# A Simple Proof of a Theorem of Milner*

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Dedicated to the memory of Eric C. Milner

A new short proof is given for the following theorem of Milner: An intersecting, inclusion-free family of subsets of an $n$-element set has at most $\binom{n}{\lceil (n+1)/2 \rceil}$ members. © 1998 Academic Press

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Let $X$ be a finite set of $n$ elements and $\mathscr{F}$ a family of its distinct subsets. We say that $\mathscr{F}$ is *intersecting* if $F, G \in \mathscr{F}$ implies $F \cap G \neq \varnothing$. On the other hand, if $F \not\subseteq G$ holds for any two distinct members of $\mathscr{F}$ then it is called *Sperner* or *inclusion-free*. Milner [1] proved the following theorem.

THEOREM 1. *An intersecting Sperner family on an n-element set has at most*

$$\binom{n}{\left\lceil \frac{n+1}{2} \right\rceil}$$

*members.*

Milner's proof was not very complicated, but it used a non-trivial theorem of the present author [2]. Here we show that this was not necessary, as our elementary proof uses the cycle method [3].

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First we give a lemma which is a weighted version of the problem on the "cycle". Let $\mathscr{C}$ be a cyclic permutation of $X$.

LEMMA 1. *Let $B_1, ..., B_r$ be an intersecting Sperner family of intervals in $\mathscr{C}$. Then*

$$\sum_{i=1}^{r} \binom{n}{|B_i|} \leqslant n \binom{n}{\left\lceil \frac{n+1}{2} \right\rceil}. \tag{1}$$

*Proof.* Consider the intervals starting from a given element to the right along $\mathscr{C}$. Obviously at most one of them can be a $B_i$, since the family is Sperner. Therefore $r \leqslant n$ holds. This settles the proof if $n$ is odd, so let us suppose that $n$ is even. Two cases will be distinguished.

1. $r = n$. Suppose that $B_i$ starts at the $i$th element along $\mathscr{C}$. The Sperner property implies $|B_i| \leqslant |B_{i+1}|$. Consequently we have $|B_1| \leqslant |B_2| \leqslant \cdots \leqslant |B_n| \leqslant |B_1|$, that is, all $B$s must have the same size $k$. It cannot be $n/2$ because of the intersecting property. Then the (equal) terms of the left side of (1) are at most $\binom{n}{\lceil (n+1)/2 \rceil}$, proving the lemma for this case.

2. $r < n$. At most one of the complementing $n/2$-element intervals can occur among the $B$s, therefore at most $n/2$ of the terms on the left side of (1) can be $\binom{n}{n/2}$, the others cannot exceed $\binom{n}{(n/2)+1}$. That is, the left hand side of (1) is at most

$$\frac{n}{2}\binom{n}{\frac{n}{2}} + \left(\frac{n}{2} - 1\right)\binom{n}{\frac{n}{2}+1}.$$

It is easy to see that the latter expression is equal to the right-hand side of (1). ∎

*Proof of the Theorem.* Let $A_1, ..., A_m$ be an intersecting Sperner family on $X$. Consider the

$$\sum_{\mathscr{C}, A_i} \binom{n}{|A_i|} \tag{2}$$

for those pairs for which $A_i$ is an interval along $\mathscr{C}$. First fix $A_i$. It is easy to see that $A_i$ is an interval in exactly $|A_i|!\,(n - |A_i|)!$ cyclic permutations of $X$. Hence (2) is equal to

$$\sum_{i=1}^{m} \sum_{\{\mathscr{C}:\, A_i \text{ is interval}\}} \binom{n}{|A_i|} = \sum_{i=1}^{m} |A_i|!\,(n - |A_i|)! \binom{n}{|A_i|} = mn!. \tag{3}$$

Fix now $\mathscr{C}$ in (2). The lemma gives an upper bound on the sum with a given $\mathscr{C}$. That is, (2) is upperbounded by

$$(n-1)!\, n \binom{n}{\left\lceil \frac{n+1}{2} \right\rceil}.$$

Comparing this with (3), the following inequality is obtained:

$$mn! \leqslant n! \binom{n}{\left\lceil \frac{n+1}{2} \right\rceil},$$

completing the proof. ▮

## REFERENCES


1. E. C. Milner, A combinatorial theorem on systems of sets, *J. London Math. Soc.* **43** (1968), 204–206.
2. G. Katona, Intersection theorems for systems of finite sets, *Acta. Math. Acad. Sci. Hungar.* **15** (1964), 329–337.
3. G. O. H. Katona, A simple proof of the Erdős–Chao Ko–Rado theorem, *J. Combin. Theory Ser. B* **13** (1972), 183–184.