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Forbidden Intersection Patterns in the Families of Subsets

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1 Introduction

Let $[n] = \{1, 2, \dots, n\}$ be a finite set, $\mathcal{F} \subset 2^{[n]}$ a family of its subsets. In the present paper $\max |\mathcal{F}|$ will be investigated under certain conditions on the family $\mathcal{F}$. The well-known Sperner theorem ([10]) was the first such result.

Theorem 1.1. *If $\mathcal{F}$ is a family of subsets of $[n]$ without inclusion ($F, G \in \mathcal{F}$ implies $F \not\subset G$) then*

$$|\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

holds, and this estimate is sharp as the family of all $\lfloor \frac{n}{2} \rfloor$-element subsets shows.

There is a very large number of generalizations and analogues of this theorem. Here we will consider only results when the condition on $\mathcal{F}$ excludes certain configurations what can be expressed by inclusion, only. That is, no intersections, unions, etc. are involved. The first such generalization was obtained by Erdős [4]. The family of k distinct sets with mutual inclusions, $F_1 \subset F_2 \subset \dots F_k$ is called a *chain of lenght* k. It will be simply denoted by P_k. Let $\mathrm{La}(n, P_k)$ denote the largest family $\mathcal{F}$ without a chain of lenght k.

Theorem 1.2 ([4]). $\mathrm{La}(n, P_{k+1})$ *is equal to the sum of the k largest bimomial coefficients of order n.*

Let V_r denote the r-*fork*, that is the following family of distinct sets: $F \subset G_1$, $F \subset G_2, \dots F \subset G_r$. The quantity $\mathrm{La}(n, V_r)$, that is, the largest family on n elements containing no V_r was first (asymptotically) determined for $r = 2$.

Theorem 1.3 ([7]).

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) \le \mathrm{La}(n, V_2) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{2}{n}\right).$$

The first result for general r is contained in the following theorem.

Theorem 1.4 ([11]).

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{r}{n} + O\left(\frac{1}{n^2}\right)\right) \le \mathrm{La}(V_{r+1}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + 2\frac{r^2}{n} + o\left(\frac{1}{n}\right)\right).$$

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The constant in the second term in the upper estimate was recently improved.

Theorem 1.5 ([1]).

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{r}{n}+O\left(\frac{1}{n^2}\right)\right) \le \mathrm{La}(V_{r+1}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+2\frac{r}{n}+o\left(\frac{1}{n}\right)\right).$$

See some remarks in Section 5 explaining why this second term is difficult to improve any more.

The aim of the present paper is to introduce some recent results and show a method, proving good upper estimates, developed recently.

2 Notations, Definitions

A *partially ordered set*, shortly *poset* P is a pair $P = (X, \le)$ where X is a (in our case always finite) set and $\le$ is a relation on X which is *reflexive* ($x \le x$ holds for every $x \in X$), *antisymmetric* (if both $x \le y$ and $x \ge y$ hold for $x, y \in X$ then $x = y$) and transitive ($x \le y$ and $y \le z$ always implies $x \le z$). It is easy to see that if $X = 2^{[n]}$ and the $\le$ is defined as $\subseteq$, then these conditions are satisfied, that is the familiy of all subsets of an n-element set ordered by inclusion form a poset. We will call this poset the *Boolean lattice* and denote it by B_n.

The definition of a *subposet* is obvious: $R = (Y, \le_2)$ is a subposet of $P = (X, \le_1)$ iff there is an injection α of Y into X is such a way that $y_1, y_2 \in Y, y_1 \le_2 y_2$ implies $\alpha(y_1) \le_1 \alpha(y_2)$. On the other hand R is an *induced subposet* of P when $\alpha(y_1) \le_1 \alpha(y_2)$ holds iff when $y_1 \le_2 y_2$. If $P = (X, \le)$ is a poset and $Y \subset X$ then the poset *spanned by Y in P* is defined as $(Y, \le^*)$ where $\le^*$ is the same as $\le$, for all the pairs taken from Y. Given a "small" poset R, $\mathrm{La}(n, R)$ denotes the maximum number of elements of $Y \subset 2^{[n]}$ (that is, the maximum number of subsets of $[n]$) such that R is not a subposet of the poset spanned by Y in B_n.

Redefine our "small" configurations in terms of posets. The chain P_k contains k elements: $a_1, \ldots, a_k$ where $a_1 < \ldots < a_k$. The *r-fork* contains $r + 1$ elements: $a, b_1, \ldots, b_r$ where $a < b_1, \ldots a < b_r$. It is easy to see that the definitions of $\mathrm{La}(n, P_k)$, $\mathrm{La}(n, V_r)$, in Sections 1 and 2 agree. In the rest of the paper we will use the two different terminology alternately. In the definition of $\mathrm{La}(n, R)$ we mean non-induced subposets, that is, if $R = V_2$ then P_3 is also excluded as a subposet.

A poset is *connected* if for any pair (z_0, z_k) of its elements there is a sequence $z_1, \ldots, z_{k-1}$ such that either $z_i < z_{i+1}$ or $z_i > z_{i+1}$ holds for $0 \le i < k$. If the poset is not connected, maximal connected subposets are called its *connected components.* Given a family $\mathcal{F}$ of subsets of $[n]$, it spans a poset in B_n. We will consider its connected components in two different ways. First as posets themself, secondly as they are represented in B_n. In the latter case the sizes of in the sets are also indicated. A *full chain* in B_n is a family of sets $A_0 \subset A_1 \subset \ldots \subset A_n$ where $|A_i| = i$. We say that a (full) chain *goes through* a family (subposet) $\mathcal{P}$ if their intersection is non-empty, that is if it "goes through" at least one member of the family.

3 Lubell's Proof of the Sperner Theorem

The number of full chains in $[n]$ is $n!$ since the choice of a full chain is equivalent to the choice of a permutation of the elements of $[n]$. On the other hand, the number of full chains going through a given set F of f elements is $f!(n-f)!$ since the chain "must grow" within F until it "hits" F and outside after that. Suppose that the family $\mathcal{F}$ of subsets of $[n]$ is without inclusion ($F, G \in \mathcal{F}$ implies $F \not\subset G$). Then a full chain cannot go through two members of $\mathcal{F}$. Therefore the set of full chains going through distinct members of $\mathcal{F}$ must be disjoint. Hence we have

$$\sum_{F \in \mathcal{F}} |F|!(n-|F|)! \leq n!.$$

Dividing the inequality by $n!$

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1 \tag{3.1}$$

is obtained. As $\binom{n}{|F|} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$, then

$$\frac{|\mathcal{F}|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq 1$$

follows, the theorem is proved.

Let us remark that inequality (3.1) is important on its own right and is called the YBLM-inequality (earlier LYM, see [3, 8, 9, 12]). □

4 The Method, Illustrated with an Old Result

Lubell's proof easily applies for Theorem 1.2, however, surprisingly it was not exploited for proving theorems of the present type. The reason might be that not the "excluded" configurations should be considered when using the idea, but the "allowed induced posets". (See later.)

Following the definition of the r-fork, let us define the *r-brush* (in a poset) which contains $r+1$ elements: $a, b_1, \ldots, b_r$ where $a > b_1, \ldots a > b_r$ and is the "complement" of the r-fork. Theorem 1.3 gives the best expected asymptotic upper bound up to the second term for V_2 in the Boolean lattice. It is easy to see that it implies the same solution for Λ_2. However the result is very different when both of them are excluded. Our notation $\mathrm{La}(n, R)$ is extended in an obvious way for the case when two subposets R_1 and R_2 are excluded: $\mathrm{La}(n, R_1, R_2)$.

Theorem 4.1 ([7]).

$$\mathrm{La}(n, V_2, \Lambda_2) = 2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

Proof. The construction giving the equality is the following:

$$\left\{F \subset [n] : \ 1 \notin F, |F| = \left\lfloor \frac{n-1}{2} \right\rfloor \right\} \cup \left\{F \subset [n] : \ 1 \in F, |F| = \left\lfloor \frac{n+1}{2} \right\rfloor \right\}.$$

The non-trivial part of the proof is the verification of the upper bound.

Let $\mathcal{F}$ be a family of subsets of $[n]$ which contains neither a V_2 nor a Λ_2 as subposet. Therefore it cannot contain a P_3 either. Consider the connected components of the poset spanned by $\mathcal{F}$. It is obvious that a connected component can be either a one element poset P_1 or a P_2. Let α_1 and α_2 be their respective numbers. Then

$$|\mathcal{F}| = \alpha_1 + 2\alpha_2. \tag{4.1}$$

We will now determine the minimum number of full chains going through a one or two-element component. Let $P(1;a)$ be a one-element component which is an a-element set. The number $c(P(1;a))$ of full chains going through $P(1;a)$ is $a!(n-a)!$. Therefore

$$\frac{c(P(1;a))}{n!} = \frac{1}{\binom{n}{a}}.$$

It takes on its minimum at the value $a = \lfloor \frac{n}{2} \rfloor$. Hence we obtained

$$\left\lfloor \frac{n}{2} \right\rfloor ! \left\lceil \frac{n}{2} \right\rceil ! \le c(P(1;a)). \tag{4.2}$$

The two-element component consisting of an a-element A and a b-element B ($A \subset B$) ($a < b$) subset is denoted by $P(2;a,b)$. The number $c(P(2;a,b))$ of full chains going through (at least one element of) $P(2;a,b)$ is

$$c(P(2;a,b)) = a!(n-a)! + b!(n-b)! - a!(b-a)!(n-b)!. \tag{4.3}$$

Divide it by $n!$.

$$\frac{c(P(2;a,b))}{n!} = \frac{1}{\binom{n}{a}} + \frac{1}{\binom{n}{b}} - \frac{1}{\binom{n}{b}\binom{b}{a}} = \frac{1}{\binom{n}{a}} + \frac{1}{\binom{n}{b}}\left(1 - \frac{1}{\binom{n}{a}}\right). \tag{4.4}$$

Suppose first that a is fixed and is $\le \lfloor \frac{n-1}{2} \rfloor$. Then (4.4) takes on its minimum for $b = \lfloor \frac{n+1}{2} \rfloor$. Fix b here and consider the following variant of (4.4):

$$\frac{c(P(2;a,b))}{n!} = \frac{1}{\binom{n}{a}} + \frac{1}{\binom{n}{b}} - \frac{1}{\binom{n}{a}\binom{n-a}{n-b}} = \frac{1}{\binom{n}{b}} + \frac{1}{\binom{n}{a}}\left(1 - \frac{1}{\binom{n-a}{n-b}}\right). \tag{4.5}$$

This is a monotone decreasing function of a in the interval $0 \le a \le \lceil \frac{n}{2} \rceil$. Therefore the pair giving the minimum in this case is $a = \lfloor \frac{n-1}{2} \rfloor, b = \lfloor \frac{n+1}{2} \rfloor$.

Suppose now that $a \ge \lfloor \frac{n}{2} \rfloor$. Then b can be chosen to be $a+1$ by (4.4), and (4.3) becomes $na!(n-a-1)!$ It achieves its minimum at $a = \lfloor \frac{n-1}{2} \rfloor$, again. We obtained

$$n \left\lfloor \frac{n-1}{2} \right\rfloor ! \left\lceil \frac{n-1}{2} \right\rceil ! \le c(P(2;a,b)). \tag{4.6}$$

Observe that a full chain cannot go through two distinct components, therefore

$$\sum_{P_1 \text{is a component}} c(P_1) + \sum_{P_2 \text{is a component}} c(P_2) \le n!$$

holds. The left hand side can be lowerestimated by (4.2) and (4.6):

$$\alpha_1 \left\lfloor \frac{n}{2} \right\rfloor ! \left\lceil \frac{n}{2} \right\rceil ! + \alpha_2 n \left\lfloor \frac{n-1}{2} \right\rfloor ! \left\lceil \frac{n-1}{2} \right\rceil ! \le n!. \tag{4.7}$$

(4.1) has to be maximized with respect to (4.7). Rewrite (4.7) a little bit:

$$\alpha_1 \left\lfloor \frac{n}{2} \right\rfloor ! \left\lceil \frac{n}{2} \right\rceil ! + 2\alpha_2 \frac{n}{2} \left\lfloor \frac{n-1}{2} \right\rfloor ! \left\lceil \frac{n-1}{2} \right\rceil ! \le n!. \tag{4.8}$$

Compare the coefficients of α_1 and $2\alpha_2$ in (4.8).

$$\left\lfloor \frac{n}{2} \right\rfloor ! \left\lceil \frac{n}{2} \right\rceil ! \ge \frac{n}{2} \left\lfloor \frac{n-1}{2} \right\rfloor ! \left\lceil \frac{n-1}{2} \right\rceil ! \tag{4.9}$$

holds (with equality for even n). Replacing the coefficient of α_1 in (4.8) using (4.9), the inequality

$$(\alpha_1 + 2\alpha_2)\frac{n}{2}\left\lfloor\frac{n-1}{2}\right\rfloor!\left\lceil\frac{n-1}{2}\right\rceil! \le n!$$

is obtained, what results in

$$|\mathcal{F}| = \alpha_1 + 2\alpha_2 \le \frac{n!}{\frac{n}{2}\left\lfloor\frac{n-1}{2}\right\rfloor!\left\lceil\frac{n-1}{2}\right\rceil!} = 2\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}.$$

□

5 A Construction = a Lower Estimate

Although we concentrate in this paper on the upper estimates, it seems to be important to show the construction serving as a lower estimate in Theorem 1.3

The consctruction for a family avoiding a V_2 is the following. Take all the sets of size $\lfloor\frac{n}{2}\rfloor$ and a family $A_1, \dots A_m$ of $\lfloor\frac{n}{2}\rfloor + 1$-element sets satisfying the condition $|A_i \cap A_j| < \lfloor\frac{n}{2}\rfloor$ for every pair $i < j$. It is easy to see that this family contains no V_2. We only have to maximize m. Since the $\lfloor\frac{n}{2}\rfloor$-element subsets of the A_is are all distinct, we have

$$m\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right) \le \binom{n}{\lfloor\frac{n}{2}\rfloor}.$$

This gives the upper estimate

$$m \le \binom{n}{\lfloor\frac{n}{2}\rfloor}\frac{2}{n}. \tag{5.1}$$

There is a very nice construction (see [5]) of such sets A_i with

$$m = \binom{n}{\lfloor\frac{n}{2}\rfloor + 1}\frac{1}{n} = \binom{n}{\lfloor\frac{n}{2}\rfloor}\left(\frac{1}{n} + O(\frac{1}{n^2})\right).$$

It is a longstanding conjecture of coding theory what the right constant is here, 1 or 2. Or if the limit exists at all?

This is why there is a disturbing factor 2 between the second terms of the lower and upper estimates in Theorem 1.3. This gap cannot be bridged without solving the problem in coding theory mentioned above.

6 The Upper Estimate in Theorem 1.3

This theorem already has two different proofs in [7] and [2], however each of these proofs needed an *ad hoc* idea. Our new method also works here. It needs some tedious calculations, but the principal idea is as easy as in the previous case.

Suppose that $\mathcal{F}$ contains no V_2 as a subposet. Then it cannot contain a P_3 either. It is easy to deduce that the components of the poset spanned by $\mathcal{F}$ are all of type Λ_r where $0 \le r$. This is a new phenomenon! The sizes of the components are unbounded. Yet, the method works.

Let $\gamma(r)$ $(0 \le r)$ denote the number of components of form Λ_r. Then

$$|\mathcal{F}| = \sum_{r=0}^{\infty}(r+1)\gamma(r). \tag{6.1}$$

is obvious. This has to be maximized under a linear condition obtained from the fact that the chains going through distinct components must be distinct.

Let $\Lambda(r; u, u_1, \ldots, u_r)$ $(u > u_1, \ldots, u_r)$ denote the component Λ_r represented by sets of sizes $u, u_1, \ldots, u_r$, respectively. $c(\Lambda(r; u, u_1, \ldots, u_r))$ denotes the number of full chains going through this component. The following lemma gives a good lower estimate on this number. For the proof see [6].

Lemma 6.1. *Suppose* $6 \leq n, 1 \leq r$. *Then*

$$u^*!(n - u^*)! + ru^*!u^*(n - u^* - 1)! \leq c(\Lambda(r; u, u_1, \ldots, u_r)) \qquad (u > u_1, \ldots, u_r)$$

holds where $u^* = u^*(n) = \frac{n}{2} - 1$ *if* n *is even,* $u^* = \frac{n-1}{2}$ *if* n *is odd and* $r - 1 \leq n$, *while* $u^* = \frac{n-3}{2}$ *if* n *is odd and* $n < r - 1$.

In the case $r = 0$ *the inequality* $\lfloor \frac{n}{2} \rfloor ! \lceil \frac{n}{2} \rceil ! \leq \Lambda(0; u)$ *holds.*

We will actually need a lower estimate on the number of full chains going through the component, divided by the number of elements of this component.

$$u^*!u^*(n - u^* - 1)! \leq \frac{u^*!(n - u^*)! + ru^*!u^*(n - u^* - 1)!}{r + 1} \; (0 \leq r) \qquad (6.2)$$

is a consequence of the lemma and the remark on the case $r = 0$.

Let $\mathcal{C}_i$ $(i \leq K = \sum_{r=0}^{\infty} \gamma(r))$ be the components spanned by $\mathcal{F}$. Each $\mathcal{C}_i$ is equal to a $\Lambda(r; u, u_1, \ldots, u_r)$ for some parameters. $c(\mathcal{C}_i)$ denotes the number of chains going through the component $\mathcal{C}_i$. The full chains going through distinct components are distinct. This implies

$$n! \geq \sum_{i=1}^{K} c(\mathcal{C}_i). \qquad (6.3)$$

Hence by Lemma 6.1 we obtain

$$n! \geq \sum_{r=1}^{K} (u^*!(n - u^*)! + ru^*!u^*(n - u^* - 1)!)\gamma(r). \qquad (6.4)$$

(6.1) has to be maximized with respect to (6.4). Slightly modify the right hand side and use (6.2):

$$\sum_{r=1}^{K} \frac{u^*!(n - u^*)! + ru^*!u^*(n - u^* - 1)!}{r + 1}(r + 1)\gamma(r)$$

$$\geq \sum_{r=1}^{K} u^*!u^*(n - u^* - 1)!(r + 1)\gamma(r). \qquad (6.5)$$

From (6.4), (6.5) and (6.1) we obtain

$$n! \geq u^*!u^*(n - u^* - 1)! \sum_{r=1}^{K} (r + 1)\gamma(r) = u^*!u^*(n - u^* - 1)!|\mathcal{F}|,$$

that is,

$$|\mathcal{F}| \leq \frac{n!}{u^*!u^*(n - u^* - 1)!}.$$

This is equal to

$$\binom{n}{\lfloor\frac{n}{2}\rfloor}\frac{\frac{n}{2}}{\frac{n}{2}-1}, \qquad \binom{n}{\lfloor\frac{n}{2}\rfloor}\frac{\frac{n+1}{2}}{\frac{n-1}{2}}, \qquad \binom{n}{\lfloor\frac{n}{2}\rfloor}\frac{\frac{n-1}{2}}{\frac{n-3}{2}}$$

in the cases $u^* = \frac{n}{2}-1$, $\frac{n-1}{2}$ and $\frac{n-3}{2}$, respectively. These are all equal to

$$= \binom{n}{\lfloor\frac{n}{2}\rfloor}\Big(1+\frac{2}{n}+O\Big(\frac{1}{n^2}\Big)\Big).$$

□

7 Excluding the N

The poset N contains 4 distinct elements a, b, c, d satisfying $a < c, b < c, b < d$. In the Boolean lattice a subposet N consists of four disticts subsets satisfying $A \subset C, B \subset C, B \subset D$. It is somewhat surprising that excluding N the result is basically the same as in the case of V_2. The goal of the present section is to sketch the proof of the following theorem.

Theorem 7.1 ([6]).

$$\binom{n}{\lfloor\frac{n}{2}\rfloor}\left(1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right) \le La(n,N) \le \binom{n}{\lfloor\frac{n}{2}\rfloor}\left(1+\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right)$$

holds.

The lower estimate is obtained from Theorem 1.3, since $\mathrm{La}(n, V_2) \le \mathrm{La}(n, N)$.

Let $\mathcal{F}$ be a family of subsets of $[n]$ contaning no four distinct members forming an N. Consider the poset $P(\mathcal{F})$ spanned by $\mathcal{F}$ in B_n. Its connected components are denoted by $\mathcal{C}_1, \ldots, \mathcal{C}_K$. and $c(\mathcal{C}_i)$ denotes the number of full chains going through $\mathcal{C}_i$. Since a full chain cannot go through two distinct components. the following inequality holds.

$$\sum_{i=1}^{K} c(\mathcal{C}_i) \le n!. \tag{7.1}$$

What can these components be? A component might be a P_3, but no component can contain a P_3 as a proper subposet, since adding one more element to P_3 an N is created no matter which element of P_3 is in relation with the new element. Furthermore, if $a < b$ are two elements of a component then a and b cannot be both comparable in the component with some other distinct elements c, d, only in the way $c < a < b < b$ what is a P_3. But one of them can be comparable with many others. Therefore the following ones are the only possible components:

$$a < b < c, \tag{7.2}$$

$$a < b_i (1 \le i \le r) \text{ where } 0 \le r, \tag{7.3}$$

$$a > b_i (1 \le i \le r) \text{ where } 0 \le r. \tag{7.4}$$

These are denoted by $P(3), V(r), \Lambda(r)$ in this order. However the numbers of full chains going through these posets depend on the sizes of the elements of the poset, that is the sizes of the members of the family $\mathcal{C}_i$. This is why we introduce the notation $P(3; u, v, w)$ for the posets of type $P(3)$ where $|a| = u < |b| = v < |c| = w$. The analogous notations are $V(r; u, u_1, \ldots, u_r)$ $(u < u_1, \ldots, u_r)$ and $\Lambda(r; v, v_1, \ldots, v_r)$ $(v < v_1, \ldots, v_r)$. All the components $\mathcal{C}_i$ in (7.1) are of this form.

An upper bound is sought for

$$|\mathcal{F}| = \sum_{i=1}^{K} |\mathcal{C}_i| \tag{7.5}$$

where $|P(3)| = 3, |V(r)| = |\Lambda(r)| = r + 1$ are obvious. This upper bound will be determined entirely on the basis of (7.1). Denote the numbers of $\mathcal{C}_i$s of type $L(3), V(r), \Lambda(r)$ by $\alpha, \beta(r), \gamma(r)$ respectively. Then (7.5) can be written in the form

$$|\mathcal{F}| = 3\alpha + \sum_{r=0}^{\infty}(r+1)\beta(r) + \sum_{r=0}^{\infty}(r+1)\gamma(r). \tag{7.6}$$

If the minima (or good lower bounds)

$$\min_{u,v,w} c(P(3; u, v, w)), \min_{u,u_1,\ldots,u_r} c(V(r; u, u_1, \ldots, u_r)), \min_{v,v_1,\ldots,v_r} c(\Lambda(r; v, v_1, \ldots, v_r)) \tag{7.7}$$

are determined then (7.1) leads to a linear combination of $\alpha, \beta(r)$, and $\gamma(r)$. That is, one linear combination, namely (7.5) has to be maximized under the condition that another combination is bounded from above. Therefore our main problem is now to determine the minima in (7.7). The first one is solved in the following lemma (for the proof see [6]).

Lemma 7.2. *$c(P(3; u, v, w))$ $(u < v < w)$ takes its minimum for the values $u = \lfloor \frac{n}{2} \rfloor - 1, v = \lfloor \frac{n}{2} \rfloor, w = \lfloor \frac{n}{2} \rfloor + 1$, that is,*

$$\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)! \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)! \left(\left\lfloor \frac{n}{2} \right\rfloor^2 - n\left\lfloor \frac{n}{2} \right\rfloor + n^2 - 1\right) \le c(L(3; u, v, w)).$$

We already have a good lower estimate for the last minimum in (7.7): Lemma 6.1. Moreover, the middle minimum in (7.7) is the same, since the excluded poset is the complement of the previous one. We obtain the following inequality from (7.1) and the lower estimates.

$$\begin{aligned} n! \ge{}& \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)! \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)! \left(\left\lfloor \frac{n}{2} \right\rfloor^2 - n\left\lfloor \frac{n}{2} \right\rfloor + n^2 - 1\right) \alpha \\ &+ \sum_{r=1}^{K}(u^*!(n-u^*)! + ru^*!u^*(n-u^*-1)!)\beta(r) \\ &+ \sum_{r=1}^{K}(u^*!(n-u^*)! + ru^*!u^*(n-u^*-1)!)\gamma(r). \end{aligned} \tag{7.8}$$

(7.6) will be maximized with respect to (7.8). For this aim, let us modify (7.8):

$$\begin{aligned} n! \ge{}& \frac{\left(\lfloor \frac{n}{2} \rfloor - 1\right)! \left(\lceil \frac{n}{2} \rceil - 1\right)! \left(\lfloor \frac{n}{2} \rfloor^2 - n\lfloor \frac{n}{2} \rfloor + n^2 - 1\right)}{3} 3\alpha \\ &+ \sum_{r=1}^{K} \frac{u^*!(n-u^*)! + ru^*!u^*(n-u^*-1)!}{r+1}(r+1)\beta(r) \end{aligned}$$

$$+\sum_{r=1}^{K}\frac{u^*!(n-u^*)!+ru^*!u^*(n-u^*-1)!}{r+1}(r+1)\gamma(r). \tag{7.9}$$

(6.2) gives a good lower estimate on the ratios in all terms, except the first one. Surprisingly, the same estimate is valid for the first term:

$$u^*!u^*(n-u^*-1)! \leq$$
$$\frac{1}{3}\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)!\left(\left\lceil\frac{n}{2}\right\rceil-1\right)!\left(\left\lfloor\frac{n}{2}\right\rfloor^2-n\left\lfloor\frac{n}{2}\right\rfloor+n^2-1\right)$$

as it can be easily checked for each u^*. All ratios can be replaced by $u^*!u^*(n-u^*-1)!$ in (7.9):

$$n! \geq u^*!u^*(n-u^*-1)!(3\alpha+\sum_{r=0}^{\infty}(r+1)\beta(r)+\sum_{r=0}^{\infty}(r+1)\gamma(r)) = |\mathcal{F}|u^*!u^*(n-u^*-1)!.$$

The proof can be finished like in the previous section. □

8 Concluding Remarks

1. Lemma 7.2 is much easier than Lemma 6.1. That is, we obtained Theorem 7.1 almost free after having the proof Theorem 1.3 with our method. This probably will happen often. The solution for a given excluded configuration can be obtained by putting together estimates for "allowed" posets, which have been already solved for other excluded patterns.

2. We see that the method can be applied for many other problems, on the other hand it does not seem to be sufficient for all excluded posets. Our research was not extensive enough to see the limits: what kind of problems can be solved, what cannot be solved using this idea.

For instance Theorem 1.5 has a quite easy proof in [1], on the other hand we do not see its proof with the present method. The family of "allowed" components seems to be too rich to handle.

3. Let us mention one more recent result. Four disticts subsets satisfying $A \subset C, A \subset D, B \subset C, B \subset D$ are called a *butterfly* and are denoted by B.

Theorem 8.1 ([2]). *Let* $n \geq 3$. *Then* $\mathrm{La}(n, B) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor+1}$.

When V_2 was excluded, the largest family consisted of the largest level, plus a $\frac{1}{n}$ part of the next level. Excluding somewhat less, the N, the result is the same. However excluding B there is a considerable jump: the largest family consists of the two largest levels. We do not see the reasons. When and why does a jump occur?

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