EXTREMAL PROBLEMS WITH EXCLUDED SUBGRAPHS IN THE n-CUBE

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1. INTRODUCTION

Let X be a finite set with n elements. 2^X denotes its power set, $\binom{X}{i}$ denotes the family of its i-element sets. The directed graph $C_n = (2^X, E)$ is defined by $E = \{(A,B) : A \subsetneq B\}$. This is what we briefly call the *n-cube* in the title but it should be called the *directed transitive n-cube*. Let $H_1, \ldots, H_k$ be directed graphs. $f_n(H_1,\ldots,H_k)$ denotes the size of a maximally sized subset $Y \subset 2^X$ of vertices of C_n under the supposition that the subgraph $C_n(Y)$ induced by Y in C_n does not contain any of $H_1, \ldots, H_k$ as a (not necessarily induced) subgraph.

The first example is the well-known old Sperner theorem [7]: $k = 1$ and H_1 is a directed edge. Then $f_n(H_1)$ is simply the size of a maximally sized family Y of subsets of X containing no comparable pairs $A \subset B$ $(A, B \in Y)$. It is known from [7] that $f_n(H_1) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ ($\lfloor x \rfloor$ is the integer part of x).

A generalization of the above example is, when H_1 is a directed path of length $l+1$ $(l \geq 0)$. This question is solved by Erdös [2]: $f_n(H_1)$ is equal to the sum of the l largest binomial coefficients $\binom{n}{i}$. An optimal construction Y consists of all i-element subsets of X with $\lfloor \frac{n-l+1}{2} \rfloor \leq i \leq \lfloor \frac{n+l-1}{2} \rfloor$. Of course, the problem posed above is too hard to solve in full generality. We will give the exact value or estimates of $f_n(H_1,\ldots,H_k)$ only for very simple graphs H_i.

Tangentially, we will consider a slightly modified problem, too. Let then $f_n^*(H_1,\ldots,H_k)$ denote the size of a maximally sized subset $Y \subset 2^X$ of vertices of C_n under the supposition that $C_n(Y)$ does not contain any of $H_1, \ldots, H_k$ as an induced subgraph. This problem has any sense only when H_i are transitively closed (if (a,b) and (b,c) are edges of H_i then (a,c) is also an edge of it). The inequality

(1) $$f_n^*(H_1,\dots,H_k) \geq f_n(H_1,\dots,H_k) \quad \text{is obvious.}$$

The case $k = 1$, $H_1 = (\{a,b\},\emptyset)$ shows that sometimes strict inequality can hold: $f_n(H_1) = 1$, $f_n^*(H_1) = n+1$.

2. NO V, NO Λ

The vertices of C_n are usually imagined to be drawn according to their sizes. The big sets are upstairs, the small ones downstairs. Consequently, the edges are directed similarly, say toward the top. Thus the graph $(\{x_1,x_2,x_3\},\{(x_1,x_2),(x_1,x_3)\})$ can be visualized as a letter V. Therefore we denote it by V. Similarly, $\Lambda = (\{x_1,x_2,x_3\},\{(x_2,x_1),(x_3,x_1)\})$.

THEOREM 1. $f_n^*(V,\Lambda) = f_n(V,\Lambda) = 2\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}$ $(n \geq 3)$.

P r o o f. 1. The inequality $\geq$ in the first place follows by (1). Let us prove the second inequality $\geq$ by a construction. Fix an element $x\in X$ and take all the $\lfloor\frac{n-1}{2}\rfloor$-element sets of $X-\{x\}$ with and without x:

$$Y = \left\{A:\ |A| = \lfloor\tfrac{n-1}{2}\rfloor,\ x\notin A\right\}\cup\left\{\{x\}\cup A:\ |A| = \lfloor\tfrac{n-1}{2}\rfloor,\ x\notin A\right\}.$$

It is easy to see that the graph induced by Y contains only edges of form $\{A,A\cup\{x\}\}$, consequently, it cannot contain adjacent edges, i.e., edges with a common vertex.

2. Let us prove now the inequality

(2) $$f_n^*(V,\Lambda) \leq 2\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}.$$

Let Y be a subset of 2^X such that $C_n(Y)$ does not contain a V or Λ as an induced subgraph. That is, $C_n(Y)$ contains adjacent edges only directed oppositely concerning the meeting-vertex. It follows that $C_n(Y)$ splits into vertex-disjoint directed complete graphs (and isolated vertices). Denote by A_i and C_i the first and last vertices of these complete graphs, resp. $(1 \leq i \leq k)$. Here $A_i\subset C_j$ iff $i = j$. Using the complements $B_i = X-C_i$, this can be restated as

(3) $$A_i\cap B_j = \emptyset \quad \text{iff } i = j \qquad (1 \leq i,j \leq k).$$

The following theorem is known for such a system of subsets:

THEOREM A. *If the sets* $A_1,\dots, A_k, B_1,\dots, B_k$ *satisfy* (3) *then*

(4) $$\sum_{i=1}^{k} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

(This theorem is a slightly modified version of a theorem of Bollobás [1]. This form is published by Tarján [8], his proof can be also found in [4].)

The number of vertices in the complete graphs from A_i to C_i can be upperbounded by $|C_i|-|A_i|+1 = n-|A_i|-|B_i|+1$. Therefore we will investigate

$$\sum_{i=1}^{k} \frac{n-|A_i|-|B_i|+1}{\binom{|A_i|+|B_i|}{|A_i|}(n-|A_i|-|B_i|+1)} \le 1 \tag{5}$$

rather than (4). Using the notation $m_i = |A_i|+|B_i|$ we have the trivial inequality

$$\binom{m_i}{\lfloor\frac{m_i}{2}\rfloor}(n-m_i+1) \ge \binom{m_i}{|A_i|}(n-m_i+1) \tag{6}$$

for the nominators of (5). We intend to show that the left hand side of (6) is minimal for $m_i = n-1$ $(0 \le m_i \le n-1)$. For later use we prove a more general inequality:

LEMMA 1. *If n, m and l are integers, $0 \le m \le n-l$, $(n,m,l) \ne (2,0,1)$, then*

$$2\binom{n-l}{\lfloor\frac{n-l}{2}\rfloor} \ge \lfloor\frac{n-m+l}{2}\rfloor \cdot \binom{m}{\lfloor\frac{m}{2}\rfloor}. \tag{7}$$

P r o o f. First, the inequality

$$\binom{m+1}{\lfloor\frac{m+1}{2}\rfloor}\lfloor\frac{n-m-1+l}{l}\rfloor \ge \binom{m}{\lfloor\frac{m}{2}\rfloor}\lfloor\frac{n-m+l}{2}\rfloor \qquad (1 \le m \le n-l-1) \tag{8}$$

will be verified. (8) is trivial for odd m. Suppose that m is even. It is obvious again if the integer parts are equal. The only case what we have to consider is $l|n-m$. In this case (8) is equivalent to

$$m(n-m-l) \ge 2l. \tag{9}$$

Here $m \ge 2$ since $m \ge 1$ and it is even. On the other hand, $n-m-l \ge l$ follows from $n-m-l \ge 1$ and $l|n-m$. (9) and (8) are proved. (7) follows if

$$2\binom{n-l}{\lfloor\frac{n-l}{2}\rfloor} \ge \lfloor\frac{n+l}{2}\rfloor.$$

This is true if $n = l+2$, and it can be easily proved for $n \ge l+2$ by induction. It is also true for $n = l$ and for $n = l+1$ except when $l = 1$, $n = 2$. The lemma is proved.□

Let us turn back to the proof of the theorem. By (5), (6) and Lemma 1 $(l = 1)$ we have

$$\sum_{i=1}^{k} \frac{n-|A_i|-|B_i|+1}{2\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}} \le 1. \tag{10}$$

Since the number of vertices of $C_n(Y)$ is $\le \sum_{i=1}^{k} (n-|A_i|-|B_i|+1)$, (10) proves (2) when $n \ge 3$. The proof is complete.□

REMARK 1. $f_2(V,\Lambda) = 2$, $f_2^*(V,\Lambda) = 3$ but the theorem is obviously true for $n = 1$.

REMARK 2. Theorem 1 is a sharpening of the Sperner theorem if n is even. A weaker condition implies the same result.

REMARK 3. P. Frankl and Z. Füredi have proved Theorem 1 independently of us (personal communication).

Our next theorem will be a slight generalization of Theorem 1. But we will formulate it in a different way. Namely, $f_n(V,\Lambda)$ can be defined equivalently, as the maximal number of different subsets of an n-element set such that there are no three sets satisfying either $A\cap B\supset C$ or $C\supset A\cup B$. Now we will add the condition that there are no two sets with

$$A\subset B, \qquad |B-A| < l$$

in the family of subsets.

In this case we may follow the proof of Theorem 1. The number of vertices along the path from A_i to C_i is at most

$$\left\lfloor\frac{|C_i|-|A_i|}{l}\right\rfloor+1 = \left\lfloor\frac{n-|A_i|-|B_i|+l}{l}\right\rfloor.$$

Therefore we have to investigate the function $\left\lfloor\frac{n-m+l}{l}\right\rfloor\binom{m}{\lfloor\frac{m}{2}\rfloor}$. This is done in Lemma 1, consequently we have

THEOREM 2. *Let $A_1,\ldots, A_m$ be a family of different subsets of an n-element set. Suppose that it contains no 3 different members with*

$$A_i\cap A_j\supset A_k \quad \text{or} \quad A_k\supset A_i\cup A_j$$

and no 2 members with

$$A_i\subset A_j \qquad |A_j-A_i| < l.$$

Then

$$m \le 2\binom{n-l}{\lfloor\frac{n-l}{2}\rfloor}$$

and this is the best possible estimate.

Although we did not formulate it in this way, the *-type result holds.

3. PUSHING TO THE MIDDLE

The graph $B_u = (\{x_0,x_1,\dots,x_u\},\{(x_1,x_0),\dots,(x_u,x_0)\})$ will be called a *u-broom* while $F_u = (\{x_0,x_1,\dots,x_u\},\{(x_0,x_1),\dots,(x_0,x_u)\})$ is a *u-fork*.

Let $Y\subset 2^X$ then Y_i denotes the family of i-element members of Y. s is defined by $|Y_s|>0$, $|Y_{s+1}| = \dots = |Y_n| = 0$.

LEMMA 2. *If $C_n(Y)$ contains no B_u as a subgraph and $s \ge \frac{n+u}{2}$ then there is an injection $f\colon Y_s \to \binom{X}{s-1}-Y_{s-1}$ ($f(A)\neq f(B)$ if $A\neq B$) with the property $f(A)\subset A$.*

P r o o f. The graph $C_n\left(Y_s \cup \left(\binom{X}{s-1}-Y_{s-1}\right)\right)$ is a bipartite graph. The lemma states that one can find $|Y_s|$ non-adjacent edges in this graph. This is true if the König-Hall condition is satisfied, that is, for any subset $Y'_s\subset Y_s$ there are at least $|Y'_s|$ elements of $\binom{X}{s-1}-Y_{s-1}$ connected with Y'_s. In other words, the number of those s-1-element subsets of the members of Y'_s which are not in Y_{s-1} is at least $|Y'_s|$. We prove now that this is true.

Any member of Y'_s contains s $(s-1)$-element subsets. At most u-1 of them belong to Y_{s-1}, so Y_s contains at least $s-u+1$ members of $\binom{X}{s-1}-Y_{s-1}$. Hence the number of elements of $\binom{X}{s-1}-Y_{s-1}$ being a subset of any member of Y'_s is at least

$$|Y'_s|\cdot\frac{s-u+1}{n-s+1}.$$

Here $(s-u+1)/(n-s+1)\ge 1$ follows by the supposition $s\ge(n+u)/2$. The König-Hall condition holds which completes the proof of the lemma.□

The next theorem says that if B_u $(u\ge 1)$ is among the exluded subgraphs then it is enough to consider the subsets with size $\le \lfloor\frac{n+u-1}{2}\rfloor$.

THEOREM 3. *$f_n(B_u,H_2,\dots,H_k)$ can be realized with a family Y of subsets A satisfying $|A|\le\lfloor\frac{n+u-1}{2}\rfloor$.*

P r o o f. Let $f(Y_s) = Y^*_{s-1}$. Since $Y_{s-1}\cap Y^*_{s-1} = \emptyset$ and $|Y^*_{s-1}| = |Y_s|$, the family $Y^* = (Y-Y_s)\cup Y^*_{s-1}$ has as many members as Y has. On the other hand $C_n(Y^*)$ does not contain H_i (or $H_1 = B_u$) as a subgraph. If, on the contrary, it contains an H_i then changing the possible vertices A in Y^*_{s-1} for $f^{-1}(A)$ it leads to a subgraph H_i in Y. This contradiction shows that $C_n(Y^*)$ does not contain H_i as a subgraph.

By repeated application of this transformation $Y\to Y^*$ we finally obtain a family $Y^{*\cdots *}$ satisfying

(i) $|Y^{*\cdots *}| = |Y|$,

(ii) it contains no H_i as a subgraph,

(iii) the sizes of its members are $\le \lfloor\frac{n+u-1}{2}\rfloor$.

The proof is complete.□

THEOREM 4. *$f_n(F_v,H_2,\ldots,H_k)$ can be realized with a family Y of subsets A satisfying* $|A| \ge \lfloor\frac{n-v+2}{2}\rfloor$.

Proof. It follows from Theorem 3 by taking the complement sets.□

THEOREM 5. *$f_n(B_u,F_v,H_3,\ldots,H_k)$ can be realized with a family Y of subsets A satisfying* $\lfloor\frac{n-v+2}{2}\rfloor \le |A| \le \lfloor\frac{n+u-1}{2}\rfloor$.

Proof. It is an easy consequence of Theorems 4 and 5.□

Let us see now some applications of the above theorems.

COROLLARY 1. *$f_n(V,\Lambda)$ can be realized with a family Y of subsets A satisfying*

$$\lfloor\tfrac{n}{2}\rfloor \le |A| \le \lfloor\tfrac{n+1}{2}\rfloor.$$

Hence the statement of Theorem 1 follows immediately for even n. However, the odd case can be obtained from the even case by a little trick: Suppose Y is a family such that $C_n(Y)$ contains neither V nor Λ. Fix an $x\in X$ and divide Y into two classes. $Y_1 = \{A\colon A\in Y,\ x\notin A\}$, $Y_2 = \{A-\{x\}\colon A\in Y,\ x\in A\}$. Neither $C_n(Y_1)$ nor $C_n(Y_2)$ contains a V or Λ. Consequently, $|Y| = |Y_1|+|Y_2| \le 4\binom{n-2}{\lfloor\frac{n-2}{2}\rfloor} = 2\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}$ holds as desired.

COROLLARY 2. *$f_n(V,B_3)$ can be realized with a family Y of subsets A satisfying* $\lfloor\frac{n}{2}\rfloor \le |A| \le \lfloor\frac{n+2}{2}\rfloor$.

It means that only the two "middle levels" should be considered when looking for the maximum. However, we were not able to determine it:

CONJECTURE 1. *$f_n(V,B_3)$ is equal to the size of the following family*

$$\left\{\{x\}\cup A\colon x,y\notin A,\ |A| = \lfloor\tfrac{n-2}{2}\rfloor\right\}\cup\left\{\{y\}\cup A\colon x,y\notin A,\ |A| = \lfloor\tfrac{n-2}{2}\rfloor\right\}\cup\left\{\{x,y\}\cup A\colon x,y\notin A,\ |A| = \lfloor\tfrac{n-2}{2}\rfloor\right\}$$

$$\cup\left\{A\colon x,y\notin A,\ |A| = \lfloor\tfrac{n}{2}\rfloor\right\}.$$

Note that for odd n, the construction of Theorem 1 has the same number of subsets.

We conjecture that the statement of Theorem 5 is true in a more general context. It is always possible to push the subsets of the optimal family to the middle:

CONJECTURE 2. *For any system of directed graphs $H_1, \ldots, H_k$ there is an integer $M = M(H_1,\ldots,H_k)$ independent of n such that $f_n(H_1,\ldots,H_k)$ can be realized with a family Y of subsets A satisfying*

$$\left| |A| - \frac{n}{2} \right| \le M.$$

There are some ways to weaken the condition of Theorem 5, but we were not able to prove Conjecture 2 for the simple case $k = 1$, $H_1 = V$. One can push the sets to the middle from below by Theorem 4, but nothing ensures the same from the top. In spite of these difficulties we have some results concerning $f_n(V)$. This is the purpose of the next section.

4. NO V

THEOREM 6.

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{1}{n} + o\left(\frac{1}{n}\right)\right) \le f_n(V) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{2}{n}\right), \tag{11}$$

P r o o f (Mixed with some remarks).

1. The construction of Theorem 1 does not contain any V or Λ, consequently, it is good here. If n is odd then

$$2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} = \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{1}{n}\right),$$

the lower estimate is proved. However, for even n it gives only $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, we need a better construction.

2. Let Y consist of all the $\lfloor \frac{n}{2} \rfloor$-element sets and some of the $\lfloor \frac{n}{2} \rfloor+1$-element ones with the property that their symmetric difference is at least 4. It is obvious, that $C_n(Y)$ contains no V for this Y. $|Y| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ + number of $\frac{n}{2}+1$-element sets in Y. Associate a 0,1-sequence with each of the $\lfloor \frac{n}{2} \rfloor+1$-element sets in the natural way. These 0,1-sequences of length n contain $\lfloor \frac{n}{2} \rfloor+1$ 1's and any two differs in at least 4 places. The number of 1's in a sequence is called its weight. In coding theory, $A(n,d,w)$ denotes the maximal number of 0,1-sequences of length n, with weight w and satisfying the property that they differ in at least d places. Using this notation, we obtained the lower estimate

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} + A\left(n, 4, \lfloor \tfrac{n}{2} \rfloor + 1\right) \le f_n(V). \tag{12}$$

Graham and Sloane [3] constructed a code proving

$$A\left(n,4,\lfloor\tfrac{n}{2}\rfloor+1\right) \geq \frac{\binom{n}{\lfloor\frac{n}{2}\rfloor+1}}{n}.$$

By (12) it proves the lower estimate of Theorem 6. On the other hand the best know upper estimate on $A\left(n,4,\lfloor\frac{n}{2}\rfloor+1\right)$ is about the double of the above lower estimate. (12) shows that we cannot hope to improve easily the right hand side of (11).

3. Now we give an example showing that none of the above constructions are always best possible. For $n = 5$ the first construction gives $2\binom{4}{2} = 12$, the second one 12 (all 2-element sets and at most two 3-element ones), again. However, the following family of subsets contains 13 sets and it does not contain any V:

$\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,3\}$, $\{2,4\}$, $\{2,5\}$, $\{3,4\}$, $\{3,5\}$, $\{1,2,3\}$, $\{1,4,5\}$, $\{2,4,5\}$, $\{3,4,5\}$.

4. Now we start to prove the upper estimate of the theorem. Let Y be an optimal family satisfying the conditions of the theorem. By Theorem 4 it can be supposed that

(14) all its members are of size $\geq \lfloor\frac{n}{2}\rfloor$.

Suppose $A,B\in Y$, $A\subset B$, $|B-A| \geq 2$. Take an arbitrary element x of $B-A$. $A\cup\{x\}$ can not be in Y because $C_n(Y)$ does not contain a V. It is easy to verify that for the family $Y' = (Y-\{B\})\cup\{A\cup\{x\}\}$ $C_n(Y')$ contains no V. Therefore we may suppose that

(15) $$A,B\in Y,\ A\subset B \text{ implies } |B-A| = 1.$$

It follows that $C_n(Y)$ contains no directed path of length ≥ 2 and the edges connect sets with difference 1. Therefore Y has the form

$$Y = \{E_1,\ldots,E_r\} \cup \bigcup_{i=1}^{r} Z_i,$$

where Z_i is a set of certain $|E_i|-1$-element subsets of E_i. Summarizing this section, we may suppose that the optimal Y satisfies (14) and (15).

5. $C = \{C_0,C_1,\ldots,C_n\}$ with $C_0\subset C_1\subset\ldots\subset C_n$, $|C_i| = i$ $(0 \leq i \leq n)$ is called a *chain*. Let $l(Z_i)$ denote the number of chains such that either $C_j = E_i$ or $C_j\in Z_i$ for some j. We will prove the inequality

(16) $$\frac{l(Z_i)}{|Z_i|+1} \quad \frac{\left(\lfloor\frac{n}{2}\rfloor+1\right)!\left(n-\lfloor\frac{n}{2}\rfloor\right)!}{\lfloor\frac{n}{2}\rfloor+2}.$$

Using the notations $|E_i| = e$, $|Z_i| = z$, it is easy to count $l(Z_i) = e!(n-e)!+$ $+z(e-1)!(n-e)!(n-e)$. Therefore we have to find the minimum of

(17) $$\frac{e!(n-e)!+z(e-1)!(n-e)!(n-e)}{z+1}$$

where $0 \le z \le e$ and either $e \ge \lfloor\frac{n}{2}\rfloor+1$ or $e = \frac{n}{2}$, $z = 0$ follows from (14). (17) can be rewritten into the form

(18) $$(e-1)!(n-e)!(n-e) + \frac{e!(n-e)!-(e-1)!(n-e)!(n-e)}{z+1}.$$

Here $e!(n-e)!-(e-1)!(n-e)!(n-e) \ge 0$ iff $2e \ge n$. The latter condition holds with one exception: n is odd, $e = \frac{n-1}{2}$, $z = 0$. In this exceptional case it is easy to check that (17) is larger than the right hand side of (16). Thus we may suppose $2e \ge n$. In this case the denominator in (18) is ≥ 0, (18) is minimal for the maximal z: $z = e$. Therefore we need the minimum

(19) $$\min_{e \ge \frac{n}{2}} \left\{\frac{e!(n-e+1)!}{e+1}\right\}$$

in place of (17). With an easy computation, one can see that the minimum in (19) is attained for $e = \lfloor\frac{n+2}{2}\rfloor$. This proves (16).

6. The total number of chains is simply $n!$. By the suppositions, if a chain contains two different elements of Y, then one of them must be an E_i and the other one be an element of Z_i. Consequently

(20) $$\sum_{i=1}^{r} l(Z_i) \le n!.$$

Now (16) and (20) imply the inequalities

$$|Y| = \sum_{i=1}^{r} (|Z_i|+1) \le \frac{\lfloor\frac{n}{2}\rfloor+2}{\left(\lfloor\frac{n}{2}\rfloor+1\right)!\left(n-\lfloor\frac{n}{2}\rfloor\right)!} \sum_{i=1}^{r} l(Z_i) \le \frac{\lfloor\frac{n}{2}\rfloor+2}{\left(\lfloor\frac{n}{2}\rfloor+1\right)!\left(n-\lfloor\frac{n}{2}\rfloor\right)!} n! =$$

$$= \binom{n}{\lfloor\frac{n}{2}\rfloor}\left(1+\frac{1}{\lfloor\frac{n}{2}\rfloor+1}\right).$$

This is somewhat stronger than the right hand side of (11). It should be mentioned that Kleitman [5] proved almost the same upper estimate under a much weaker condition: If Y is a family of different subsets of an n-element set X and it contains no three different members with $A \cap B = C$ then $|Y| \le \binom{n}{\frac{n}{2}} + \frac{2^n}{n} + o\left(\binom{n}{n/4}\right)$.□

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