On the Number of Unions in a Family of Sets[a]

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The aim of this paper is to prove the following theorem, which was posed as a conjecture in [1].

THEOREM: Let $\mathscr{A} = (A_1, A_2, \ldots, A_n)$ be an n-member family of subsets of an n-element set X and $\mathscr{U} = \{\bigcup_{i \in I} A_i\colon I \subseteq \{1, 2, \ldots, n\}\}$ the set of unions. If $|\mathscr{U}| > 2^{n-1}$, then $|\mathscr{U}| = 2^{n-1} + 2^m$ for some $m = 0, 1, \ldots, n-1$.

A somewhat related result is a special case of a theorem of Tverberg [4]: If the family $\mathscr{A} = (A_1, \ldots, A_f)$ of subsets of the n-element set X has $f \geq n+1$ members, then there exist disjoint nonempty $I_1, I_2 \subseteq \{1, \ldots, f\}$ such that $\bigcup_{i \in I_1} A_i = \bigcup_{i \in I_2} A_i$ (see also Lindström [2]). Recently, Lindström [3] has found that for $f \geq n+2$ even $\bigcup_{i \in I_1} A_i = \bigcup_{i \in I_2} A_i$ and $\bigcap_{i \in I_1} A_i = \bigcap_{i \in I_2} A_i$ simultaneously can be achieved. Their proofs use linear algebraic technique.

Our proof is straightforward. We determine all possible families of sets for which the hypotheses of the theorem are satisfied:

SUPPLEMENT TO THE THEOREM: $|\mathscr{U}| > 2^{n-1}$ if and only if there exist a p-element subset $C \subseteq X$ $(0 \leq p \leq n)$ and a p-member family $\mathscr{C}$ of subsets of C (called the core of $\mathscr{A}$) such that $\mathscr{C}$ is one of the families listed in TABLE 1 and $\mathscr{A}$ consists of the members of $\mathscr{C}$ and the $n-p$ one-element subsets of $X \setminus C$.

The strong constraint for the number of unions is quite surprising if contrasted to the many ways the possible values can be attained.

COROLLARY: The number of essentially different families for which $|\mathscr{U}| = 2^{n-1} + 2^{n-k}$ is

$$\frac{k^2 + k - 4}{2} \quad \text{for} \quad 6 \leq k \leq n,$$

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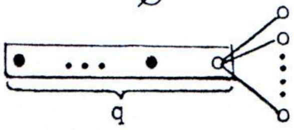
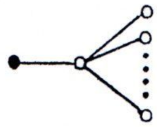
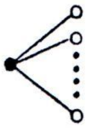
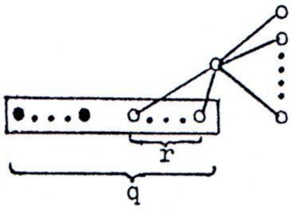
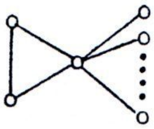
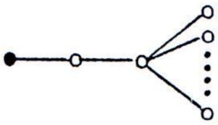
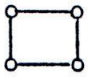
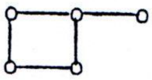
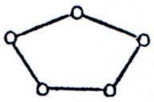
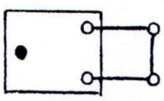
TABLE 1. Possible Cores

Name	Family[a]	Number of Points/Sets (p)	Number of Unions[b] (u)	$2^{-p}u$
∅	∅	0	1	1
$\mathscr{C}_1(p, q)$, $p \geq 3$, $p \geq q \geq 3$		p	$2^{p-1} + 1$	$\frac{1}{2} + \frac{1}{2^p}$
$\mathscr{C}_1(p, 2)$, $p \geq 3$		p	$2^{p-1} + 1$	$\frac{1}{2} + \frac{1}{2^p}$
$\mathscr{C}_1(p, 1)$, $p \geq 2$		p	$2^{p-1} + 1$	$\frac{1}{2} + \frac{1}{2^p}$
$\mathscr{C}_2(p, q, r)$, $p \geq 4$, $p - 1 \geq q \geq 3$, $q \geq r \geq 1$, except $p - 1 = q$, $r = 1$		p	$2^{p-1} + 1$	$\frac{1}{2} + \frac{1}{2^p}$
$\mathscr{C}_2(p, 2, 2)$, $p \geq 3$		p	$2^{p-1} + 1$	$\frac{1}{2} + \frac{1}{2^p}$
$\mathscr{C}_2(p, 2, 1)$, $p \geq 4$		p	$2^{p-1} + 1$	$\frac{1}{2} + \frac{1}{2^p}$
$\mathscr{C}_3$		4	10	$\frac{1}{2} + \frac{2}{16}$
$\mathscr{C}_4$		4	9	$\frac{1}{2} + \frac{1}{16}$
$\mathscr{C}_5$		5	17	$\frac{1}{2} + \frac{1}{32}$
$\mathscr{C}_6$		5	17	$\frac{1}{2} + \frac{1}{32}$
$\mathscr{C}_7$		5	17	$\frac{1}{2} + \frac{1}{32}$

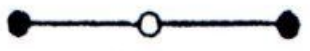
[a]Notation: • one-element set; o——o two-element set; [o...o] (q) q-element set ($q \geq 3$).

[b]The number of unions can be determined by direct calculations.

and for $1 \le k \le 5$ it is given by the following table:

k	1	2	3	4	5
Number of families	1	1	5 if $n \ge 4$ 4 if $n = 3$	9	16

REMARK: In the theorem it is an important assumption that the number of members of $\mathscr{A}$ is the same as the number of elements of X, as shown by the following examples.

Family	The Number of Points	The Number of Sets	The Number of Unions
	3	4	7
	4	3	7

We start with four simple lemmas.

LEMMA 1: No member of $\mathscr{A}$ is the union of some other members of $\mathscr{A}$. In particular, the members of $\mathscr{A}$ are pairwise different and none of them is the empty set.

Proof: If $A_j = \bigcup_{i \in I} A_i$ for some $1 \le j \le n$, $I \subseteq \{1, 2, \ldots, n\}$, $j \notin I$, then in any union A_j can be substituted by $\bigcup_{i \in I} A_i$. Hence, $|\mathscr{U}| \le 2^{n-1}$. □

LEMMA 2: Let $\mathscr{B} = (A_{i_1}, \ldots, A_{i_s})$ $(1 \le i_1 < \cdots < i_s \le n)$ be a subfamily of $\mathscr{A}$ and $\mathscr{V} = \{\bigcup_{i \in I} A_i : I \subseteq \{i_1, \ldots, i_s\}\}$ the set of unions in $\mathscr{B}$. Then

$$2^{-s}|\mathscr{V}| \ge 2^{-n}|\mathscr{U}|.$$

Proof: As $\mathscr{U} = \{V \cup \bigcup_{j \in J} A_j : V \in \mathscr{V},\ J \subseteq \{1, \ldots, n\} \backslash \{i_1, \ldots, i_s\}\}$ we have $|\mathscr{U}| \le 2^{n-s}|\mathscr{V}|$. □

LEMMA 3: If $|\mathscr{U}| > 3 \cdot 2^{n-2}$, then $\mathscr{A}$ consists of the one-element subsets of X and so $|\mathscr{U}| = 2^n$.

Proof: If $|A_1| \ge 2$, then $|\mathscr{U}| \le |\{Y \subseteq X : Y \supseteq A_1\}| + |\{\bigcup_{i \in I} A_i : I \subseteq \{2, \ldots, n\}\}| \le 2^{n-2} + 2^{n-1} = 3 \cdot 2^{n-2}$. □

LEMMA 4: If $|\mathscr{U}| > 2^{n-1}$, then $|A_i| \ge 3$ for at most one member A_i of $\mathscr{A}$.

Proof: If $|A_1| \ge 3$ and $|A_2| \ge 3$, then $|\mathscr{U}| \le |\{Y \subseteq X : Y \supseteq A_1\}| + |\{Y \subseteq X : Y \supseteq A_2\}| + |\{\bigcup_{i \in I} A_i : I \subseteq \{3, \ldots, n\}\}| \le 2^{n-3} + 2^{n-3} + 2^{n-2} = 2^{n-1}$. □

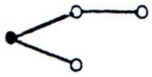
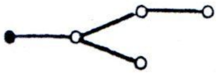
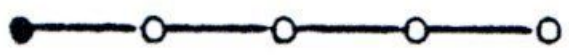
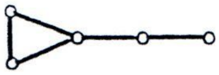
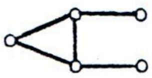
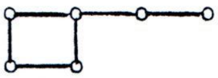
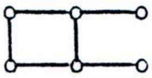
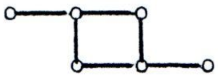
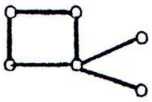
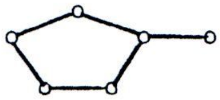
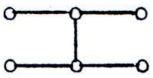
Now we turn to the proof of the theorem and its supplement. In virtue of Lemma 4 we shall distinguish two cases:

Case 1: $|A_i| \le 2$ for each $i = 1, \ldots, n$;

Case 2: $|A_1| = q \ge 3$ and $|A_i| \le 2$ for $i = 2, \ldots, n$.

(Here and in the forthcoming considerations we freely use renumbering of members and subfamilies of $\mathscr{A}$.)

TABLE 2. Some Families with Few Unions

Name	Family	Points (p)	Number of Sets (s)	Unions (u)	$2^{-s}u$
$\mathscr{B}_1$		4	4	8	$\leq \frac{1}{2}$
$\mathscr{B}_2$		5	5	15	$\leq \frac{1}{2}$
$\mathscr{B}_3$		5	5	16	$\leq \frac{1}{2}$
$\mathscr{B}_4$		5	5	15	$\leq \frac{1}{2}$
$\mathscr{B}_5$		5	5	15	$\leq \frac{1}{2}$
$\mathscr{B}_6$		6	6	28	$\leq \frac{1}{2}$
$\mathscr{B}_7$		6	6	28	$\leq \frac{1}{2}$
$\mathscr{B}_8$		6	6	28	$\leq \frac{1}{2}$
$\mathscr{B}_9$		6	6	31	$\leq \frac{1}{2}$
$\mathscr{B}_{10}$		6	6	29	$\leq \frac{1}{2}$
$\mathscr{B}_{11}$		5	4	12	$\leq \frac{3}{4}$
$\mathscr{B}_{12}$		6	5	23	$\leq \frac{3}{4}$

Referring to Case 1, let us regard the two-element members of $\mathscr{A}$ as edges of a graph. First suppose that this graph is connected. Then it has at least $n - 1$ edges; hence there are two possibilities:

Possibility 1: $|A_1| = 1$ and $|A_i| = 2$ for $i = 2, \ldots, n$;
Possibility 2: $|A_i| = 2$ for each $i = 1, \ldots, n$.

In Possibility 1 $\mathscr{A}$ is a rooted tree. Owing to Lemma 2, it cannot contain a subfamily $\mathscr{B}_1$, $\mathscr{B}_2$, or $\mathscr{B}_3$ (see TABLE 2). We prove by induction on n that this forces $\mathscr{A}$ to be $\mathscr{C}_1(n, 1)$, $\mathscr{C}_1(n, 2)$, or $\mathscr{C}_2(n, 2, 1)$ (see TABLE 1). If $n = 2$, then $\mathscr{A} = \mathscr{C}_1(2, 1)$.

TABLE 3. Rooted Tree on n Vertices

Distance of x from the Root	The Remaining Rooted Tree on $n-1$ Vertices				
	$\mathscr{C}_1(2, 1)$	$\mathscr{C}_1(n-1, 1)$ $(n-1 \geq 3)$	$\mathscr{C}_1(3, 2)$	$\mathscr{C}_1(n-1, 2)$ $(n-1 \geq 4)$	$\mathscr{C}_2(n-1, 2, 1)$ $(n-1 \geq 4)$
1	$\mathscr{C}_1(3, 1)$	$\mathscr{C}_1(n, 1)$	$\mathscr{B}_1$	$\supseteq\mathscr{B}_1$	$\supseteq\mathscr{B}_1$
2	$\mathscr{C}_1(3, 2)$	$\supseteq\mathscr{B}_1$	$\mathscr{C}_1(4, 2)$	$\mathscr{C}_1(n, 2)$	$\supseteq\mathscr{B}_2$
3	—	—	$\mathscr{C}_2(4, 2, 1)$	$\supseteq\mathscr{B}_2$	$\mathscr{C}_2(n, 2, 1)$
4	—	—	—	—	$\supseteq\mathscr{B}_3$

Let $n \geq 3$. Deleting a nonroot vertex x of valency 1 and the edge incident to it, we obtain a rooted tree on $n-1$ vertices. By the induction hypothesis it is $\mathscr{C}_1(n-1, 1)$, $\mathscr{C}_1(n-1, 2)$, or $\mathscr{C}_2(n-1, 2, 1)$. Then the rooted tree on n vertices is given by TABLE 3.

In Possibility 2 the graph is connected and it has n edges; hence, it contains a unique circuit. If the length of the circuit is 3, then $\mathscr{A}$ cannot contain $\mathscr{B}_4$ or $\mathscr{B}_5$, so $\mathscr{A}$ is $\mathscr{C}_2(n, 2, 2)$. If the length is 4, then the exclusion of $\mathscr{B}_6$, $\mathscr{B}_7$, $\mathscr{B}_8$, and $\mathscr{B}_9$ means that $\mathscr{A}$ can only be $\mathscr{C}_3$ or $\mathscr{C}_5$. If the length is 5, then $\mathscr{A}$ is the pentagon, $\mathscr{C}_6$, as it cannot contain $\mathscr{B}_{10}$.

Now we show that for any circuit of length $p \geq 6$ the number of unions is less than 2^{p-1}; hence, by Lemma 2, it cannot be contained in $\mathscr{A}$. Let $u(p)$ denote the number of unions for the circuit of length p, and $v(p)$ for the path of length p (with p vertices, $p-1$ edges). Let the vertices of the path be in order $x_1, x_2, x_3, \ldots, x_p$. Grouping the possible unions V into three sets according to $x_1 \notin V$, $x_1, x_2 \in V$, but $x_3 \notin V$ or $x_1, x_2, x_3 \in V$, we obtain the recurrence formula

$$v(p) = v(p-1) + v(p-3) + (v(p-1) - v(p-2))$$

$$= 2v(p-1) - v(p-2) + v(p+3), \qquad (1)$$

for $p \geq 4$. Now consider a circuit of length p: $x_0, x_1, x_2, \ldots, x_{p-2}, x_{p-1}$. Count the unions U in the following five groups: (i) $x_0 \notin U$, (ii) $x_0 \in U$, and $U \backslash \{x_0\}$ is a union of some edges of the path on $\{x_1, \ldots x_{p-1}\}$, (iii) $x_0, x_1 \in U$, $x_2 \notin U$, and $U \backslash \{x_0, x_1\}$ is a union of some edges of the path on $\{x_3, \ldots, x_{p-1}\}$, (iv) $x_0, x_{p-1} \in U$, $x_{p-2} \notin U$, and $U \backslash \{x_0, x_{p-1}\}$ is a union of some edges of the path on $\{x_1, \ldots, x_{p-3}\}$, (v) x_{p-1}, $x_0, x_1 \in U$, $x_{p-2}, x_2 \notin U$. Then we obtain

$$u(p) = v(p-1) + (v(p-1) - v(p-3)) + v(p-3) + v(p-3) + v(p-5)$$

$$= 2v(p-1) + v(p-3) + v(p-5), \qquad (2)$$

for $p \geq 6$. Combining **(1)** and **(2)**, we get a recurrence formula for $u(p)$ as well:

$$u(p) = 2u(p-1) - u(p-2) + u(p-3), \qquad (3)$$

for $p \geq 9$. Clearly, for $p = 1, 2, 3$ we have $v(p) = 2^{p-1}$. For $p \geq 4$ we can use **(1)** to obtain

p	1	2	3	4	5	6	7	...
$v(p)$	1	2	4	7	12	21	37	...

For $p = 3, 4, 5$ $u(p)$ can be determined by direct calculation (cf. TABLE 1) and then (2) can be applied to get:

p	3	4	5	6	7	8	···
$u(p)$	5	10	17	29	51	90	···

By induction (3) yields $u(p) > u(p-1)$ and $u(p) < 2u(p-1)$; hence, $u(p) < 2^{p-1}$ for $p \geq 6$, as we have claimed. (We remark that $u(p) = z_1^p + z_2^p + z_3^p$, where $z_1 = 1.75488$, $z_2 = 0.12256 + 0.74486i$, $z_3 = 0.12256 - 0.74486i$ are the roots of $z^3 - 2z^2 + z - 1 = 0$.)

So far we have determined the structure of $\mathscr{A}$ if the graph formed by the two-element members of $\mathscr{A}$ is connected. Now suppose that the graph is disconnected. Let the connected components be $X_1, \ldots, X_c$ ($c \geq 2$), and let $n_j = |X_j|$, $\mathscr{A}_j$ be the family of those members (including the one-element sets) of $\mathscr{A}$ that are contained in X_j and $\mathscr{U}_j$ be the set of unions of $\mathscr{A}_j$, $j = 1, \ldots, c$. Obviously, $\mathscr{U} = \{Y_1 \cup \cdots \cup Y_c: Y_1 \in \mathscr{U}_1, \ldots, Y_c \in \mathscr{U}_c\}$; hence,

$$2^{-n}|\mathscr{U}| = \prod_{j=1}^{c} 2^{-n_j}|\mathscr{U}_j|. \tag{4}$$

By assumption, $2^{-n}|\mathscr{U}| > \frac{1}{2}$; hence, each factor $2^{-n_j}|\mathscr{U}_j > \frac{1}{2}$. Thus $|\mathscr{U}_j| > 2^{n_j-1}$, so $|\mathscr{A}_j| \geq n_j$. Now

$$n = |\mathscr{A}| = \sum_{j=1}^{c} |\mathscr{A}_j| \geq \sum_{j=1}^{c} n_j = n$$

implies that $|\mathscr{A}_j| = n_j$ for each $j = 1, \ldots, c$. Hence the assumptions of the theorem are satisfied by the n_j-member family $\mathscr{A}_j$ on the n_j-element set X_j, $j = 1, \ldots, c$. Our previous considerations tell us that any connected component $\mathscr{A}_j$ is either one of $\mathscr{C}_1(n_j, 1)$ (for $n_j \geq 2$), $\mathscr{C}_1(n_j, 2)$ (for $n_j \geq 3$), $\mathscr{C}_2(n_j, 2, 1)$ (for $n_j \geq 4$), $\mathscr{C}_2(n_j, 2, 2)$ (for $n_j \geq 3$), $\mathscr{C}_3$ (for $n_j = 4$), $\mathscr{C}_5$ (for $n_j = 5$), and $\mathscr{C}_6$ (for $n_j = 5$) or $n_j = 1$. Thus the factors on the right-hand side of (4) have the form $1/2 + 1/2^{n_j}$, except for $\mathscr{A}_j = \mathscr{C}_3$, when $2^{-n_j}|\mathscr{U}_j| = \frac{5}{8}$ (see TABLE 1). Their product is greater than one-half only in the following cases: $|\mathscr{U}_1| > 2^{n_1-1}$ and $|\mathscr{U}_j| = 2^{n_j}$ for $j = 2, \ldots, c$, or $|\mathscr{U}_1| = 3 \cdot 2^{n_1-2}$, $|\mathscr{U}_2| = 3 \cdot 2^{n_2-2}$, and $|\mathscr{U}_j| = 2^{n_j}$ for $j = 3, \ldots, c$, since $\frac{3}{4} \cdot \frac{5}{8} < \frac{1}{2}$ and $\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} < \frac{1}{2}$. By Lemma 3, $|\mathscr{U}_j| = 2^{n_j}$ implies $n_j = 1$. Moreover, $|\mathscr{U}_j| = 3 \cdot 2^{n_j-2}$ for only $n_j = 2$, $\mathscr{A}_j = \mathscr{C}_1(2, 1)$. Hence, in Case 1 the core of $\mathscr{A}$ can be $\varnothing$, $\mathscr{C}_1(p, 1)$ ($2 \leq p \leq n$), $\mathscr{C}_1(p, 2)$ ($3 \leq p \leq n$), $\mathscr{C}_2(p, 2, 1)$ ($4 \leq p \leq n$), $\mathscr{C}_2(p, 2, 2)$ ($3 \leq p \leq n$), $\mathscr{C}_3$ (for $n \geq 4$), $\mathscr{C}_5$ (for $n \geq 5$), $\mathscr{C}_6$ (for $n \geq 5$), and $\mathscr{C}_4$ (for $n \geq 4$) (the last one yielded by the second possibility). If the core has p elements, then

$$|\mathscr{U}| = \begin{cases} 2^{n-1} + 2^{n-p}, & \text{except when the core is } \varnothing \text{ or } \mathscr{C}_3, \\ 2^{n-1} + 2^{n-1}, & \text{if the core is } \varnothing, \\ 2^{n-1} + 2^{n-3}, & \text{if the core is } \mathscr{C}_3. \end{cases} \tag{5}$$

Referring to Case 2, let $|A_1| = q \geq 3$ and $|A_i| \leq 2$ for $i = 2, \ldots, n$. We shall consider the subfamily $\mathscr{A}' = \{A_2, \ldots, A_n\}$ and the set of its unions $\mathscr{U}' = \{\bigcup_{i \in I} A_i: I \subseteq \{2, \ldots, n\}\}$. Since $2^{n-1} < |\mathscr{U}| \leq |\mathscr{U}'| + |\{Y \subseteq X: Y \supseteq A_1\}| \leq |\mathscr{U}'| + 2^{n-3}$, it

TABLE 4. Result of Adding a New Vertex and Edge

Distance of the New Vertex from the Leftmost Vertex	The Tree on $n-1$ Vertices				
	$\mathscr{G}_1(1)$	$\mathscr{G}_1(2)$	$\mathscr{G}_1(n-1)$ $(n-1 \geq 3)$	$\mathscr{G}_2(4)$	$\mathscr{G}_2(n-1)$ $(n-1 \geq 5)$
1	$\mathscr{G}_1(2)$	$\mathscr{G}_1(3)$	$\mathscr{G}_1(n)$	$\mathscr{B}_{11}$	$\supseteq \mathscr{B}_{11}$
2	—	$\mathscr{G}_1(3)$	$\mathscr{G}_2(n)$	$\mathscr{G}_2(5)$	$\supseteq \mathscr{B}_{12}$
3	—	—	—	$\mathscr{G}_2(5)$	$\mathscr{G}_2(n)$
4	—	—	—	$\mathscr{B}_{11}$	$\supseteq \mathscr{B}_{11}$

follows that $|\mathscr{U}'| > 3 \cdot 2^{n-3}$. Similarly, as in Case 1, we consider the graph formed by the two-element members of $\mathscr{A}'$. If this graph is connected, then it is a tree. By Lemma 2 this tree cannot contain $\mathscr{B}_{11}$ or $\mathscr{B}_{12}$ (see TABLE 2). Then we can prove by induction on n that $\mathscr{A}'$ is one of the two graphs shown in FIGURE 1. Indeed, adding a new vertex and edge yields the result contained in TABLE 4.

If the graph formed by the two-element members of $\mathscr{A}'$ is disconnected, then let the connected components be $X_1, \ldots, X_c$ $(c \geq 2)$. Let $n_j = |X_j|$, $\mathscr{A}'_j$ be the family of those members (including the one-element sets) of $\mathscr{A}'$ that are contained in X_j, and $\mathscr{U}'_j$ be the set of unions of members of $\mathscr{A}'_j$, $j = 1, \ldots, c$. Obviously, $\mathscr{U}' = \{Y_1 \cup \cdots \cup Y_c \colon Y_1 \in \mathscr{U}'_1, \ldots, Y_c \in \mathscr{U}'_c\}$; hence,

$$2^{-n}|\mathscr{U}'| = \prod_{j=1}^{c} 2^{-n_j}|\mathscr{U}'_j|. \tag{6}$$

By assumption $\frac{1}{2} \geq 2^{-n}|\mathscr{U}'| > \frac{3}{8}$. Each factor in the right-hand side is ≤ 1. If $|\mathscr{A}'_i| \leq n_i - 2$, then $2^{-n_i}|\mathscr{U}'_i| \leq \frac{1}{4}$, and similarly, if $|\mathscr{A}'_i| = n_i - 1$ and $|\mathscr{A}'_k| = n_k - 1$ $(i \neq k)$, then $2^{-n_i}|\mathscr{U}'_i| \leq \frac{1}{2}$ and $2^{-n_k}|\mathscr{U}'_k| \leq \frac{1}{2}$; hence, in both cases $\prod_{j=1}^{c} 2^{-n_j}|\mathscr{U}'_j| \leq \frac{1}{4}$, so these cases cannot occur. On the other hand, $\mathscr{A}'$ has $n - 1 = \sum_{j=1}^{c} n_j - 1$ members; hence, we infer that $|\mathscr{A}'_1| = n_1 - 1$ and $|\mathscr{A}'_j| = n_j$ for $j = 2, \ldots, c$. Then $2^{-n_1}|\mathscr{U}'_1| \leq \frac{1}{2}$; thus, $2^{-n_j}|\mathscr{U}'_j| > \frac{3}{4}$ $(j = 2, \ldots, c)$. Lemma 3 applied to the n_j-member family $\mathscr{A}'_j$ on the n_j-element set X_j yields that $n_j = 1$ for $j = 2, \ldots, c$. Hence $2^{-n_1}|\mathscr{U}'_1| = 2^{-n}|\mathscr{U}| > \frac{3}{8}$. From the previous considerations it follows that $\mathscr{A}'_1$ is a tree of the form $\mathscr{G}_1(n_1)$ (for $1 \leq n_1 \leq n$; $\mathscr{G}_1(1)$ is an empty family) or $\mathscr{G}_2(n_1)$ (for $4 \leq n_1 \leq n$).

Now we take into account the set A_1 $(|A_1| = q \geq 3)$ as well. For notational convenience, let $k = n_1$. If the tree $\mathscr{A}'_1$ is $\mathscr{G}_1(k)$ for some $1 \leq k \leq n$, then already $|\mathscr{U}'| = 2^{n-1}$ and A_1 can be any set not belonging to $\mathscr{U}'$ (cf. Lemma 1). If $\mathscr{A}'_1$ is $\mathscr{G}_1(1)$, then $A_1 \supseteq X_1$, and we obtain that the core of $\mathscr{A}$ is $\mathscr{C}_1(q, q)$ $(3 \leq q \leq n)$ (see TABLE

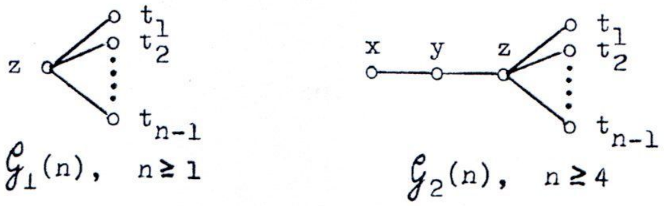


FIGURE 1.

TABLE 5.

	The Value of N		The value of $2^m N$	
$A_1 \cap \{x, y, z\}$	If $r = 0$	If $r > 0$	If $r = 0$	If $r > 0$
$\varnothing$	0	2	0	$2^{n-q-k+r+1}$
$\{x\}$	2^{k-3}	$2^{k-3-r} + 2$	2^{n-q-2}	$2^{n-q-2} + 2^{n-q-k+r+2}$
$\{y\}$	1	2	$2^{n-q-k+1}$	$2^{n-q-k+r+2}$
$\{z\}$	1	0	$2^{n-q-k+1}$	0
$\{x, y\}$	0	1	0	$2^{n-q-k+r+2}$
$\{x, z\}$	2^{k-3}	2^{k-3-r}	2^{n-q-1}	2^{n-q-1}
$\{y, z\}$	0	0	0	0
$\{x, y, z\}$	0	0	0	0

1). If $\mathscr{A}'_1$ is $\mathscr{G}_1(2)$, then $|A_1 \cap X_1| = 1$ and the core of $\mathscr{A}$ is $\mathscr{C}_1(q + 1, q)$ $(3 \leq q \leq n - 1)$. If $\mathscr{A}'_1$ is $\mathscr{G}_1(k)$ for $k \geq 3$, then let $r = |A_1 \cap \{t_1, \ldots, t_{k-1}\}|$ (for the notation, see FIG. 1). We have two possibilities: (i) $A_1 \cap X_1 = \{z\}$, and (ii) $z \notin A_1$, $r > 0$. In the first case, the core of $\mathscr{A}$ is $\mathscr{C}_1(q + k - 1, q)$ $(3 \leq q \leq n - k + 1)$; in the second, it is $\mathscr{C}_2(q + k - r, q, r)$ $(3 \leq q \leq n - k + r,\ 1 \leq r \leq q,\ r \leq k - 1)$. In any case, if the core has p elements, then

$$|\mathscr{U}'| = 2^{n-1} + 2^{n-p} \tag{7}$$

(cf. TABLE 1).

Now suppose $\mathscr{A}'_1 = \mathscr{G}_2(k)$, $4 \leq k \leq n$ (see FIG. 1). Then let $q = |A_1|$, $r = |A_1 \cap \{t_1, \ldots, t_{k-3}\}|$, $m = |X \backslash (A_1 \cup X_1)|$. Direct calculations show that $|\mathscr{U}'| = (3 \cdot 2^{k-3} + 1)2^{n-k}$, and for the number of unions that can be obtained only by using A_1, we obtain $|\mathscr{U} \backslash \mathscr{U}'| = N \cdot 2^m$, where N depends on $A_1 \cap X_1$, as shown in TABLE 5.

Hence we have

$$|\mathscr{U}| = (3 \cdot 2^{k-3} + 1)2^{n-k} + N \cdot 2^m. \tag{8}$$

Except for the case $A_1 \cap \{x, y, z\} = \{x\}$, $r > 0$, we have $2^m N \leq 2^{n-4}$ as $q \geq 3$ and $k - r \geq 3$. Then by (**8**) and $k \geq 4$,

$$|\mathscr{U}| \leq 3 \cdot 2^{n-3} + 2^{n-4} + 2^{n-4} = 2^{n-1}.$$

So the only remaining possibility is $A_1 \cap \{x, y, z\} = \{x\}$ and $r > 0$. Then

$$|\mathscr{U}| = 3 \cdot 2^{n-3} + 2^{n-k} + 2^{n-q-2} + 2^{n-q-k+r+2}.$$

Here $2^{n-k} \leq 2^{n-4}$, $2^{n-q-2} \leq 2^{n-5}$, $2^{n-q-k+r+2} \leq 2^{n-4}$. The sum is greater than 2^{n-1} if and only if $n - k = n - 4$ and $n - q - k + r + 2 = n - 4$, that is, $k = 4$, $r = 1$, $q = 3$. Thus we obtain the five-element core $\mathscr{C}_7$. For this

$$|\mathscr{U}| = 2^{n-1} + 2^{n-5} \tag{9}$$

(cf. TABLE 1). This concludes the proof of the theorem. □

Proof of the Corollary: If the core of $\mathscr{A}$ is empty, then the number of unions is $2^n = 2^{n-1} + 2^{n-1}$. If the core has p members, $1 \leq p \leq n$, then $|\mathscr{U}| = 2^{n-1} + 2^{n-p}$,

TABLE 6.

k	$\lvert\mathcal{U}\rvert = 2^{n-1} + 2^{n-k}$ for the Cores
1	$\varnothing$
2	$\mathcal{C}_1(2, 1)$
3	$\mathcal{C}_1(3, q)$, $q = 1, 2, 3$; $\mathcal{C}_2(3, 2, 2)$; $\mathcal{C}_3$ (if $n \geq 4$)
4	$\mathcal{C}_1(4, q)$, $q = 1, 2, 3, 4$; $\mathcal{C}_2(4, 3, r)$, $r = 2, 3$; $\mathcal{C}_2(4, 2, r)$, $r = 1, 2$; $\mathcal{C}_4$
5	$\mathcal{C}_1(5, q)$, $q = 1, 2, 3, 4, 5$; $\mathcal{C}_2(5, 4, r)$, $r = 2, 3, 4$; $\mathcal{C}_2(5, 3, r)$, $r = 1, 2, 3$; $\mathcal{C}_2(5, 2, r)$, $r = 1, 2$; $\mathcal{C}_5$; $\mathcal{C}_6$; $\mathcal{C}_7$
≥ 6	$\mathcal{C}_1(k, q)$, $q = 1, \ldots, k$; $\mathcal{C}_2(k, k-1, r)$, $r = 2, \ldots, k-1$; $\mathcal{C}_2(k, q, r)$, $q = 2, \ldots, k-2$, $r = 1, \ldots, q$

except for the core $\mathcal{C}_3$ when $\lvert\mathcal{U}\rvert = 2^{n-1} + 2^{n-3}$ [see **(5)**, **(7)**, and **(9)**]. The possible cores listed in TABLE 1 are pairwise different. Hence we have TABLE 6. Thus the number of possibilities is as given in the corollary. □

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