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Monochromatic components in edge-colored complete uniform hypergraphs

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Monochromatic components in edge-colored complete uniform hypergraphs

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Abstract

Let K_n^r denote the complete r-uniform hypergraph on vertex set $V = [n]$. An f-coloring is a coloring of the edges with colors $\{1, 2, \dots, f\}$, it defines monochromatic r-uniform hypergraphs $H_i = (V, E_i)$ for $i = 1, \dots, f$, where E_i contains the r-tuples colored by i. The connected components of hypergraphs H_i are called monochromatic components. For $n > rk$ let $f(n, r, k)$ denote the maximum number of colors, such that in any f-coloring of K_n^r, there exist k monochromatic components covering V. Moreover let $f(r, k) = \min_{n>rk} f(n, r, k)$. A reformulation (see [5]) of an important special case of Ryser's conjecture states that $f(2, k) = k + 1$ for all k. This conjecture is proved to be true only for $k \le 4$, so the value of $f(2, 5)$ is not known. On the contrary, in this paper we show that for $r > 2$ we can determine $f(r, k)$ exactly, and its value is rk.

Keywords: Hypergraphs, edge coloring, Ryser's conjecture.

1 Introduction

An r-uniform hypergraph $H = (V, E)$ is called r-partite, if the vertex set is partitioned into r classes: $V = V_1 \cup \ldots \cup V_r$, such that for each edge $e \in E$ and for each $1 \le i \le r$ we have $|e \cap V_i| = 1$. Let $\nu(H)$ denote the size of the maximum matching in H, i.e., the maximum number of pairwise disjoint edges, and let $\tau(H)$ denote the size of the minimum cover of H, i.e., the size of the smallest subset $T \subseteq V$, such that T intersects every edge.

A hyperwalk in H is a sequence $v_1, e_1, v_2, e_2, \dots, v_{t-1}, e_{t-1}, v_t$, where for all $i < t$ we have $v_i \in e_i$ and $v_{i+1} \in e_i$. We say that $v \sim w$, if there is a hyperwalk from v to w. The relation $\sim$ is an equivalence relation, its classes are called the connected components of the hypergraph H.

Let K_n^r denote the complete r-uniform hypergraph on vertex set $V = [n]$. An f-coloring is a coloring of the edges with colors $\{1, 2, \dots, f\}$, it defines monochromatic r-uniform hypergraphs $H_i = (V, E_i)$ for $i = 1, \dots, f$, where E_i contains the r-tuples

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colored by i. The connected components of a hypergraph H_i are called monochromatic components having color i. By *monochromatic components* we mean all monochromatic components having any color.

For $n > rk$ let $f(n, r, k)$ denote the maximum number of colors, such that in any f-coloring of K_n^r there exist k monochromatic components covering V. Moreover let $f(r, k) = \min_{n>rk} f(n, r, k)$.

A famous conjecture (usually called Ryser's conjecture), appeared in the Thesis of his student, J. R. Henderson [7], states that for an r-uniform r-partite hypergraph H the inequality $\tau(H) \leq (r-1)\cdot\nu(H)$ always holds.

This conjecture is widely open, except for the case $r = 2$, when it is equivalent to Kőnig's theorem [8], and for the case $r = 3$, which was proved by Aharoni [1], using topological results from [2]. We mention also some related results. Henderson [7] showed that the conjecture cannot be improved, if $r-1$ is a prime power. Füredi [4] proved that the fractional covering number is always at most $(r-1)\cdot\nu(H)$, and Lovász [9] proved that the fractional matching number is always at least $\frac{2}{r}\cdot\tau(H)$.

Here we concentrate on the special case of $\nu = 1$, i.e., when H is intersecting. Even for this special case, not too much is known. Gyárfás in [5] showed that this special case of the conjecture is equivalent to saying that $f(2, k) = k + 1$, and he also proved this conjecture for $k = 2, 3$ (for $k = 1$ this is an easy observation of Erdős and Rado), and later Tuza [11] announced a proof for $k = 4$. For $k > 4$ this conjecture is also widely open. Some recent papers study this special case, e.g., see [3, 10].

In this paper we concentrate the hypergraph generalization of this reformulation. Gyárfás [6] asked how sharp lower and upper bounds can be given for $f(r, k)$. For $r = 2$ it is clear that $k \leq f(2, k) \leq k + 1$. For bigger values of r it seems that no bounds were published, Gyárfás [6] showed that $5 \leq f(3, 2) \leq 8$.

Surprisingly enough, we show that for any $r \geq 3$ and $k \geq 1$, the value of $f(r, k)$ is exactly rk. Both the proof of $f(r, k) \geq rk$ and the construction showing $f(r, k) \leq rk$ are simple.

2 Construction

Theorem 2.1. *If $r > 2$ then $f(r, k) \leq rk$, i.e., for $n = \binom{rk+1}{k}$ we can color the edges of the r-uniform complete hypergraph K_n^r by $rk+1$ colors, such that no k monochromatic components can cover the vertex set.*

Proof. Let $X = \{1, 2, \ldots, rk + 1\}$ be a set, and let V consist of all the k-element subsets of X, and $n := |V| = \binom{rk+1}{k}$. Let $K_n^r = (V, E)$ be the complete r-uniform hypergraph on V. Each edge $e \in E$ consists of r k-tuples, so it avoids at least one element of X, color e by the smallest $i \in X$, such that e avoids i. This colors all edges of K_n^r by $rk + 1$ colors.

We claim that V cannot be covered by k monochromatic components. For suppose that the edges colored by $1, 2, \ldots, k$ cover the whole V. However in this case $v^* = \{1, 2, \ldots, k\} \in V$ is clearly an uncovered element. □

3 Main Theorem

Theorem 3.1. *If $r > 2$ then $f(r, k) = rk$.*

Proof. It remained to prove that for any $n > rk$, if we color all the r-tuples of V (i.e., the edges of K_n^r) by at most rk colors, then V can be covered by k monochromatic components. We will prove more, namely, that we can cover V by at most k monochromatic components, with the additional property, that no two of them have the same color.

A coloring is *wasteful*, if there is a color i, such that for any r-tuple colored by i is contained in a component of H_j for some $j \neq i$. In this case each r-tuple R colored by i can be recolored by the appropriate color $j \neq i$, where R is contained in a component of H_j. For each $j \neq i$ the components of H_j remain the same, and finally color i is unused. If we can cover V by at most k monochromatic components now, then we can cover V by the same k monochromatic components in the original colored hypergraph.

Therefore we may assume that the coloring we are dealing with is not wasteful, so we have an r-tuple R colored by 1, such that R is not contained in any monochromatic component having color $j > 1$. For a subset $R' \subseteq R$ let $col(R')$ denote the set of colors of all those r-tuples that contain R'. Suppose $R_1, R_2 \subseteq R$ having size $|R_1| = |R_2| = r - 1$, and $R_1 \neq R_2$. As $r \geq 3$, using the assumption above, we have $col(R_1) \cap col(R_2) = \{1\}$. As we have at most rk colors, by the pigeonhole principle there is a subset $R' \subseteq R$ with size $|R'| = r - 1$, such that $|col(R')| \leq k$. In this case the $|col(R')| \leq k$ monochromatic components containing R' covers the whole V. □

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References

[1] R. Aharoni Ryser's conjecture for tripartite 3-graphs, *Combinatorica* **21** (2001), pp. 1–4.

[2] R. Aharoni, P. Haxell Hall's theorem for hypergraphs, *J. Graph Theory* **35** (2000), pp. 83–88.

[3] D. S. Altner, J. P. Brooks Coverings and matchings in r-partite hypergraphs *Optimization online* (2010) www.optimization-online.org/DB_HTML/2010/06/2666.html

[4] Z. Füredi Maximum degree and fractional matchings in uniform hypergraphs, *Combinatorica* **1** (1981), pp. 155–162.

[5] A. Gyárfás Partition coverings and blocking sets in hypergraphs, *Commun. Comput. Autom. Inst. Hungar. Acad. Sci.* **71** (1977)

[6] A. Gyárfás Talk at Rényi Institute, (November 11, 2010)

[7] J. R. Henderson Permutation Decompositions of $(0,1)$-matrices and decomposition transversals, *Thesis, Caltech* (1971)
thesis.library.caltech.edu/5726/1/Henderson_jr_1971.pdf

[8] D. Kőnig Theorie der endlichen und unendlichen Graphen, *Leipzig* (1936)

[9] L. Lovász On minimax theorems of combinatorics, *Matematikai Lapok* **26** (1975), pp. 209–264.

[10] T. Mansour, C. Song, R. Yuster A comment on Ryser's conjecture for intersecting hypergraphs *Graphs and Combinatorics* **25** (2009), pp. 101–109.

[11] Zs. Tuza On special cases of Ryser's conjecture, *manuscript.*