Largest families without an r-fork

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1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$ be a finite set, $\mathcal{F} \subset 2^{[n]}$ a family of its subsets. In the present paper max $|\mathcal{F}|$ will be investigated under certain conditions on the family $\mathcal{F}$. The well-known Sperner theorem ([8]) was the first such theorem.

Theorem 1.1 *If $\mathcal{F}$ is a family of subsets of $[n]$ without inclusion ($F, G \in \mathcal{F}$ implies $F \not\subset G$) then*

$$|\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

holds, and this estimate is sharp as the family of all $\lfloor \frac{n}{2} \rfloor$-element subsets shows.

There is a very large number of generalizations and analogues of this theorem. Here we will mention only some results when the condition on $\mathcal{F}$ excludes certain configurations what can be expressed by inclusion, only. That is, no intersections, unions, etc. are involved. The first such generalization was obtained by Erdős [3]. The family of k distinct sets with mutual inclusions, $F_1 \subset F_2 \subset \ldots F_k$ is called a *chain of lenght k*. It will be simply denoted by P_k. Let $\mathrm{La}(n, P_k)$ denote the largest family $\mathcal{F}$ without a chain of lenght k.

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Theorem 1.2 [3] $\mathrm{La}(n, P_{k+1})$ *is equal to the sum of the k largest bimomial coefficients of order n.*

Let V_r denote the *r-fork*, that is the following family of distinct sets: $F \subset G_1, F \subset G_2, \dots F \subset G_r$. The quantity $\mathrm{La}(n, V_r)$, that is, the largest family on n elements containing no V_r was first (asymptotically) determined for $r = 2$.

Theorem 1.3 [5]

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right) \le \mathrm{La}(n, V_2) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{2}{n}\right).$$

[9] has correctly determined the main term for V_r (in a somewhat more general form), proving the following theorem.

Theorem 1.4 [9]

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{r}{n}+O\left(\frac{1}{n^2}\right)\right) \le \mathrm{La}(V_{r+1}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+2\frac{r^2}{n}+o\left(\frac{1}{n}\right)\right).$$

The main aim of the present paper (Section 2) is to improve the (upper estimate of the) second term. For the sake of completeness, we repeat the proof of the lower estimate, too.

Theorem 1.5

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{r}{n}+O\left(\frac{1}{n^2}\right)\right) \le \mathrm{La}(V_{r+1}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+2\frac{r}{n}+O\left(\frac{1}{n^2}\right)\right).$$

In Section 4 we show how the upper bound in the general form of Tran's theorem [9] can be attacked by the usage of Theorem 1.5. In most cases a better second term is obtained.

In Section 3 we give estimates for the maximum size of a family $\mathcal{F} \subset 2^{[n]}$ containing no $r + s$ distinct members satisfying $A_1, \dots, A_s \subset B_1, \dots, B_r$.

2 An auxiliary inequality

If $\mathcal{F} \subset 2^{[n]}$ is a family let f_i denote the number of its i-element members. The Sperner theorem, has the following sharpening, known as the YBLM-inequality ([10], [1], [6], [7]).

Theorem 2.1 *Let $\mathcal{F}$ be a family of subsets of $[n]$ without inclusion. If $f_i = |\{F :\ F \in \mathcal{F}, |F| = i\}|$ then*

$$\sum_{i=0}^{n} \frac{f_i}{\binom{n}{i}} \leq 1 \tag{2.1}$$

holds.

The main ingredient of our Theorem 1.5 is given below.

Theorem 2.2 *Suppose that the family $\mathcal{F}$ contains no $r+1$-fork $(0 < r)$ and $[n] \notin \mathcal{F}$. Then*

$$\sum_{i=0}^{n-1} \frac{f_i}{\binom{n}{i}} \left(1 - \frac{r}{n-i}\right) \leq 1. \tag{2.2}$$

Proof. A *chain* is a family $\mathcal{C} = \{C_0, C_1, \ldots, C_n\},\ C_0 \subset C_1 \subset \ldots \subset C_n$ where $|C_i| = i$ $(0 \leq i \leq n)$. We say that a chain $\mathcal{C}$ *goes through* a family $\mathcal{F}$, if $\mathcal{C} \cup \mathcal{F} \neq \emptyset$. Let $C(F)(F \in \mathcal{F})$ be the set of all chains going through F. We have $|C(F)| = |F|!(n-|F|)!$. Similarly, let $C(F_1, F_2)$ denote the set of those chains which go through both F_1, F_2. This set is empty unless one of them includes the other one.

The following easy lemma, which is actually a primitive sieve, will be applied for chains.

Lemma 2.3 *If $X_1, \ldots, X_u$ are subsets of a set X, then*

$$|X_1| + \cdots + |X_u| \leq |X| + \sum_{i<j} |X_i \cap X_j| \tag{2.3}$$

holds.

Proof. An element of X which is outside of all X_i is not counted on the left hand side, but it is counted on the right hand side. An element which belongs to exactly one X_i is counted exactly once on both sides. Finally, if an element belongs to exactly $2 \leq v$ of X_is then it is counted v times on the left hand side and $1 + \binom{v}{2}$ times on the right hand side. The obvious inequality $v \leq 1 + \binom{v}{2} (2 \leq v)$ completes the proof. $\square_{\mathrm{L}}$

Let X be the set of all chains in $[n]$, while the X_is be the chains going through a given member of $\mathcal{F}$. (2.3) becomes

$$\sum_{F \in \mathcal{F}} |C(F)| - \sum_{F_1, F_2 \in \mathcal{F}, F_1 \subset F_2} |C(F_1, F_2)| \leq n!. \tag{2.4}$$

Introduce the notation $\mathcal{U}(F) = \{G : \ G \in \mathcal{F}, F \subset G\}$. Rewrite (2.4) using this notation.

$$\sum_{F \in \mathcal{F}} |C(F)| - \sum_{F \in \mathcal{F}} \sum_{G \in \mathcal{U}(F)} |C(F, G)| \leq n!. \tag{2.5}$$

Here $|C(F, G)| = |F|!|G - F|!(n - |G|)!$, (2.5) can be written in the following form.

$$\sum_{F \in \mathcal{F}} |F|!(n - |F|)! - \sum_{F \in \mathcal{F}} \sum_{G \in \mathcal{U}(F)} |F|!|G - F|!(n - |G|)! \leq n!.$$

Divide it by $n!$.

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} - \sum_{F \in \mathcal{F}} \sum_{G \in \mathcal{U}(F)} \frac{1}{\binom{n}{|F|}\binom{n-|F|}{|G|-|F|}} = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \left(1 - \sum_{G \in \mathcal{U}(F)} \frac{1}{\binom{n-|F|}{|G|-|F|}} \right) \leq 1. \tag{2.6}$$

Observe that

$$\binom{n - |F|}{|G| - |F|} \geq \binom{n - |F|}{1}$$

since $1 \leq |G| - |F| < n - |F|$. Moreover, since $\mathcal{F}$ contains no V_r, the inequality $|\mathcal{U}(F)| \leq r$ must hold. Substituting these facts in (2.6) we obtain

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \left(1 - \frac{r}{n - |F|} \right) \leq 1.$$

To finish the proof we only have introduce f_i.

$\square_{\mathrm{T}}$

3 Proof of Theorem 1.5

Upper bound. (2.2) in the form

$$\sum_{i=0}^{n-1} \frac{f_i}{\binom{n}{i}\frac{n-i}{n-i-r}} \leq 1 \tag{3.1}$$

suggests that one has to find the maximum of

$$b(i) = \binom{n}{i}\frac{n-i}{n-i-r}$$

in i.

Lemma 3.1 *Suppose* $6r + \frac{3}{2} < n$. *Then*

$$\binom{n}{i}\frac{n-i}{n-i-r} \leq \binom{n}{\lceil \frac{n}{2} \rceil}\frac{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor - r}$$

holds for $0 \leq i < n-r$.

Proof. Consider the "derivative" of the function $b(i)$ $(0 \leq i < n-r)$, that is, compare two consecutive values $(1 \leq i < n-r)$:

$$f(i-1) = \frac{n!}{(i-1)!(n-i)!(n-i+1-r)} < f(i) = \frac{n!}{i!(n-i-1)!(n-i-r)}.$$

This is equivalent to $i(n-i-r) < (n-i)(n-i+1-r)$ and $0 < 2i^2 - (3n-2r+1)i + n^2 - nr + n$. The discriminant of the corresponding quadratic equation is $(3n-2r+1)^2 - 8(n^2-nr+n) = n^2+4r^2-4nr-2n-4r+1 = (n-2r)^2 - 2n - 4r + 1$. The following inequalities are obvious for $6r + \frac{3}{2} < n$.

$$(n-2r-2)^2 < (n-2r)^2 - 2n - 4r + 1 < (n-2r-1)^2 \tag{3.2}$$

Let $\alpha_1 < \alpha_2$ be the roots of the equation. Using (3.2) we obtain the following bound for the roots.

$$\frac{n}{2} + \frac{1}{2} < \alpha_1 < \frac{n}{2} + \frac{3}{4}$$

and

$$n - r - \frac{1}{4} < \alpha_2 < n - r.$$

This shows that $f(i)$ is growing until $\lceil \frac{n}{2} \rceil$ and is decreasing from this point to $n-r-1$. $\square_{\mathrm{L}}$

Partition the family $\mathcal{F}$ according to the sizes of its members: $\mathcal{F}_1 = \{F : F \in \mathcal{F}, |F| < n-r\}, \mathcal{F}_2 = \{F : F \in \mathcal{F}, |F| \geq n-r\}$. Apply Theorem 2.2 for $\mathcal{F}_1$ and use Lemma 3.1.

$$1 \geq \sum_{i=0}^{n-r-1} \frac{f_i}{\binom{n}{i}} \left(1 - \frac{r}{n-i}\right) = \sum_{i=0}^{n-r-1} \frac{f_i}{\binom{n}{i}\frac{n-i}{n-i-r}} \geq \sum_{i=0}^{n-r-1} \frac{f_i}{\binom{n}{\lceil \frac{n}{2} \rceil}} \cdot \frac{\lfloor \frac{n}{2} \rfloor - r}{\lfloor \frac{n}{2} \rfloor}$$

$$= \frac{1}{\binom{n}{\lceil \frac{n}{2} \rceil}} \cdot \frac{\lfloor \frac{n}{2} \rfloor - r}{\lfloor \frac{n}{2} \rfloor} \sum_{i=0}^{n-r-1} f_i = \frac{1}{\binom{n}{\lceil \frac{n}{2} \rceil}} \cdot \frac{\lfloor \frac{n}{2} \rfloor - r}{\lfloor \frac{n}{2} \rfloor} |\mathcal{F}_1|.$$

Hence we obtain the right upper bound for $\mathcal{F}_1$. On the other hand,

$$|\mathcal{F}_2| \leq \sum_{n-r}^{n} \binom{n}{i}$$

shows

$$|\mathcal{F}_2| = O(n^r) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} O\left(\frac{1}{n^2}\right).$$

$\mathcal{F}_1| + |\mathcal{F}_2| = |\mathcal{F}|$ finishes the proof. $\square_{\mathrm{Up}}$

Lower bound. (See [9].) For a fixed k and a take all the subsets $\{x_1, \ldots, x_k\}$ of $[n]$ satisfying $x_1 + \cdots + x_k \equiv a \pmod{\lceil \frac{n}{r} \rceil}$. ($x_i$s are different.) Suppose that some sets have the same $k-1$-element intersection, say $x_1 + \cdots + x_{k-1}$. Then the equations $x_1 + \cdots + x_{k-1} + y_1 \equiv a$ and $x_1 + \cdots + x_{k-1} + y_2 \equiv a$ imply $y_1 \equiv y_2 \pmod{\lceil \frac{n}{r} \rceil}$. It is obvious that there are at most r such numbers $\pmod{n}$.

Choose a maximizing the number of solutions. For this a the number of solutions is at least

$$\frac{\binom{n}{k}}{\lceil \frac{n}{r} \rceil} \geq \binom{n}{k} \frac{r}{n+r} \geq \binom{n}{k} \left(\frac{r}{n} - \frac{r^2}{n^2}\right).$$

Take $k = \lceil \frac{n+1}{2} \rceil$. Then the family consisting of all $\lfloor \frac{n}{2} \rfloor$-element sets and the $\lceil \frac{n+1}{2} \rceil$-element ones constructed above will contain no $r+1$-fork. The number of sets is as it is given in the theorem. $\square_{\mathrm{Lo}}$

Remark. This construction is a slight generalization of the case when r is 1 [4]. It was shown that one can find approximately $\frac{1}{n}$ part of all the sets of size $\lceil \frac{n+1}{2} \rceil$ without an intersection of size $\lfloor \frac{n}{2} \rfloor$. On the other hand, it is trivial that there is an upper bound which is approximately twice as much. It is an old open problem of coding theory which one is the right constant. Or neither one? This is why to get rid of the factor 2 in the second term in Theorems 1.3, 1.5, 4.1-4.2 and 5.1 seems to be difficult.

Another approach. Earlier we had a more complicated proof for the upper bound. It contained an inequality what might have some interest in its own right.

Theorem 3.2 *Suppose that the family $\mathcal{F}$ contains no $r+1$-fork ($0 < r$) and the sizes of all members of $\mathcal{F}$ are at most m where $m < n-r$. Then*

$$\sum_{i=0}^{m} \frac{f_i}{\binom{n}{i}} \leq 1 + \frac{r}{n-m} + \frac{r^2}{(n-m-r)^2}. \tag{3.3}$$

Proof. Replacing i by m in the coefficient, the statement of Theorem 2.2 becomes

$$\sum_{i=0}^{m} \frac{f_i}{\binom{n}{i}} \leq \frac{1}{1-\frac{r}{n-m}}.$$

The easy inequality

$$\frac{1}{1-\frac{r}{n-m}} \leq 1 + \frac{r}{n-m} + \frac{r^2}{(n-m-r)^2}$$

finishes the proof. $\square_{\mathrm{T}}$

The proof of Theorem 1.5 can be finished by choosing m to be somewhat more than $\frac{n}{2}$. Then Theorem 3.2 ensures that the number of members of $\mathcal{F}$ with size $\leq m$ cannot exceed the desired bound. On the other hand, the number of sets in $\mathcal{F}$ is at most the sum of the binomial coefficients from m to n what is much less than the largest binomial coefficient, if m is chosen properly. But this is only a sketch, the details need some theorems from the asymptotical theory of binomial coefficients and tedious calculations.

Problem. What is the maximum of the left hand side of (3.3) under the condition $F \in \mathcal{F}$ implies $|F| \leq m$? If there is no upper bound for the sizes in $\mathcal{F}$ then the family consisting of all i-element sets and $[n]$ gives 2 while the family contains no 2-fork. If $[n]$ is excluded, that is, $m = n - 1$ then the following family gives asymptotically $1 + \frac{1}{4}$. Suppose n is even and divide $[n]$ into two equal parts, $[n/2]$ and its complement. Take all $n - 1$-element sets containing $[n/2]$ and all $n-2$-element sets not containing it. This family contains no 2-fork. Is this the asymptotically best for family without 2-forks?

4 Tran's theorem and its partial improvement

An *r-fork with a k-shaft* is a family of distinct subsets $F_1, F_2, \ldots, F_k$, $G_1, G_2, \ldots, G_r$ such that $F_1 \subset F_2 \subset \ldots \subset F_k,\ F_k \subset G_1, F_k \subset G_2, \ldots, F_k \subset G_r$. It is a combination of a path and a fork, it is denoted by ${}_kV_r$. Tran's Theorem in its full generality was the following.

Theorem 4.1 [9] *Let* $1 \le r, 1 \le k$ *be given integers. Then*

$$\sum_{i=\lfloor \frac{n-k+1}{2} \rfloor}^{\lfloor \frac{n+k-1}{2} \rfloor} \binom{n}{i} + \binom{n}{\lfloor \frac{n+k+1}{2} \rfloor} \left(\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right)$$

$$\le \mathrm{La}({}_kV_{r+1}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + 2\frac{r\binom{k+r-1}{k}}{n} + o\left(\frac{1}{n}\right)\right).$$

We are going to prove a somewhat stronger upper bound in most cases, namely the following statement.

Theorem 4.2

$$\mathrm{La}({}_kV_{r+1}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + 2\frac{\lfloor \frac{k^2}{4} \rfloor + r}{n} + O\left(\frac{1}{n^2}\right)\right).$$

Proof. Suppose that the family $\mathcal{F}$ contains no ${}_kV_r$. Let $\mathcal{F}_1$ denote the family of those members F of $\mathcal{F}$ for which there is no chain of lenght $k - 1$ below F, that is, there are no distinct sets $F_1, F_2, \ldots, F_{k-1} \in \mathcal{F}$ such that $F_1 \subset F_2 \subset \ldots \subset F_{k-1} \subset F$. On the other hand, $\mathcal{F}_2 = \mathcal{F} - \mathcal{F}_1$. It is easy to see that $\mathcal{F}_1$ contains no chain of length k, therefore

$$|\mathcal{F}_1| \le \sum_{i=\lfloor \frac{n-k+2}{2} \rfloor}^{\lfloor \frac{n+k-2}{2} \rfloor} \binom{n}{i} \qquad (4.1)$$

holds by Theorem 1.2. On the other hand, $\mathcal{F}_2$ contains no V_r, therefore

$$|\mathcal{F}_2| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right). \tag{4.2}$$

(4.1) and (4.2) imply

$$|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| \le \sum_{i=\lfloor \frac{n-k+2}{2} \rfloor}^{\lfloor \frac{n+k-2}{2} \rfloor} \binom{n}{i} + \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right). \tag{4.3}$$

Comparing the lower estimate in Theorem 4.1 and (4.3) we see that $\binom{n}{\lfloor \frac{n+k}{2} \rfloor}$ is replaced by $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Let us study their ratio

$$\frac{\binom{n}{\lfloor \frac{n}{2} \rfloor}}{\binom{n}{\lfloor \frac{n+k}{2} \rfloor}} = \prod_{i=0}^{K-1} \frac{\lfloor \frac{n+k}{2} \rfloor - i}{\lceil \frac{n}{2} \rceil - i} \tag{4.4}$$

where K is a notation for $\lfloor \frac{n+k}{2} \rfloor - \lfloor \frac{n}{2} \rfloor$. Observe that $K = \frac{k}{2}$ when both n and k are even, it is $\frac{k-1}{2}$ if n is even, k is odd, it is $\frac{k}{2}$ when n is odd, k is even, and it is $\frac{k+1}{2}$ in the last case when n and k are both odd. One factor of (4.4) can be rewritten in the form

$$\frac{\lfloor \frac{n+k}{2} \rfloor - i}{\lceil \frac{n}{2} \rceil - i} = 1 + \frac{K'}{\lceil \frac{n}{2} \rceil - i} \tag{4.5}$$

where K' denotes $\lfloor \frac{n+k}{2} \rfloor - \lceil \frac{n}{2} \rceil$ and is actually equal to $\frac{k}{2}, \frac{k-1}{2}, \frac{k-2}{2}$ and $\frac{k-1}{2}$ following the order at K. Since the dependence on n should be avoided, take the trivial upper bounds $K \le \lfloor \frac{k+1}{2} \rfloor$ and $K' \le \lfloor \frac{k}{2} \rfloor$. It is easy to see that (4.5) can be upperbounded by

$$1 + \frac{\lfloor \frac{k}{2} \rfloor}{\lceil \frac{n}{2} \rceil - i} = 1 + \frac{2\lfloor \frac{k}{2} \rfloor}{n} + O(\frac{1}{n^2}).$$

Using this bound for all factors in (4.4), the following upper bound is obtained for the ratio:

$$\prod_{i=0}^{K-1}\left(1 + \frac{2\lfloor \frac{k}{2} \rfloor}{n} + O(\frac{1}{n^2})\right) = 1 + \frac{2\lfloor \frac{k+1}{2} \rfloor \lfloor \frac{k}{2} \rfloor}{n} + O\left(\frac{1}{n^2}\right).$$

Therefore $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ can be replaced in (4.3) by

$$\binom{n}{\lfloor \frac{n+k}{2} \rfloor} \left(1 + \frac{2\left\lfloor \frac{k+1}{2} \right\rfloor \left\lfloor \frac{k}{2} \right\rfloor}{n} + O\left(\frac{1}{n^2}\right)\right)$$

$$= \binom{n}{\lfloor \frac{n+k}{2} \rfloor} \left(1 + \frac{2\left\lfloor \frac{k^2}{4} \right\rfloor}{n} + O\left(\frac{1}{n^2}\right)\right).$$

□

Let us see now that

$$\left\lfloor \frac{k^2}{4} \right\rfloor + r \le r\binom{k+r-1}{k}$$

holds for $3 \le r$. The right hand side is not increased by replacing r by 3 in the binomial coefficient. The remaining inequality

$$\left\lfloor \frac{k^2}{4} \right\rfloor + r \le r\binom{k+2}{k}$$

is really easy to prove. That is, our upper bound is stronger whenever $3 \le n$. However, this is not true in general for $r = 1, 2$.

However we strongly believe that $\left\lfloor \frac{k^2}{4} \right\rfloor$ can be completely deleted from the the second term of the upper estimate.

Conjecture 1

$$\mathrm{La}({}_kV_{r+1}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right).$$

5 Complete two level posets

In this section we are trying to maximize the size of a family $\mathcal{F}$ containing no $r+s$ distinct members satisfying $A_1, \ldots, A_s \subset B_1, \ldots, B_r$. Let $T_{r,s}$ denote the poset with two levels, s element on the lower, r elements on the upper level, every lower one is in relation with every upper one. It is easy to see that our condition can be formulated in the way that we are looking for the maximum number of the elements in the Boolean lattice of subsets of $[n]$ (defined by inclusion) without containing $T_{r,s}$ as a subposet. Let the maximum be denoted by $\mathrm{La}(n, P)$ for an arbitrary poset P.

Theorem 5.1 *Suppose that $2 \le s, 3 \le r$ and $s \le r$ hold. Then*

$$\binom{n}{\lceil \frac{n}{2} \rceil} + \binom{n}{\lceil \frac{n}{2} \rceil - 1} + \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(2\frac{r+s-4}{n} + O\left(\frac{1}{n^2}\right)\right)$$

$$\le \mathrm{La}(n, T_{r,s}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(2 + 2\frac{r+s-3}{n} + O\left(\frac{1}{n^2}\right)\right).$$

Proof. Upper estimate.

Lemma 5.2 *If a poset P contains no $T_{r,s}$ as a subposet, then it can be partitioned into posets P_1 and P_2 so that P_1 contains no $T_{r-1,1}$ and P_2 contains no $T_{1,s}$.*

Proof. Let P_1 be the set of those elements a of P for which the number of elements $b \in P$ satisfying $a < b$ is at most $r-2$. Then it is obvious that the poset induced by theses elements contains no $T_{r-1,1}$. Let $P_2 = P - P_1$. Suppose that in contrast to our statement, P_2 contains a $T_{1,s}$ as a subposet: $a_1, \ldots a_s, b \in P_2, a_1 < b, \ldots a_s < b$. Since b is not in P_1, there are some $c_1, \ldots, c_{r-1}$ in P_1 such that $b < c_1, \ldots, c_{r-1} < b$ holds. It is easy to see that all these elements, $a_1, \ldots a_s, b, c_1, \ldots, c_{r-1}$ form a $T_{r,s}$ in P $\square_{\mathrm{L}}$

Apply the lemma for the poset spanned by the family containing no sets forming a $T_{r,s}$. The two families obtained are denoted by $\mathcal{F}_1$ and $\mathcal{F}_2$. Theorem 1.5 can be directly used for $\mathcal{F}_1$. Since complementation preserves inclusion, Theorem 1.5 can also be used for $\mathcal{F}_2$. Adding up the two upper estimates, the upper estimate of Theorem 5.1 is obtained. $\square_{\mathrm{Up}}$

Lower estimate. Let $\mathcal{F}_1$ be a family of sets of size $\lceil \frac{n}{2} \rceil - 2$ such that the size of the union of any $s-1$ members of $\mathcal{F}_1$ is at least $\lceil \frac{n}{2} \rceil$, and let $\mathcal{F}_2$ be a family of sets of size $\lceil \frac{n}{2} \rceil + 1$ such that the size of the intersection of any $r-1$ of them is at most $\lceil \frac{n}{2} \rceil - 1$. We denote by $\mathcal{F}$ the family containing all members of $\mathcal{F}_1$ and $\mathcal{F}_2$ along with all sets of size $\lceil \frac{n}{2} \rceil - 1$ and all those of size $\lceil \frac{n}{2} \rceil$. The family $\mathcal{F}$ contains no $r+s$ distinct members $A_1, \ldots, A_s, B_1, \ldots, B_r$ satisfying $A_1, \ldots, A_s \subset B_1, \ldots, B_r$. Indeed, it is possible to see that the size of the union of any s members of $\mathcal{F}$ is at least $\lceil \frac{n}{2} \rceil$ whereas the intersection of any r members of $\mathcal{F}$ has size at most $\lceil \frac{n}{2} \rceil - 1$.

Let $A_1, \ldots, A_s \in \mathcal{F}$. If at least two sets among $A_1, \ldots, A_s$ have size at least $\lceil \frac{n}{2} \rceil - 1$ then the union of these two sets has a size at least $\lceil \frac{n}{2} \rceil$. Otherwise, if at most one set among $A_1, \ldots, A_s$ has size larger than or equal to $\lceil \frac{n}{2} \rceil - 1$ then at least $s-1$ sets among $A_1, \ldots, A_s$ belong to $\mathcal{F}_1$. By construction the size of the union of these $s-1$ sets is at least $\lceil \frac{n}{2} \rceil$. On the other hand let us consider r members $B_1, \ldots, B_r$ of $\mathcal{F}$. If at least two sets among $B_1, \ldots, B_r$ have size at most $\lceil \frac{n}{2} \rceil$ then the intersection of these two sets is at most $\lceil \frac{n}{2} \rceil - 1$. If at most one set among $B_1, \ldots, B_r$ has size at most $\lceil \frac{n}{2} \rceil$, then at least $r-1$ sets among $B_1, \ldots, B_r$ belong to $\mathcal{F}_2$ and consequently their intersection is at most $\lceil \frac{n}{2} \rceil - 1$.

By using the construction in the proof of Theorem 1.5 $\mathcal{F}_2$ can be made as large as

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \left(\frac{r-2}{n} + O\left(\frac{1}{n^2} \right) \right).$$

By the same construction we obtain

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \left(\frac{s-2}{n} + O\left(\frac{1}{n^2} \right) \right)$$

sets of size $\lceil \frac{n}{2} \rceil + 2$ such that the intersection of any $s-1$ of them is at most $\lceil \frac{n}{2} \rceil$. The family $\mathcal{F}_1$ is obtained by taking the complement of each of those sets. Adding up the sizes of the four families, the lower estimate of the theorem is obtained. $\square_{\text{Up}}$

Remarks. 1. In the lower estimate the main term is the sum of the two largest binomial coefficients of order n. In the upper estimate it is the double of the largest one. They are the same if n is odd, but different for even ns. If one of the numbers $\binom{n}{\frac{n}{2}}$ is replaced by the second largest binomial coefficient $\binom{n}{\frac{n}{2}-1}$ then we loose a little. Then $r+s-3$ should be replaced by $r+s-2$.

2. The proofs work for $r = s = 2$ as well. However the upper estimate is too week. It has been proved [2] that $\mathrm{La}(n, T_{2,2})$ is the sum of the two largest binomial coefficients.

3. We believe that the lower estimate is the (asymptotically) correct one up to the second term.

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