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Bill Jackson, Tibor Jordán, Brigitte Servatius, and Herman Servatius

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Bill Jackson*, Tibor Jordán**, Brigitte Servatius***, and Herman Servatius‡

Abstract

A bar-and-joint framework in the plane with degree of freedom 1 is called a mechanism. It is well-known that the operations of 0-extension and 1-extension, the so called Henneberg moves, can always be performed on a framework so that its degree of freedom is preserved. It was conjectured by the first and second author in 2012 that for a mechanism in generic position these operations can be performed without restricting its motion. In this note we provide a counterexample.

1 Introduction

A 2-dimensional *(bar-and-joint) framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^2$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^2$. A *flexing* of the framework (G, p) is a continuous function $\pi : [0, 1] \times V \to \mathbb{R}^2$ such that $\pi_0 = p$, and such that the corresponding edge lengths in frameworks (G, p) and (G, π_t) are the same for all $t \in [0, 1]$, where $\pi_t : V \to \mathbb{R}^2$ is defined by $\pi_t(v) = \pi(t, v)$ for all $v \in V$. The flexing π is *trivial* if the frameworks (G, p) and (G, π_t) are congruent for all $t \in [0, 1]$. A framework is said to be *rigid* if it has no non-trivial flexings. It is called a *mechanism* if it has degree of freedom 1, that is, if it is not rigid but can be made rigid by inserting one additional bar.

A realization of a graph is *generic* if there are no algebraic dependencies between the coordinates of the vertices. It is known, see [7], that the rigidity of frameworks (and their degree of freedom) in $\mathbb{R}^2$ is a generic property, that is, the rigidity of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic.

*School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, England. email: `B.Jackson@qmul.ac.uk`

**Department of Operations Research, Eötvös University, and the MTA-ELTE Egerváry Research Group on Combinatorial Optimization, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary. email: `jordan@cs.elte.hu`

***Mathematical Sciences, Worcester Polytechnic Institute, Worcester MA 01609-2280, USA. email: `bservat@wpi.edu`

‡Mathematical Sciences, Worcester Polytechnic Institute, Worcester MA 01609-2280, USA. email: `hservat@wpi.edu`

We say that the graph G is *rigid* in $\mathbb{R}^2$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^2$ is rigid. See [1, 7] for more details.

The 0-*extension* operation on vertices x, y in a graph H adds a new vertex z and new edges xz, yz to H. The *1-extension* operation [2] (on edge xy and vertex w) deletes an edge xy from a graph H and adds a new vertex z and new edges zx, zy, zw for some vertex $w \in V(H) - \{x, y\}$. See Figure 1. It is known that these extension operations, the so-called *Henneberg moves*, preserve rigidity [6]. A graph $G = (V, E)$ is *minimally rigid* if G is rigid, but $G - e$ is not rigid for all $e \in E$. We say that graph H is a *mechanism* if $H = G - e$ for some minimally rigid graph G.

It is well-known that a graph is minimally rigid if and only if it can be constructed from an edge by a sequence of 0-extensions and 1-extensions. Similarly, if K, G are mechanisms with $K \subset G$ and such that K is contained in no rigid subgraph of G, then G can be obtained from K by a sequence of 0-extensions and 1-extensions.

Let $G = (V, E)$ be a graph. We shall consider realisations (G, p) of G in $\mathbb{R}^2$ which are in *standard position with respect to two given vertices* v_1, v_2 i.e. $p(v_1) = (0, 0)$ and $p(v_2)$ lies on the 'y-axis'. We will suppress the coordinates of v_1, v_2 which are fixed at zero and take $p \in \mathbb{R}^{2|V|-3}$. We say that such a realisation (G, p) is *generic* if the $2|V| - 3$ coordinates of p are algebraically independent over $\mathbb{Q}$.

Given a realisation (G, p) of a mechanism G we refer to the set of all frameworks (G, q) which are in standard position with respect to (v_1, v_2) and can be reached by a flexing of (G, p) as *the flex of* (G, p). Let $\Theta(G, p) = \{q \in \mathbb{R}^{2|V|-3} : (G, q) \text{ is in the flex of } (G, p)\}$. It is known that $\Theta(G, p)$ is diffeomorphic to a circle when (G, p) is generic.

Suppose K, G are mechanisms with $K \subset G$ and $v_1, v_2 \in V(K)$. Put $\Theta(K, G, p) = \{q|_K : q \in \Theta(G, p)\}$. Then $\Theta(K, G, p) \subseteq \Theta(K, p|_K)$ but we may have $\Theta(K, G, p) \neq \Theta(K, p|_K)$ since the edges of G which do not belong to K may place additional constraints on how $(K, p|_K)$ flexes inside of (G, p). When $\Theta(K, G, p) \neq \Theta(K, p|_K)$, $\Theta(K, G, p)$ will be a closed 1-manifold with boundary i.e. will be diffeomorphic to a closed line segment. This can occur, for example, when K is contained in a rigid subgraph of G, in which case $\Theta(K, G, p)$ will contain the single point $p|_K$.

Motivated by a problem concerning globally linked pairs of vertices in graphs, it was conjectured in a recent paper [4] that if K, G are mechanisms with $K \subseteq G$ and such that K is contained in no rigid subgraph of G then there exists a generic realisation (G, p) of G such that $\Theta(K, G, p) = \Theta(K, p|_K)$. This conjecture is still open.

The inductive construction of mechanisms from submechanisms mentioned above leads to the idea of proving this conjecture recursively. It can be seen that if G is obtained from H by a 0-extension, and $\Theta(K, H, p) = \Theta(K, p|_K)$ for some generic realisation (H, p) of H, then p can be extended to a generic realisation $(G, \tilde{p})$ of G such that $\Theta(K, G, \tilde{p}) = \Theta(K, \tilde{p}|_K) = \Theta(K, p|_K)$. This led the first and second author of this note to conjecture an analogous result for 1-extensions. The conjecture was posed at a workshop on rigidity held at BIRS (Banff, Canada) in 2012. A similar idea was outlined previously by Owen and Power [5, Problem 2].

This conjecture was subsequently disproved by the third and fourth author. The goal of this note is to present a small (in fact, the smallest possible) counterexample together with a simple analysis. We shall prove the following:

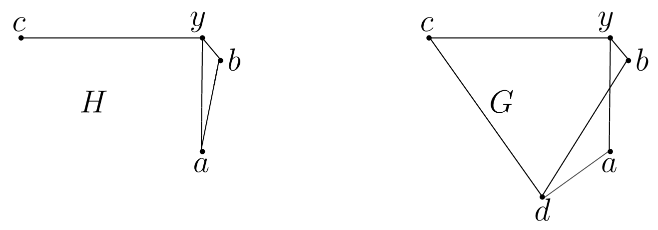

Figure 1: Let K be the subgraph of G and H with $V(K) = \{a, y, c\}$ and $E(K) = \{ay, yc\}$. The graph G is obtained from H by a 1-extension which deletes the edge ab and adds the vertex d and the edges da, db, dc.

Theorem 1.1. *There exist mechanisms K, H, G with $K \subset H$ and such that K is contained in no rigid subgraph of G and G is obtained from H by a 1-extension, for which $\Theta(K, H, p) = \Theta(K, p|_K)$ for some generic realisation (H, p) of H and for which p cannot be extended to a generic realisation $(G, \tilde{p})$ of G for which $\Theta(K, G, \tilde{p}) = \Theta(K, \tilde{p}|_K) = \Theta(K, p|_K)$.*

We shall prove that the graph in Figure 1 is a counterexample. Based on this fact it is not hard to construct an infinite family of counterexamples. We need some new notation and two lemmas. The first lemma is a special case of [3, Lemma 3.3].

Lemma 1.2. *[3] Let (G, p) be a generic realization of a mechanism G in standard position. Then the set of edge lengths of (G, p) are algebraically independent over the rationals.*

Given a framework (G, p) and an edge uv of G let $\ell_p(uv)$ denote the length of uv in (G, p). We suppress the subscript p when it is obvious which realisation we are referring to.

Lemma 1.3. *Let (C_4, p) be a realisation of the 4-cycle $C_4 = aybda$. Suppose that $\ell(ay) > 2\ell(by)$. Consider the flex $\Theta(C_4, p)$ with a pinned at the origin and y pinned on the y-axis and suppose that d transcribes a circle around a in this flex. Then $\ell(ad) \leq 2\ell(by)$*

Proof. Note that since ay is an edge, y must also remain fixed throughout the flex. Applying the triangle inequality when d is at the point furthest away from y in the flex, we deduce that

$$\ell(db) - \ell(by) \leq \ell(ad) + \ell(ay) \leq \ell(db) + \ell(by). \tag{1}$$

Consider the case when $\ell(ad) \geq \ell(ay)$. Applying the triangle inequality when d is at the point nearest to y in the flex, we deduce that

$$\ell(db) - \ell(by) \leq \ell(ad) - \ell(ay) \leq \ell(db) + \ell(by). \tag{2}$$

Adding (1) and (2) gives

$$\ell(db) - \ell(by) \leq \ell(ad) \leq \ell(db) + \ell(by)$$

and hence $\ell(db) - \ell(by) + \ell(ay) \leq \ell(ad) + \ell(ay)$. Inequality (1), now gives

$$\ell(db) - \ell(by) + \ell(ay) \leq \ell(ad) + \ell(ay) \leq \ell(db) + \ell(by).$$

This gives $\ell(ay) \leq 2\ell(by)$ and contradicts the hypothesis that $\ell(ay) > 2\ell(by)$.

Hence $\ell(ad) < \ell(ay)$. Applying the triangle inequality when d is at the point nearest to y in the flex, we deduce that

$$\ell(db) - \ell(by) \leq -\ell(ad) + \ell(ay) \leq \ell(db) + \ell(by). \tag{3}$$

Adding (1) and (3) gives $\ell(db) - \ell(by) \leq \ell(ay) \leq \ell(db) + \ell(by)$ and hence $\ell(db) - \ell(by) + \ell(ad) \leq \ell(ad) + \ell(ay)$. Inequality (1), now gives

$$\ell(db) - \ell(by) + \ell(ad) \leq \ell(ad) + \ell(ay) \leq \ell(db) + \ell(by)$$

and hence $\ell(ad) \leq 2\ell(by)$ as required. •

Proof of Theorem 1.1. Consider the graphs in Figure 1. Choose a generic realisation (H, p) in which $\ell(cy) > \ell(ay) > 4\ell(by)$. Extend this to a realisation (G, p) of G by placing d at some point in the plane. Consider the flexes of (G, p) and $(H, p|_H)$ which keep a fixed at the origin and y on the 'y-axis'. Note that since ay is an edge this implies that y also remains fixed throughout each flex. It is easy to see that c transcribes a circle about y during the flex of $(H, p|_H)$ and hence that $\Theta(K, H, p|_H) = \Theta(K, p|_K)$.

Assume that c transcribes a circle around y in $\Theta(G, p)$. Let (G, p_t) be the position of G at time t in $\Theta(G, p)$. The triangle inequality gives $|\ell(da) - \ell(dc)| \leq \|p_t(c) - p_t(a)\| \leq \ell(da) + \ell(dc)$ for all $t \in [0, 1]$. Since a is pinned at the origin this gives

$$|\ell(da) - \ell(dc)| \leq \min_t\{\|p_t(c)\|\} = \ell(cy) - \ell(ay) \tag{4}$$

and

$$\ell(da) + \ell(dc) \geq \max_t\{\|p_t(c)\|\} = \ell(cy) + \ell(ay). \tag{5}$$

We first consider the case when d transcribes a circle around a in $\Theta(G, p)$. Lemma 1.3 applied to the 4-cycle $aybda$ implies that $\ell(ad) \leq 2\ell(by)$. This gives $2\ell(ad) \leq 4\ell(by) < \ell(ay)$. Since c transcribes a circle around y, we may now apply Lemma 1.3 to the 4-cycle $adcya$ (with the roles of a and y reversed) to deduce that $\ell(yc) \leq 2\ell(ad) \leq 4\ell(by)$. This contradicts the fact that $\ell(cy) > 4\ell(by)$.

Hence d cannot transcribe a circle around a in $\Theta(G, p)$. Since the flex of G is continuous and $p(a)$ is inside the circle transcribed by c, the points $p_{t_0}(c), p_{t_0}(d), p_{t_0}(a)$ must be collinear for some $t_0 \in [0, 1]$. Hence we have $|\ell(da) - \ell(dc)| = \|p_{t_0}(c)\|$ or $\ell(da) + \ell(dc) = \|p_{t_0}(c)\|$. Thus equality must hold in either (4) or (5). Either alternative implies that $p_{t_0}(d), p_{t_0}(c), p(a), p(y)$ are collinear. Hence all four points lie on the y-axis and the edge lengths of the four-cycle of (G, p) on vertex set $\{d, c, y, a\}$ are algebraically dependent over the rationals. Lemma 1.2 now implies that (G, p) cannot be generic.

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