove the lemma in an indirect way. Suppose that none of the M_i's has n independent 1's. Thus, by the marriage principle, in every M_i ($1 \leqslant i \leqslant m$) there are k_i rows with their logical sum t_i having less than k_i 1's. That is, t_i has at least $n - k_i + 1$ 0's. Form n sets from t_i's in the following way: Let the j-th set contain all the t_i's which have 0 in the j-th place, taking into account only the first $n - k_i + 1$ 0's in every t_i. Thus, every t_i is contained by exactly $n - k_i + 1$ sets, and the logical sum t_i of the t_i's contained by the i-th set has 0 in the i-th place. The matrix formed from $z_1, z_2, \ldots, z_n$ as rows has 0's in the main diagonal. This is a contradiction with property (F_2); the lemma is proved.

Proof of Sufficiency of Theorem 3. Let the condition (F_1) be fulfilled. Consider a fixed edge b_i ($1 \leqslant i \leqslant n$) of B. Choose an arbitrary set of edges of the A_i's, and form the greatest common divisor of its elements. Let $f_{i1}, \ldots, f_{iq_i}$ be the sequence of the possible greatest common divisors which divide b_i. Remember that $f_{ij} = e_k$ is the greatest common divisor of certain d_l's and d_l's are the greatest common divisors of certain edges of A_i's, that is, f_{ij} is the greatest common divisor of certain edges of A_i's (using $((a, b), c) = (a, b, c)$). We can use Lemma 6, since

$$f_{i1} \mid b_i, f_{i2} \mid b_i, \ldots, f_{iq_i} \mid b_i,$$

and the inequality condition of Lemma 6 also holds:

$$b_i \geqslant 3^{nm \cdot 2^{nm}} a^{2^{nm}+1} \geqslant 3^{\sum_{j=1}^{q_i} s_j} a^{q_i+1}.$$

(The first inequality is a condition of Theorem 3; the second inequality is a consequence of the fact that $f_{ij} = e_k$ is a greatest common divisor of at most nm edges of A_i's ($s_j \leqslant nm$) and that we can form at most 2^{nm} such greatest common divisors from nm elements ($q_i \leqslant 2^{nm}$).)

Thus Lemma 6 gives that we can divide b_i into parts

$$b_i = \sum b_i(l) \tag{22}$$

so that

(23) every $b_i(l)$ is divisible by at least one A-edge from every f_{ij} $(1 \leqslant j \leqslant q_i)$.

Now we divide B into parts on the basis of the partition (22) of its edges and we prove that it is possible to fill up these parts of B with

$A_1, A_2, \ldots, A_m$. Denote by $\bar{B}$ a part of B with edges $b_1(l_1), b_2(l_2), \ldots, b_n(l_n)$. If we prove for arbitrary $\bar{B}$

(24) $$(A_1, A_2, \ldots, A_m) \mid \bar{B},$$

then Theorem 3 is already proved too.

It is easy to see that for the matrices

$$M(\bar{B}, a_{11}, \ldots, a_{1n}), \ldots, M(\bar{B}, a_{m1}, \ldots, a_{mn})$$

the condition (F_2) is a consequence of (F_1) and (23), because, if $e_i \mid b_j$, then $b_j(l_j)$ is divisible by at least one of a's from e_i; thus z_i (see (F_2)) has 1 in the j-th place. The reason is that z_i is constructed similarly to e_i. That shows, if in a place in the matrix $M(B, e_1, \ldots, e_n)$ stays a 1, then in the matrix formed from the rows $z_1, z_2, \ldots, z_n$ also stays a 1. Hence it follows that the latter matrix cannot contain n independent 0's because $M(B, e_1, \ldots, e_n)$ also has none.

Now we use Lemma 7 for the matrices

$$M(\bar{B}, a_{11}, \ldots, a_{1n}), \ldots, M(\bar{B}, a_{m1}, \ldots, a_{mn}).$$

There is an i such that $M(\bar{B}, a_{i1}, \ldots, a_{in})$ has n independent 1's, that is, we can order to every edge of $\bar{B}$ an a_{ij} $(1 \leqslant j \leqslant n)$ to different edges different a_{ij} which is divisor of it. Moreover in this case trivially (Definition C)

$$A_i \mid^p \bar{B};$$

(24) and the theorems are proved.

Now we should like to consider some interesting special cases:

THEOREM 4. *In the case of*

$$b_i \geqslant 3^{n, 2^n} \cdot a^{2^n+1}$$

(*where a is the maximum of edges of A*)

$$A \mid B$$

holds if and only if

(F_3) *choosing $k (1 \leqslant k \leqslant n)$ edges of A in arbitrary manner their greatest common divisor d is a divisor of at least k edges of B.*

Proof. We have to verify only that in this special case (F_1) leads to (F_3).

In our case (F_1) has the following form: choose in arbitrary manner k edges $(1 \leqslant k \leqslant n)$ of A, denote by d their greatest common divisor, form n sets with only element d, in such a way that let d be an element of $n - k + 1$ sets. Thus in our case $n - k + 1$ sets have one element and $k - 1$ sets will be void.

The greatest common divisors of the elements of one set are the following:

$$e_1 = e_2 = \cdots = e_{n-k+1} = d, \qquad e_{n-k+2} = \cdots = e_n = \infty.$$

That is, the matrix $M(B, e_1, ..., e_n)$ has $n - k + 1$ identical rows, and $k - 1$ 0 rows. A matrix of this type has no n independent 0's if and only if the number of 0's in the first row is less or equal to $n - k$. However, this means that d is a divisor at least k edges of B. The latter condition is just (F_3).

The next theorem gives the answer for the problem of de Bruijn [1]. It has a simpler proof [2] but here it is an easy consequence of our former results.

THEOREM 5. *A has the property "$A \mid B$ if and only if $A \mid^p B$" if and only if from any two edges of A one of them is the divisor of the other.*

Proof. Suppose the edges of A are indexed monotonically. If A has the above property then $a_1 \mid a_2 \mid \cdots \mid a_n$. We can use Theorem 4; the greatest common divisor of any k edges of A is the least one of them. Choosing the k largest edges we obtain by (F_3) that at least k edges of B are divisible by a_{n-k+1}. For $k = 1, 2, ..., n$, we obtain that there are 1 edge (say b_{i_1}) of B divisible by a_n, 2 edges (one is different from b_{i_1}, say b_{i_2}) of B divisible by a_{n-1}, 3 edges (one is different from b_{i_1}, b_{i_2}, say b_{i_3}) of B divisible by $a_{n-2}, ...,$ and n edges (one is different from $b_{i_1}, ..., b_{i_{n-1}}$, that is b_{i_n}) of B divisible by a_1. Here $i_1, i_2, ..., i_n$ is a permutation of $1, 2, ..., n$; thus $a_1 \mid b_{i_n}, ..., a_n \mid b_{i_1}$ means $A \mid^p B$. The first part of the theorem is proved.

If A does not have the property $a_1 \mid a_2 \mid \cdots \mid a_n$, then there are two edges a_i and a_j $(i < j)$ for which $a_i \nmid a_j$ holds. Now we construct a B for which $A \mid B$ does, but $A \mid^p B$ does not, hold. Let p be a prime number greater than $3^{n2^n} \cdot a_n^{2^n+1}$ and let

$$\begin{aligned} &b_1 = a_1 p, ..., b_{i-1} = a_{i-1} p, \qquad && b_i = (a_i, a_j)p, \\ (25) \qquad &b_{i+1} = a_{i+1} p, ..., b_{j-1} = a_{j-1} p, && b_j = [a_i, a_j]p, \\ &b_{j+1} = a_{j+1} p, ..., b_n = a_n p \end{aligned}$$

be the edges of B. We first verify $A \mid B$ by Theorem 4. If we choose k

edges $a_{i_1},\ldots, a_{i_k}$ and there is neither a_i nor a_j among them, then for $d = (a_{i_1},\ldots, a_{i_k})$

$$d \mid b_{i_1},\ldots, d \mid b_{i_k}$$

trivially hold. If there is only one of a_i and a_j among them, e.g., $i_1 = i$ or j, then $d \mid b_{i_2},\ldots, d \mid b_{i_k}, d \mid b_j$ ensures the condition (F_3). Similarly, if $i_1 = i$, $i_2 = j$, then $d \mid b_{i_3},\ldots, d \mid b_{i_k}, d \mid b_i, d \mid b_j$ hold.

On the other hand, $A \mid^p B$ does not hold. Here $(a_i, a_j) < a_i$. Let m be the largest index such that $a_m < a_i$. If we want to do the one-to-one ordering between the edges of A and B, we cannot order $b_1, b_2,\ldots, b_m$ and b_i to $a_{m+1}, a_{m+2},\ldots, a_n$ because the divisors of $b_1, b_2,\ldots, b_m$ and b_i are less than a_i or greater than a_n, and $a_i \leqslant a_{m+1}, a_{m+2},\ldots, a_n \leqslant a_n$. Thus, to $a_{m+1},\ldots, a_n$ we can order the edges $b_{m+1},\ldots, b_{i-1}, b_{i+1},\ldots, b_n$ but they are few, the ordering is impossible, and the theorem is proved.

In Theorem 5 we have shown that it is true only in a special case that we can fill up something only if we can fill it up in a "regular" way. However, we may define the word "regular" in a wider sense:

DEFINITION 5. A filling up $A \mid B$ is regular if we can reach this filling up by cuts, where "cut" is the operation in which we divide the whole parallelepiped by an $n - 1$ dimensional hyperplane.

After this definition we can formulate the following theorem, the validity of which is clear from the proof of Theorem 3.

THEOREM 6. $(A_1, A_2,\ldots, A_m) \mid B$ *if and only if it is possible regularly, too (Assuming* (5)).

Another interesting special case of Theorem 3 is if we have n-dimensional cubes with relative prime edges.

THEOREM 7. *Let* $C_1, C_2,\ldots, C_m$ *be n-dimensional cubes with edges* $c_1, c_2,\ldots, c_m$ *satisfying* $(c_i, c_j) = 1$ $(i \neq j)$ *and let*

$$b_j > 3^{nm2^{nm}} \cdot c^{2^{nm}+1} \qquad (1 \leqslant j \leqslant n)$$

where $c = \max(c_1,\ldots, c_m)$. *Then*

$$(C_1,\ldots, C_m) \mid B$$

holds if and only if the $m \times n$ *matrix* $M(B, c_1,\ldots, c_m)$ *has no m independent* 0's.

Proof. We apply Theorem 3 and first reformulate (F_1) for this case.

Choosing k_i $(1 \leqslant k_i \leqslant n)$ edges from C_i, c_i is their greatest common divisor. We form sets and every c_i will contained by $n - k_i + 1 \geqslant 1$ sets.

(a) CASE $m > n$. In this case there is obviously a set of at least two different elements. The greatest common divisor of the number of this set will be 1, because they are relative prime numbers. Thus $M(B, e_1, ..., e_n)$ cannot have n independent 0's because it contains a row consisting of 1's, that is, in this case $(C_1, ..., C_m) \mid B$ always holds.

(b) CASE $m \leqslant n$. In this case it is also sufficient to consider the one-element sets. The most important case is

$$e_1 = c_1, \quad e_2 = c_2, ..., e_m = c_m, \quad e_{m+1} = \cdots = e_n = \infty.$$

We have the condition from (F_1) that $M(B, c_1, ..., c_m, \infty, ..., \infty)$ cannot contain n independent 0's, or $M(B, c_1, ..., c_m)$ cannot contain m independent 0's. It is easy to see that the other cases (if one c_i is contained in more than one set) do not give new condition; the theorem is proved.

However, we may modify the above theorem by using the König-Egerváry theorem [4]:

KÖNIG-EGERVÁRY THEOREM. *In a* 0, 1 *matrix the maximal number of independent* 0*'s is equal to the minimal number of rows and columns containing all the* 0*'s.*

In our case $M(B, c_1, ..., c_m)$ has at most $m - 1$ independent 0's, by the König-Egerváry theorem there are $m - 1$ rows and columns containing all the 0's. We separate two cases:

(α) There are $m - 1$ columns containing all the 0's. In this case there are $n - m + 1$ columns consisting of 1's, that is, B has $n - m + 1$ edges divisible by all the c_i's.

(β) There are p $(p \geqslant 1)$ rows and $m - 1 - p$ columns containing all the 0's. For example, let the first row be one of the above rows. Then the matrix $M(B, c_2, ..., c_m)$ cannot have $m - 1$ independent 0's, that is, $(C_2, ..., C_m) \mid B$.

We have obtained the following modified form of Theorem 7.

THEOREM 7A. *Under the condition of Theorem 7* $(C_1, ..., C_m) \mid B$ *if and only if*

(a) $m > n$,

(b) $m \leqslant n$, *and there are* $n - m + 1$ *edges of* B *divisible by all the numbers* $c_1, ..., c_m$, *or we can fill up* B *by less than* m *of* C_i*'s.*

2. Coverings with Knight Figures

Let us consider the two-dimensional lattice points. For the sake of visuality the elementary lattice squares, corresponding to the lattice points, will be called *fields*. We define a graph G; its vertices are the fields of the whole plane and two vertices are connected with an edge, if the corresponding fields can be covered with a knight figure of type $a \times b$ (i.e., with $K(a, b)$). Now to every set S of fields there corresponds a subgraph G_S of G, the vertices of which are the elements of S and the edges of which are those edges of G, which connect elements of S.

We say that a field is *white* (or *black*), if the sum of its coordinates is even (or odd). Let A_S denote the set of the white fields of S. If a subset $\tilde{A}$ of A_S is given, then the elements of $\tilde{A}$ will be called *knights*. We say that a knight *attacks* a field of S if they are connected with an edge in G_S. So we can speak about the set of fields attacked by the elements of $\tilde{A}$. We denote by B_S the set of the fields, attacked by elements of A_S. First we consider an infinite stripe S, consisting of the fields (k, l), for which

$$0 \leqslant k \leqslant m - 1,$$
$$-\infty < l < +\infty,$$

where m is a positive integer. We mark each row of S by the common second coordinate of its fields. We denote the r-th row of S by S^r. We introduce some notations

$$A^r = A_S \cap S^r,$$
$$B^r = B_S \cap S^r,$$
$$A^r(p) = \{(k, l) : (k, l) \in A^r,\ k \equiv p \ (\mathrm{mod}\ 2b)\},$$
$$B^r(p) = \{(k, l) : (k, l) \in B^r,\ k \equiv p \ (\mathrm{mod}\ 2b)\},$$

where p runs over the residual classes mod $2b$. Obviously $A^r(p) = 0$ if $p \not\equiv r \ (\mathrm{mod}\ 2)$ and $B^r(p) = 0$ if $p \not\equiv r + a + b \ (\mathrm{mod}\ 2)$.

Let us fix the value of r and let $\tilde{A}$ be an arbitrary subset of A^r. Let $\tilde{C}$ be the set of fields, attacked by elements of $\tilde{A}$, and let

$$\tilde{A}(p) = A^r(p) \cap \tilde{A},$$
$$\tilde{C}^{r,r+a} = B^{r+a} \cap \tilde{C}, \qquad \tilde{C}^{r,r+b} = B^{r+b} \cap \tilde{C},$$
$$\tilde{C}^{r,r+a}(p) = B^{r+a}(p) \cap \tilde{C}, \qquad \tilde{C}^{r,r+b}(p) = B^{r+b}(p) \cap \tilde{C}.$$

We say that the r-th row is *a-defective* if

$$|\tilde{C}^{r,r+a}| < |\tilde{A}|$$

and *b-defective* if

$$|\tilde{C}^{r,r+b}| < |\tilde{A}|.$$

REMARK. The notions of defectivity will be useful because our proofs will have the following outline: our aim is to give necessary and sufficient conditions for the coverability of a lattice rectangular R. The difficulty lies in the proof of sufficiency. Here we shall use the marriage principle, which is formulated in the previous section in the language of matrices.

MARRIAGE PRINCIPLE. *Let G be a bipartite graph, that is, the vertices of G can be decomposed for two disjoint classes A and C so that within a class there is no connection. Then a necessary and sufficient condition for the existence of a one-to-one correspondence between A and C along the edges of the graph, i.e., a decomposition for factors of degree* 1, *is the following*: $|A| = |C|$ *and*

> *condition* (C): *If $\tilde{A}$ is an arbitrary subset of A, and $\tilde{C}$ denotes the set of vertices, connected with at least one element of $\tilde{A}$, then*
>
> $$|\tilde{A}| \leqslant |\tilde{C}|.$$

Now if we could give a one-to-one transformation of the set of rows of R onto itself so that, if in the r-th row of R, there are g_r knights, then they attack at least g_r knights in the image of the r-th row, then we could apply the marriage principle, and the existence of a covering would be proved. Naturally the image of the r-th row may be only one of the $r - b$-th, $r - a$-th, $r + a$-th, and $r + b$-th rows. Unfortunately it may happen that a row is defective, but we can prove that, if the r-th row is, e.g., a-defective, then the difference $|\tilde{C}^{r,r+b}| - |\tilde{A}|$ is relatively big, and this fact makes it possible to use the marriage principle.

We say that the residue p (modulo $2b$) for which $p \equiv r \pmod 2$ is *deficient*, if

$$|B^{r+a}(p + b)| = |A^r(p)| - 1;$$

it is *profitable*, if

$$|B^{r+a}(p + b)| = |A^r(p)| + 1;$$

and *neutral* if

$$|B^{r+a}(p + b)| = |A^r(p)|.$$

LEMMA 8. *If p is the least non-negative element of the corresponding residual class and $p \equiv r \pmod 2$, then*

(i) *p is deficient, if and only if* $0 \leqslant p \leqslant b - 1$ *and*

$$\frac{m-1-p}{2b} - \left[\frac{m-1-p}{2b}\right] < \frac{1}{2}.$$

(ii) *p is profitable, if and only if* $b \leqslant p \leqslant 2b - 1$ *and*

$$\frac{m-1-p}{2b} - \left[\frac{m-1-p}{2b}\right] \geqslant \frac{1}{2}.$$

(iii) *p is neutral if and only if it is neither deficient nor profitable.*

The proof of the lemma is trivial.

The following lemma is also easily provable:

LEMMA 9. *If p is profitable and* $\tilde{A}(p)$ *is not empty, then*

$$|\tilde{C}^{r,r+a}(p+b)| \geqslant |\tilde{A}(p)| + 1.$$

If p is neutral, then

$$|\tilde{C}^{r,r+a}(p+b)| \geqslant |\tilde{A}(p)|,$$

while, if p is deficient, then

$$|\tilde{C}^{r,r+a}(p+b)| \geqslant |\tilde{A}(p)|$$

unless $\tilde{A}(p) = A^r(p)$, *and in this case*

$$|\tilde{C}^{r,r+a}(p+b)| = |\tilde{A}(p)| - 1.$$

LEMMA 10. *If m is even, then the number of profitable p's is equal to the number of the deficient p's* ($r \equiv p \pmod 2$).

Proof. If b is even, then p is deficient if and only if $p + b$ is profitable. Really $r \equiv p \equiv p + b \pmod 2$ and $0 \leqslant p \leqslant b - 1$ if and only if $b \leqslant p + b \leqslant 2b - 1$. Further, if

$$\frac{m-1-p}{2b} - \left[\frac{m-1-p}{2b}\right] < \frac{1}{2},$$

then

$$\frac{m-1-(p+b)}{2b} - \left[\frac{m-1-(p+b)}{2b}\right]$$
$$= \frac{m-1-p}{2b} - \frac{1}{2} - \left[\frac{m-1-p}{2b} - \frac{1}{2}\right]$$
$$= \frac{m-1-p}{2b} - \frac{1}{2} - \left(\left[\frac{m-1-p}{2b}\right] - 1\right)$$
$$= \frac{m-1-p}{2b} - \left[\frac{m-1-p}{2b}\right] + \frac{1}{2} \geqslant \frac{1}{2}.$$

The contrary direction is provable similarly.

If b is odd, then we suppose that r is even (the case when r is odd can be reduced to this case). In this case we can consider only the even p's. We assert that it is impossible that both 0 and $b-1$ be deficient. If they were, then we should have

$$(26) \qquad \frac{m-1}{2b} - \left[\frac{m-1}{2b}\right] < \frac{1}{2} \quad \text{and}$$
$$\frac{m-1-(b-1)}{2b} - \left[\frac{m-1-(b-1)}{2b}\right] < \frac{1}{2}.$$

The latter inequality leads to

$$(27) \qquad \frac{m}{2b} - \left[\frac{m}{2b} - \frac{1}{2}\right] < 1 \quad \text{and} \quad \frac{m}{2b} - \left[\frac{m}{2b}\right] \geqslant \frac{1}{2}.$$

Both (26) and (27) can be true if and only if

$$\frac{m}{2b} = \left[\frac{m}{2b}\right] + \frac{1}{2}.$$

But in this case

$$m = 2b\left[\frac{m}{2b}\right] + b.$$

We supposed that m is even, so this equality may not be true, because b is odd.

Now applying again Lemma 8 we can show that, if $b-1$ is not deficient, then p is deficient if and only if $p+b+1$ is profitable, and, if 0 is not deficient, then p is deficient if and only if $p + b - 1$ is profitable. The lemma is proved.

LEMMA 11. *If m is even,* $(a, b) = 1$ *and the r-th row is a-defective, then*

$$|\tilde{C}^{r,r+b}| \geqslant |\tilde{A}| + \frac{m}{2b} - \frac{a}{2} - b.$$

Similarly, if the r-th row is b-defective, then

$$|\tilde{C}^{r,r+a}| \geqslant |\tilde{A}| + \frac{m}{2a} - \frac{b}{2} - a.$$

Proof. If the r-th row is a-defective, then by Lemmas 9 and 10 there is at least one deficient p', for which $\tilde{A}(p') = A^r(p')$, and at least one profitable p'', for which $\tilde{A}(p'') = 0$. We remark that from the proof of Lemma 10 it follows that $b > 1$. For p' we have

$$|\tilde{C}^{r,r+b}(p' + a)| = B^{r+b}(p' + a)$$

and

$$|\tilde{C}^{r,r+b}(p' - a)| = B^{r+b}(p' - a);$$

therefore

$$|\tilde{C}^{r,r+b}(p' + a)| \geqslant \left[\frac{m}{2b}\right] \tag{28}$$

and

$$|\tilde{C}^{r,r+b}(p' - a)| \geqslant \left[\frac{m}{2b}\right], \tag{29}$$

while

$$|\tilde{A}(p')| \leqslant \left[\frac{m}{2b}\right] + 1. \tag{30}$$

Using $(a, b) = 1$ we obtain that the sequence

$$p', p' + 2a, p' + 2 \cdot 2a, \ldots, p' + (b - 1) \cdot 2a$$

of residual classes modulo $2b$ consists of disjoint classes, because if, for some t_1 and t_2 $(0 \leqslant t_1 < t_2 < b)$,

$$p' + t_1 \cdot 2a \equiv p' + t_2 \cdot 2a \pmod{2b},$$

that is,

$$(t_1 - t_2)\, 2a = s \cdot 2b$$

is valid, then from $(a, b) = 1$ we have $a \mid s$, and so $a \leqslant s$, but in this case

$$(t_1 - t_2)a < b \cdot s = (t_1 - t_2)a,$$

and this is a contradiction. Further, this sequence contains every residue for which $A^r(p) \neq 0$. Thus, p'' is an element of the sequence, for example,

$$p'' \equiv p' + s_0 \cdot 2a \pmod{2b}$$

$(1 \leqslant s_0 \leqslant b - 1)$. Now, obviously, if $1 \leqslant s < s_0$, then

$$(31) \qquad |\tilde{C}^{r,r+b}(p' + s \cdot 2a + a)| \geqslant |\tilde{A}(p' + s \cdot 2a)| - \left(\left[\frac{a}{2b}\right] + 1\right)$$

and, if $s_0 < s \leqslant b - 1$, then

$$(32) \qquad |\tilde{C}^{r,r+b}(p' + s \cdot 2a - a)| \geqslant |\tilde{A}(p' + s \cdot 2a)| - \left(\left[\frac{a}{2b}\right] + 1\right).$$

Using (23), (29), (31), and (32):

$$\begin{aligned} |\tilde{C}^{r,r+b}| &= |\tilde{C}^{r,r+b}(p' + a)| + |\tilde{C}^{r,r+b}(p' - a)| \\ &\quad + \sum_{s=1}^{s_0-1} |\tilde{C}^{r,r+b}(p' + s \cdot 2a + a)| \\ &\quad + \sum_{s=s_0+1}^{b-1} |\tilde{C}^{r,r+b}(p' + s \cdot 2a - a)| \\ &\geqslant 2\left[\frac{m}{2b}\right] + \sum_{s=1}^{s_0-1} |\tilde{A}(p' + s \cdot 2a)| \\ &\quad + \sum_{s=s_0+1}^{b-1} |\tilde{A}(p' + s \cdot 2a)| - (b-2)\left(\left[\frac{a}{2b}\right] + 1\right); \end{aligned}$$

so from (30) we have

$$\begin{aligned} |\tilde{C}^{r,r+b}| &\geqslant |\tilde{A}(p')| + \sum_{s=1}^{s_0-1} |\tilde{A}(p' + s \cdot 2a)| + \sum_{s=s_0+1}^{b-1} |\tilde{A}(p' + s \cdot 2a)| \\ &\quad + \left[\frac{m}{2b}\right] - (b-2)\left(\left[\frac{a}{2b}\right] + 1\right) - 1 \\ &= |\tilde{A}| + \left[\frac{m}{2b}\right] - (b-2)\left(\left[\frac{a}{2b}\right] + 1\right) - 1. \end{aligned}$$

Now the assertion of the lemma follows easily by using $b > 1$.

THEOREM 10. *If* $(a, b) = 1$ *and both* a *and* b *are odd numbers, then a rectangular* R *with large enough sizes* ($m \geqslant m_0(a, b)$, $n \geqslant n_0(a, b)$) *is coverable with knight figures of type* $a \times b$ *if and only if the edge lengths of* R *are even.*

Proof. We restrict our considerations for G_R.

To prove the necessity we remark that the fields of an arbitrary knight figure belong to rows, the marks of which are incongruent modulo 2, because a and b are odd. If R is coverable, then it must have the same number of fields in rows with even marks and in rows with odd marks. This is true if and only if the number of rows is even. The same is true for the number of columns, so the necessity is proved.

Turn to the proof of sufficiency. $a + b$ is odd, so the fields of an arbitrary knight figure are identically colored. Therefore it is sufficient to prove that the white fields of R are coverable. We define a bipartite graph G^* as follows: each of the disjoint classes A and C contains $|A_R|$ elements, the elements of A represent the elements of A_R and the elements of C represent the elements of $B_R = A_R$; an element of A and an element of C are connected if the corresponding fields can be covered with a knight figure. We hope that it does not lead to misunderstanding if we identify the vertices of the graph with the corresponding fields (this is inaccurate, because in this case A and C are not disjoint).

Let $\tilde{A}$ be an arbitrary subset of A_R, and $\tilde{C}$ the set of fields, attacked by elements of $\tilde{A}$ ($\tilde{C} \subset B_R$). We assert that for G^* the condition (C) of the marriage principle is satisfied, that is,

$$|\tilde{A}| \leqslant |\tilde{C}|.$$

Suppose that m and n are even, $n \geqslant 4ab$. Let d_a and d_b denote the number of a-defective (or b-defective) rows of R. We can assume that $d_a \leqslant d_b$. If $d_a > d_b$ then a similar argument proves the theorem. We use the following simple

LEMMA 12. *If* α, β, γ *are positive integers,* $(\alpha, \beta) = 1$, *and*

$$\gamma > \alpha\beta - \alpha - \beta,$$

then the Diophantine equation

$$\gamma = x\alpha + y\beta$$

has non-negative solution x_0, y_0 *with the property*

$$y_0 \leqslant \alpha - 1.$$

Proof. Let x and y be an arbitrary solution, and l an integer such that $0 \leqslant y + l\alpha \leqslant \alpha - 1$. The desired solution: $x_0 = x - l\beta$, $y_0 = y + l\alpha$. We have to verify only $x_0 \geqslant 0$. However, $y_0\beta \leqslant (\alpha - 1)\beta = \alpha\beta - \beta$ and

$$x_0\alpha = \gamma - y_0\beta \geqslant \gamma - \alpha\beta - \beta > -\alpha$$

results in

$$x_0 > -1.$$

Let now $\gamma = n/2$, $\alpha = a$, $\beta = b$. We divide the rows into $x_0 + y_0$ classes. First we define x_0 classes. Each of them contains $2a$ rows so that the i-th class $(0 \leqslant i \leqslant x_0 - 1)$ contains the rows

$$i \cdot 2a,\ i \cdot 2a + 1,\ i2a + 2,\ldots,\ i2a + 2a - 1.$$

In each of these classes we pair two rows if the difference of their marks is equal to a. Similarly each of the y_0 classes contains $2b$ rows; the j-th class $(0 \leqslant j \leqslant y_0 - 1)$ contains the rows

$$x \cdot 2a + j2b,\ x_0 2a + j2b + 1,\ldots,\ x_0 2a + (j + 1)\, 2b - 1.$$

In each of these classes we pair two rows if the difference of their marks is b. Let the set R_a contain the rows that belong to one of the x_0 classes and R_b contain the other rows.

The pairs define a one-to-one transformation of the set of the rows onto itself.

If $d_b = 0$, then obviously $|\tilde{A}| \leqslant |\tilde{C}|$. If $d_b > 0$, then we cannot assert that the knights of a row attack at least the same number of fields in the pair of the corresponding row, but we can estimate the number of the attacked fields.

In R_b we have $y_0 2b$ rows. From Lemma 9 and Lemma 8 we obtain that the loss of one row of R_b (i.e., the difference of the number of the (white) fields attacked by the elements of the row in the pair of the corresponding row) is at most $a/2$, so the total loss of rows of R_b is not more than

$$y_0 2b\,\frac{a}{2} \leqslant (a - 1)\, ba \leqslant a^2 b.$$

We can suppose that R_a contains at least one b-defective row because, if it did not, then we could reflect R so that the r-th row should go over to the $(n - 1 - r)$-th row, and for this rectangular our assumption will already be valid, using the fact that, for the number of rows of R_b, $y_0 2b < 2ab \leqslant n/2$ holds.

Let us denote by $d_b{}^*$ the number of b-defective rows of $R_a(d_b{}^* \geqslant 1)$. From Lemma 11 it follows that the excess of a b-defective row of R_a is at least

$$\frac{m}{2a} - \frac{b}{2} - a.$$

For the number $d_a{}^*$ of a-defective rows in R_a we have

$$d_a{}^* \leqslant d_a \leqslant d_b \leqslant d_b{}^* + y_0 2b \leqslant d_b{}^* + 2ab.$$

Thus, their loss is at most $(d_b{}^* + 2ab)(b/2)$; so, if

$$d_b{}^* \left(\frac{m}{2a} - \frac{b}{2} - a\right) \geqslant a^2 b + (d_b{}^* + 2ab)\,\frac{b}{2}$$

is valid, then the condition (C) is satisfied. But, if $m \geqslant m_0(a, b)$, then this inequality is true. A rough estimate for $m_0(a, b)$ is given by

$$m_0(a, b) = 2(a + b)(ab + 1) \max(a, b). \tag{33}$$

Applying the marriage principle we have that G^* can be decomposed for factors of degree 1. But this decomposition of G^* does not give directly a covering of A_R, because, if a and b are odd numbers, then $A_R = B_R$.

Now we keep only the edges that are factors in the decomposition, guaranteed by marriage principle, and join the vertices of G^* that correspond to the same field of A_R. So we obtain a graph of degree 2, and it is easy to see that this graph is the union of disjoint circles, each of them consisting of an even number of edges because G_0 is a bipartite graph. Leaving each second edge in all circles, we have really a decomposition of G_R for factors of degree 1, and the theorem is proved.

THEOREM 11. *If* $(a, b) = 1$ *and* $a \not\equiv b \pmod 2$, *then a rectangular with large enough sizes* $(m \geqslant m_0(a, b), n \geqslant n_0(a, b))$ *is coverable with knight figures of type* $a \times b$ *if and only if one of its sizes is even.*

Proof. The necessity of the conditions follows from the fact that each knight figure covers two fields. So, if R is coverable, then it must contain an even number of fields.

If $\tilde{A}$ is an arbitrary subset of A_R and $\tilde{C}$ is the set of fields, attacked by elements of $\tilde{A}(\tilde{C} \subset B_R)$, then it is now completely sufficient to prove that

$$|\tilde{A}| \leqslant |\tilde{C}| \tag{34}$$

because in this case a knight figure covers differently colored fields, and the application of the marriage problem gives the theorem.

If m and n are even, then the idea used in the proof of Theorem 10 can be applied without change. In what follows, we get rid of the assumption that n is even, so let n be odd. Without any loss of generality we can suppose that a is even. Let $a > 2$ and $n \geqslant n_0(a, b) = 10ab$, $m \geqslant m_0(a, b)$, where $m_0(a, b)$ is equal to (33). R will be decomposed into four parts. The rectangular R_1 consists of the rows

$$0, 1, 2, \ldots, 4ab - 1;$$

the rectangular R_2 consists of the rows

$$4ab, 4ab + 1, \ldots, 6ab - 1;$$

R_3 consists only of the row of mark $6ab$; and R_4 contains the remaining rows of R.

If for a row of R_2 with an even mark r we have

$$| \tilde{A}^r | \leqslant | \tilde{C}^r |, \tag{35}$$

where $\tilde{A}^r = A^r \cap \tilde{A}$, $\tilde{C}^r = B^r \cap \tilde{C}$, then decomposing R for three parts R_1^*, R_2^*, R_3^*, so that R_1^* consists of the rows

$$0, 1, \ldots, r - 1,$$

R_2^* of the row r, and R_3^* of the remaining rows of R, the condition (C) is trivially satisfied because R_1^* and R_3^* have even (and large enough) sizes, so we can apply the results, proved above.

If for each row of R_2 with even mark (35) is not true, then we decompose R_2 into b blocks, each of them containing $2a$ consecutive rows. This can be done because R_2 contains $2ab$ rows. In a block we pair two rows if their marks differ with a. The marks of a pair are congruent modulo 2, so we can speak about even and odd pairs. If the r-th and $r + a$-th rows form an even pair and no one of them is a-defective, then

$$| \tilde{C}^{r, r+a} | \geqslant | \tilde{A}^r |$$

and

$$| \tilde{C}^{r+a, r} | \geqslant | \tilde{A}^{r+a} |.$$

If, for example, $| \tilde{A}^r | \leqslant | \tilde{A}^{r+a} |$, then

$$| \tilde{C}^r | \geqslant | \tilde{C}^{r+a, r} | \geqslant | \tilde{A}^{r+a} | \geqslant | \tilde{A}^r |,$$

so for the r-th row (35) is valid, and we can apply the idea used there.

In R_2 we have $b \cdot a/2$ even pairs, and if each of them contains a-defective row, then R_2 contains at least $b \cdot a/2$ a-defective rows. Now we decompose R_2 into a blocks, each of them containing $2b$ consecutive rows. In a block we pair two rows, if their marks differ with b. We assert that R_2 gives an excess more than $m/2$. Really the excess of the a-defective rows is at least

$$b \cdot \frac{a}{2}\left(\frac{m}{2b} - \frac{a}{2} - b\right),$$

while the loss of the remaining rows of R_2 is at most

$$\left(2ab - b \cdot \frac{a}{2}\right)\frac{a}{2} = \frac{3}{4}a^2b.$$

So R_2 gives at least an excess of

$$b \cdot \frac{a}{2}\left(\frac{m}{2b} - \frac{a}{2} - b\right) - \frac{3}{4}a^2b,$$

and if

(36) $$m \geqslant 4\,\frac{a^2b + ab^2}{a - 2}$$

then this is more than $m/2$. (From $m \geqslant m_0(a, b)$, (36) already follows). The loss of R_3 is at most $m/2$ and for R_1 and R_4 we can apply the results, already proved, because their sizes are even, and greater than the given lower bounds. So for R we have

$$|\tilde{A}| \leqslant |\tilde{C}|$$

and so, in the case $a > 2$, the theorem is proved.

If $a = 2$, then R_2 gives an excess greater than $m/2$ only if it contains $8b$ rows and not only $4b$ rows. Thus in this case we must suppose that $n \geqslant n_0(2, b) = 24b$, and the proof can be obtained in the same way as above. Q.E.D.

Now we are able to state the main theorem of this section, which contains Theorems 10 and 11 as a special case:

THEOREM 12. *Let* $(a, b) = d$.

1. *If*

$$\frac{a}{d} \equiv \frac{b}{d} \pmod 2,$$

then a rectangular with large enough sizes ($m \geqslant m_1(a, b), n \geqslant n_1(a, b)$ *is coverable with knight figures of type* $a \times b$ *if and only if both* m *and* n *are divisible by* $2d$.

2. *If*

$$\frac{a}{d} \not\equiv \frac{b}{d} \pmod 2,$$

then a rectangular with large enough sizes ($m \geqslant m_1(a, b), n \geqslant n_1(a, b)$ *is coverable with knight figures of type* $a \times b$ *if and only if either* m *or* n *is divisible by* $2d$.

Proof. We take the coordinates of the fields of R modulo d, and we represent every residual class by its least non-negative element. We mark each row (column) by the common second (first) coordinates of its elements. It is obvious that two fields, covered by an arbitrary knight figure of type $a \times b$, have the same coordinates modulo d.

We consider only those fields of R which have the same coordinates (i, j) modulo d $(0 \leqslant i, j \leqslant d - 1)$. We denote the set of these fields by $R(i, j)$. From these fields we can form a new rectangular $R^*(i, j)$, if we transform the field $(kd + i, ld + j)$ into the field (k, l). The covering of $R(i, j)$ with knight figures of type $a \times b$ is equivalent to the covering of $R^*(i, j)$ with knight figures of type $(a/d) \times (b/d)$. A necessary condition for the coverability of R is that each $R^*(i, j)$ should contain an even number of fields.

If m is not divisible by $2d$, then necessarily there exists a residue i_0 such that it is the mark of an odd number of columns, and similarly, if n is not divisible by $2d$, then there exists a residue j_0 such that it is the mark of an odd number of rows. If neither m or n is divisible by $2d$, then $R^*(i_0, j_0)$ contains an odd number of fields, and this is a contradiction.

So generally at least one size of R must be divisible by $2d$. If

$$\frac{a}{d} \not\equiv \frac{b}{d} \pmod 2,$$

then this condition is also sufficient, because in this case the rectangulars $R^*(i, j)$ have an even number of fields for every i and j, and we can apply Theorem 11.

$$\left(m_1(a, b) = dm_0\left(\frac{a}{d}, \frac{b}{d}\right), \qquad n_1(a, b) = dn_0\left(\frac{a}{d}, \frac{b}{d}\right)\right).$$

So 2 is proved.

Turn to assertion 1. If, for example, m was not divisible by $2d$, then

$R^*(i_0, j)$ would not have even sizes, in contradiction with Theorem 10. From Theorem 10 the sufficiency of the condition follows too. (The bounds for the sizes can be chosen in the same way as in Part 2.)

Q.E.D.

The two simplest knight figures are those of types $a \times 0$ and $a \times a$. Their n-dimensional generalizations are the knight figures of type $a \times 0 \times \cdots \times 0$ (or $a \times a \times \cdots \times a$). For these cases we can prove the following theorems:

THEOREM 13. *An n-dimensional parallelepiped R can be covered with knight figures of type* $a \times 0 \times \cdots \times 0$ *if and only if one of its sizes is divisible by* $2a$.

THEOREM 14. *An n-dimensional parallelepiped R can be covered with knight figures of type* $a \times a \times \cdots \times a$ *if and only if its sizes are divisible by* $2a$.

Proof of Theorem 13. Necessity. Suppose that R consists of the lattice fields of coordinates $(k_1, k_2, \ldots, k_n)$, where $1 \leqslant k_i \leqslant m_i$. Now we take the coordinates modulo a. Every knight figure consists of fields having identical coordinates modulo a, so R must contain an even number of fields of any fixed coordinates. If m_i is not divisible by $2a$, then there exists a residue r_i such that $[(m_i - r_i)/a]$ is an odd number. If no size of R is divisible by $2a$, then it is easy to see that R contains

$$\prod_{i=1}^{n} \left[\frac{m_i - r_i}{a}\right]$$

fields with coordinates $(r_1, r_2, \ldots, r_n)$, so the covering is impossible.

The *sufficiency* of the condition is obvious.

Proof of Theorem 14. Necessity. Suppose that R consists of the lattice fields of coordinates $(k_1, k_2, \ldots, k_n)$ where $0 \leqslant k_i \leqslant m_i$. Now we say that a field $(k_1, k_2, \ldots, k_n)$ of R is white (black), if $[k_1/a]$ is even (odd). It is obvious that every knight figure covers differently colored fields. Therefore the numbers of white and black fields must be identical. This is possible if and only if m_1 is divisible by $2a$. The same must be valid for the other sizes of R, so the necessity is proved. The sufficiency follows from the fact that a parallelepiped $2a \times 2a \times \cdots \times 2a$ can be covered trivially in a unique manner.

CONSTRUCTIONS

The conditions of Theorem 12 guarantee the existence of a covering. For knight figures of type $1 \times b$ we can also give constructions for the covering.

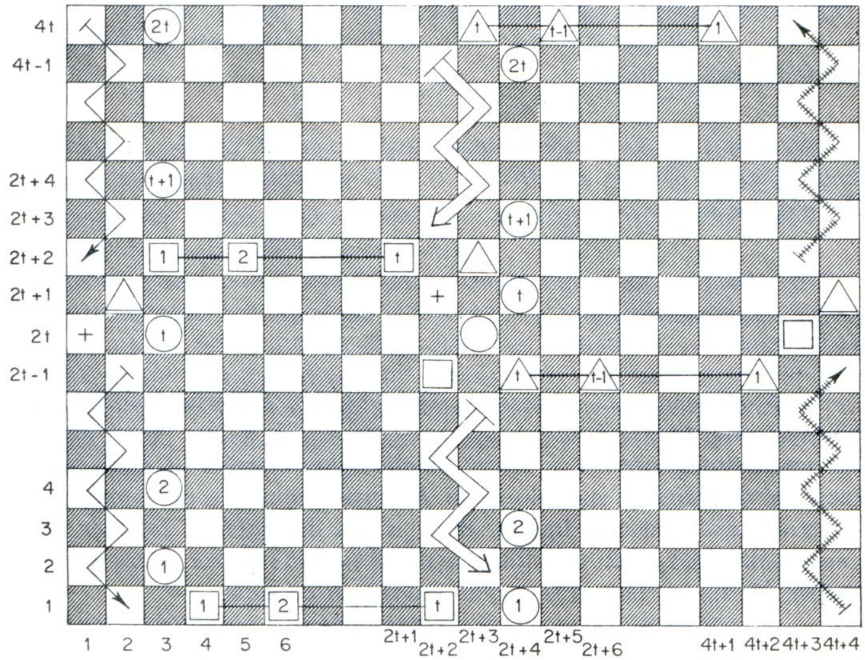

FIGURE 1

CASE $b = 2t + 1$. Figure 1 shows schematically the covering of a rectangular $4t \times (4t + 4)$. We covered only the white fields. The covering of black fields is symmetrical. Identically marked fields belong to the same knight figure. The covering of columns $1, 2, 2t + 2, 2t + 3, 4t + 3, 4t + 4$ is clear. Columns 3 and $2t + 4$, 4 and $2t + 5, \ldots, 2t + 1$ and $4t + 2$ can be covered almost pairwise as it is shown by the figure for columns 3 and $2t + 4$.

For a rectangular $2 \times (4t + 2)$ we have a trivial covering. Theorem 3 asserts that a rectangular $2m \times 2n$, where $m \geqslant M$, $n \geqslant N$, can be covered with rectangulars of type $2 \times (4t + 2)$ and $4t \times (4t + 4)$. However, in this special case the proof is very similar and we can reach smaller boundaries $M = (2t - 1)\, 2t$ and $N = 2t(2t + 1)$.

This is the reason for repeating the proof for this case.

The rectangular $4t \times (4t+2)$ is obviously coverable (using rectangulars $2 \times (4t+2)$) and, if

$$n > (2t+1)(2t+2) - (2t+1) - (2t+2) = 2t(2t+1) - 1,$$

then a rectangular $4t \times 2n$ can be obtained as a union of rectangulars of types $4t \times (4t+2)$ and $4t \times (4t+4)$ dividing the side of length $2n$ into parts of length $4t+2$, $4t+4$ on the basis of Lemma 12. Similarly, if

$$m > 2t(2t+1) - 2t - (2t+1) = (2t-1)\,2t - 1,$$

then a rectangular $2m \times 2n$ can be obtained as a union of rectangulars of types $4t \times 2n$ and $(4t+2) \times 2n$ where the latter rectangular coverable with rectangulars of type $2 \times (4t+2)$.

CASE $b = 2t$. Figure 2 shows the covering of the rectangular $(2t+1) \times 4t$. The covering of columns $1, 2, 2t+1, 2t+2$ is marked on the figure and the other columns can be covered in the same manner.

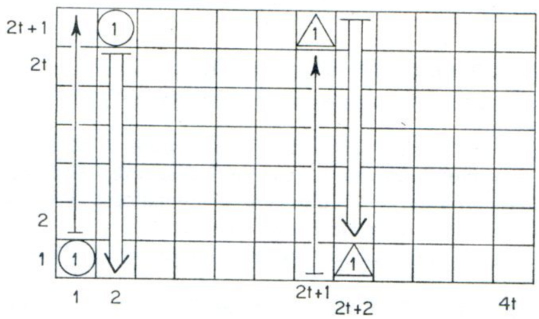

FIGURE 2

Figure 3 shows the covering of a rectangular $(4t-1) \times (4t+2)$. The striped rectangular is a $(2t-2) \times 4t$, which is trivially coverable, because a rectangular $2 \times 4t$ has a trivial covering. The covering of columns $3, 4, 2t+3, 2t+4$ is shown on the figure; the covering of the non-marked columns can be obtained in the same way.

Apply the same method as before. We have that, if

$$m > 4t(4t-3) + 1 \qquad \text{and} \qquad n > (2t-1)\,2t + 1,$$

then any rectangular $m \times 2n$ is the union of disjoint rectangulars of types $2 \times 4t$, $(2t+1) \times 4t$, $(4t-1) \times (4t+2)$.

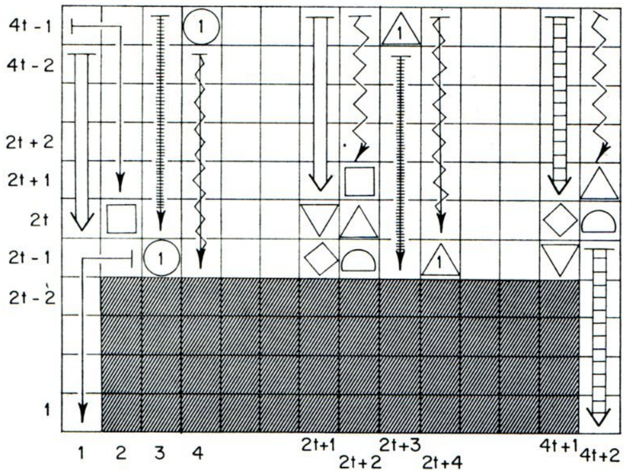

FIGURE 3

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