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Characterizing and Recognizing Generalized Polymatroids

András Frank, Tamás Király, Júlia Pap, and David Pritchard

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Abstract

Generalized polymatroids are a family of polyhedra with several nice properties and applications. One property of generalized polymatroids used widely in existing literature is "total dual laminarity;" we make this notion explicit and show that *only* generalized polymatroids have this property. Using this we give a polynomial-time algorithm to check whether a given linear program defines a generalized polymatroid, and whether it is integral if so. Additionally, whereas it is known that the intersection of two integral generalized polymatroids is integral, we show that no larger class of polyhedra satisfies this property.

1 Introduction

The joint history of matroids and linear programming dates back to the late 1960s. Edmonds [7] found an explicit inequality description for the independent set polytope of matroids, and showed that its dual linear program is "uncrossable." Building on this, he proved [6] a combinatorial min-max theorem for the maximum weight of a common independent set of two matroids.

Edmonds [6] observed that his techniques and results immediately extended from independent set polytopes to the more general class of *polymatroids* — a packing linear program (LP) with a nonnegative, monotone non-decreasing submodular upper bound, roughly corresponding to removing the subcardinality restriction from the rank function of matroids. The techniques of [6] also extend in a straightforward way when we replace one or both of the polymatroids by a *contrapolymatroid* — a covering LP with a supermodular lower bound. A common generalization was introduced by Hassin [16, Section VII] who developed a greedy algorithm for polyhedra constrained simultaneously by a nonnegative, monotone non-decreasing submodular function from above and by a nonnegative, monotone non-decreasing supermodular function from below, satisfying a certain cross-inequality linking the two functions

*MTA-ELTE Egerváry Research Group, Dept. of Operations Research, Eötvös Loránd University, Budapest, {frank,tkiraly,papjuli}@cs.elte.hu. Supported by grant no. CK80124 from the National Development Agency of Hungary, based on a source from the Research and Technology Innovation Fund.

**Department of Computer Science, Princeton University, dp6@cs.princeton.edu.

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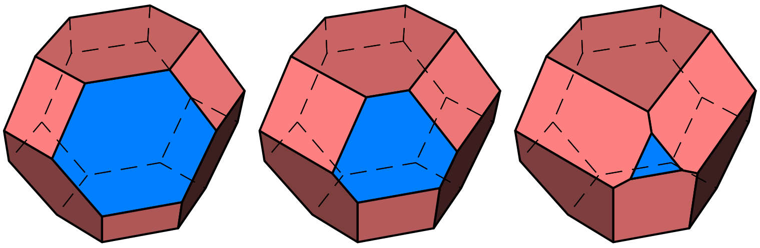

Figure 1: Left: an illustration of a g-polymatroid. Its vertices are all ordered distinct 3-tuples from $\{0,1,2,3\}$. Its facet-defining inequalities are $\binom{|S|}{2} \le x(S) \le 3|S| - \binom{|S|}{2}$ for each nonempty $S \subsetneq [3]$. The facet defined by $x_1 + x_2 + x_3 \le 6$ is highlighted in blue. Center and right: the polytopes obtained by increasing the right-hand side of the constraint $x_1 + x_2 + x_3 \le 6$ to 6.5 and 7.2 respectively. The center polytope is still a g-polymatroid, but the rightmost is not.

(see Definition 1.1 below). Finally, the slightly more general concept of *generalized polymatroids* (g-polymatroids for short) — when assumptions on nonnegativity, finiteness, and monotonicity of the constraining set functions are omitted — was introduced and investigated in [9] to unify objects like polymatroids, contra-polymatroids, base-polyhedra, and submodular polyhedra.

For arbitrary set-functions p, b with $p : 2^{[n]} \to \mathbb{R} \cup \{-\infty\}$ and $b : 2^{[n]} \to \mathbb{R} \cup \{+\infty\}$, let $Q(p, b)$ denote the packing-covering polyhedron

$$Q(p, b) := \{x \in \mathbb{R}^n \mid \forall S \subseteq [n] : p(S) \le x(S) \le b(S)\}. \tag{1}$$

Note that infinities mean absent constraints. In this paper, we treat $\pm\infty$ as "integers" for convenience.

Definition 1.1 (Paramodular, g-polymatroid)**.** The pair (p, b) is defined to be *paramodular* if p is supermodular, b is submodular, $p(\varnothing) = b(\varnothing) = 0$, and the "cross-inequality" $b(S) - p(T) \ge b(S \setminus T) - p(T \setminus S)$ holds for all $S, T \subseteq [n]$. A *g-polymatroid* is either $\varnothing$, or any polyhedron $Q(p, b)$ where (p, b) is paramodular.

Any g-polymatroid defined by a paramodular pair was shown in [9] to be non-empty, and $\varnothing$ is included just for convenience.

Figure 1 shows two examples of g-polymatroids, and one non-example.

Several properties of polymatroids were proved to hold also for g-polymatroids in [9]. A g-polymatroid is integral if and only if p and b are integral (a polyhedron is *integral* if each face contains an integral point; equivalently [8], every integral objective function yields an integer optimal value). Moreover, even the linear system $\{p_i(S) \le x(S) \le b_i(S)$ for every $S \subseteq [n], i = 1, 2\}$ describing the intersection of two g-polymatroids is totally dual integral, and hence the intersection is integral (a linear system is *totally dual integral (TDI)* if for each integral primal objective with finite optimal value, some optimal dual solution is integral). See also the surveys [12, 13] and the books [11, 15] as references.

A further important property proved in [9] is that distinct paramodular pairs define distinct g-polymatroids, or in other words, a non-empty g-polymatroid uniquely determines its defining paramodular pair. However, $Q(p, b)$ may be a g-polymatroid even if (p, b) is not paramodular. In fact, there are various relaxations of the notion of paramodularity that still define g-polymatroids, for example intersecting paramodularity. These kinds of weaker forms are important in several applications because they help recognizing polyhedra given in specific forms to be g-polymatroids. The main question we are led to consider is: what exactly is necessary and sufficient to define a g-polymatroid? Also, does there exist a polynomial algorithm that given a linear system, decides if the polyhedron described by it is a(n integral) g-polymatroid? We will answer these questions in Section 4.

Consider a packing-covering polyhedron, where every constraint is of the form $x(S) \geq \beta$ or $x(S) \leq \beta$: it is of the form $Q(p, b)$ for some p, b. In LP duality each such constraint gives rise to a dual variable corresponding to S. Let y^ℓ resp. y^u be the dual variable vector corresponding to the lower resp. upper bound constraints. If in the primal problem we want to maximize cx over $Q(p, b)$, then the dual is:

$$\{\min y^u b - y^\ell p \mid y^u, y^\ell \geq \mathbf{0}, (y^u - y^\ell)\chi = c\}, \tag{2}$$

where χ denotes the matrix whose rows are the characteristic vectors χ_S of the subsets S of $[n]$. As a technicality, when $b(S) = +\infty$ (or likewise $p(S) = -\infty$) for some S, the dual variable y^u_S does not really exist, but the notation (2) still accurately represents the dual provided that y^u_S is fixed at 0 and the constant $y^u_S b(S)$ term in the objective is ignored — all duals we deal with will have finite objective value, so $y^u_S = 0$ is without loss of generality.

1.1 Results

The *support* of a dual solution is the set system consisting of all sets for whom at least one dual variable is nonzero. A set system is *laminar* if for every two sets S_i, S_j in it, either $S_i \subseteq S_j$, or $S_j \subseteq S_i$, or $S_i \cap S_j = \varnothing$. A dual solution is *laminar* if its support is laminar.

Definition 1.2 (TDL)**.** The pair (p, b) is *totally dual laminar* (TDL) if for every primal objective with finite optimal value, some optimal dual solution to (2) is laminar.

The TDL property is already ubiquitous in the literature, but we think it is useful to make it explicit and give it an idiomatic name.

One of our main results, Theorem 2.2, is to show that if (p, b) is totally dual laminar, then the polyhedron $Q(p, b)$ is a g-polymatroid. If in addition p and b are integral, then $Q(p, b)$ is an integral g-polymatroid. This, together with Theorem 2.1, characterizes g-polymatroids as the set of all polyhedra that have at least one TDL formulation. As a negative result, we show in Section 2.4 that testing if a given system is TDL is NP-hard.

In Section 4 we show that there is a polynomial-time algorithm, which for a given system of linear inequalities, determines whether the polyhedron it describes is a

$$
\begin{array}{lll}
 & (p,b) \text{ paramodular} & \in \mathsf{P} \\
 & \Downarrow & \\
 & (p,b) \text{ intersecting paramodular} & \in \mathsf{P} \\
 & \Downarrow & \\
 & (p,b) \text{ near paramodular} & \in \mathsf{P} \\
 & \Downarrow & \\
 & (p,b) \text{ truncation paramodular} & \in \mathsf{P} \\
 & \Downarrow & \\
\begin{array}{c}\text{equivalent if } Q(p,b) \\ \text{is full-dimensional}\end{array} \left\{ \begin{array}{l} \\ \\ \\ \\ \\ \end{array} \right. & \begin{array}{l} (p,b) \text{ TDL} \\ \Downarrow \\ Q(p,b) \text{ integer g-polymatroid} \\ \Downarrow \\ Q(p,b) \text{ g-polymatroid} \end{array} & \begin{array}{l} \mathsf{NP}\text{-hard} \\ \\ \in \mathsf{P} \\ \\ \in \mathsf{P} \end{array}
\end{array}
$$

Figure 2: Summarizing most of our results, where (p, b) is an integer-valued pair whose finite values are given explicitly as an input. The pre-existing notions of *intersecting* and *near* paramodularity are defined in Section 2.2 and Section 6, respectively.

g-polymatroid (Theorem 4.1). Despite that testing for TDL is NP-hard, the proof uses Theorem 2.2, uncrossing methods, and a decomposition theorem for non-full-dimensional g-polymatroids. The method also gives a polynomial-time algorithm to tell whether a g-polymatroid is integral, see Theorem 4.16. In contrast, testing an arbitrary polyhedron for integrality [20] or TDI-ness is coNP-complete [5], the latter even for cones [19].

One might ask for a g-polymatroid P if it is true that *every* (p, b) such that $Q(p, b) = P$ satisfies that (p, b) is TDL? This is, in fact, false, as Example 4.12 shows. But it is a consequence of Theorem 4.4 that it holds in the special case when P is full-dimensional.

Edmonds' polymatroid intersection theorem was shown in [9] to extend to integral g-polymatroids as well. In Theorem 5.1 we prove the following converse statement: if the intersection of a polyhedron P with each integral g-polymatroid is integral, then P is an integral g-polymatroid. By combining this with the g-polymatroid intersection theorem, one obtains that a polyhedron P is an integral g-polymatroid if and only if its intersection with every integral g-polymatroid is integral. In other words, the family of integral g-polymatroids is maximal subject to integral pairwise intersections.

In Section 6 we give a relaxation of paramodularity, called truncation-paramodularity, that guarantees total dual laminarity, and can be verified in polynomial time if the finite values of the functions are given as an input. This relaxation enables us to give a short proof of a mild generalization of Schrijver's supermodular colouring theorem. The relations that we obtain between truncation-paramodularity and related versions of paramodularity are summarized in Figure 2 on page 4.

Recently there have been several interesting studies of a class of polyhedra called *generalized permutahedra* [1, 2, 21, 22]. By slightly extending this line of work we get one more interesting characterization, illustrated in Figure 3 on page 5. Let χ_i denote the ith standard unit basis vector, χ_0 the origin, and for $S \subseteq \{0\} \cup [n]$, let $\triangle_S$ denote conv.hull$\{\chi_i \mid i \in S\}$. The following result will be proved in Section 7.

Theorem 1.3. *A polyhedron P is a nonempty bounded g-polymatroid if and only if there is an equality of Minkowski sums*

$$P + \sum_{i=1}^{\ell} \lambda_i \triangle_{L_i} = \sum_{j=1}^{r} \rho_j \triangle_{R_j}$$

for some choice of positive multipliers λ, ρ and nonempty subsets L_i, R_j of $\{0\} \cup [n]$. Moreover, each nonempty bounded g-polymatroid P has exactly one such representation (up to order) such that the L_i and R_j are mutually distinct and the trivial $\triangle_{\{0\}}$ is not used.

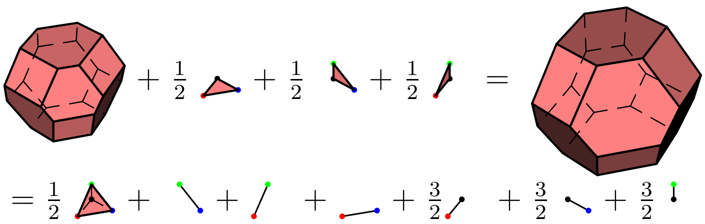

Figure 3: Illustrating Theorem 1.3 for the g-polymatroid shown in the center of Figure 1, where + indicates the Minkowski sum. The 4 coloured points denote the origin (black) and three standard unit basis vectors. The polyhedron in the right of Figure 1 does not admit such a decomposition.

Preliminary definitions. The *direct product* or *Cartesian product* of two polyhedra $P \subseteq \mathbb{R}^A$ and $Q \subseteq \mathbb{R}^B$ is $P \times Q := \{(x, y) \in \mathbb{R}^{A \cup B} \mid x \in P, y \in Q\}$ (we assume A and B are disjoint). A *subpartition* of a set X is a family of pairwise disjoint nonempty subsets of X; i.e. it is a partition of a subset of X. The *1-norm* $\|v\|_1$ of a vector v is the sum of the absolute values of its coordinates, $\|v\|_1 = \sum_i |v_i|$.

1.2 Related Work

Why are natural characterizations of g-polymatroids important? Many other general classes of polyhedra with somewhat esoteric definitions have been studied: e.g. lattice polyhedra [17], submodular flow polyhedra [8], bisubmodular polyhedra [26, §49.11d], and M-convex functions [18]. In some cases the definitions are chosen to be precisely as general as possible while allowing the proof techniques to go through, e.g. Schrijver's framework for total dual integrality with cross-free families [26, §60.3c][23]. Simpler characterizations of such classes are more likely to arise naturally, and can be easier to understand. Relations amongst these complex classes are known: Schrijver [24] showed that P is a submodular flow polyhedron iff P is a lattice polyhedron for a distributive lattice; and Frank and Tardos [13] showed that P is a submodular flow polyhedron iff P is the projection along coordinate axes of the intersection of two g-polymatroids.

A few characterizations of g-polymatroids are known. One uses *base polyhedra*, which generalize the convex hull of the bases of a matroid. A base polyhedron is a set $\{x \in \mathbb{R}^n \mid \forall S \subset [n], x(S) \leq b(S); x([n]) = b([n])\}$ where b is submodular with $b(\varnothing) = 0$. So each base polyhedron is a subset of the hyperplane $x([n]) = c$ for some constant c. An important relation, whose proof is short and originally due to Fujishige [14], is:

Theorem 1.4. *$B \subseteq \{x : x([n]) = c\}$ is a base polyhedron if and only if the projection $\{(x_1, \dots, x_{n-1}) \mid x \in B\}$ is a nonempty g-polymatroid.*

To prove Theorem 5.1, we exploit another known characterization, implicitly by Tomizawa [30] (see proof and discussion in [15, Thm. 17.1]):

Theorem 1.5. *A polyhedron in $\mathbb{R}^n$ is a g-polymatroid if and only if for each x, its tangent cone at x has a generating set which is a subset of $\{\pm\chi_i \mid i \in [n]\} \cup \{\chi_i - \chi_j \mid i, j \in [n]\}$.*

A result of Danilov and Koshevoy [4] is related to Theorem 5.1 but is slightly weaker. They call a collection of rational linear subspaces a *pure system* if the following holds: if P_1 and P_2 are two integer polyhedra with the property that the affine hull of any face is a translation of a subspace in the collection, then $P_1 \cap P_2$ is an integer polyhedron. It is shown in [4, Example 9] that the pure system generated by the possible faces of n-dimensional g-polymatroids is a maximal pure system.

A useful property [12, 13] is that for a g-polymatroid P defined by an unknown paramodular pair, the minima and maxima

$$i(S) := \min_{x \in P} x(S) \qquad \text{and} \qquad a(S) := \max_{x \in P} x(S) \tag{3}$$

yield the unique defining paramodular pair, i.e. $P = Q(i, a)$. This implies that when (p, b) and (p', b') are paramodular and distinct, $Q(p, b)$ and $Q(p', b')$ are also distinct.

The family of g-polymatroids is closed under translation, reflection of all coordinates, box-intersection, taking faces, direct products, and many other operations [12, 13]. Linear optimization over a bounded g-polymatroid is possible with an iterative greedy algorithm [13]; conversely, bounded P is a g-polymatroid iff for every objective $\max\{c \cdot x \mid x \in P\}$, the following iterative greedy algorithm is always correct: iteratively maximize the coordinates with positive c-coefficients in decreasing c-order, minimize those with negative c-coefficients similarly, and interleave the maximizations and minimizations arbitrarily [27].

One notable application of g-polymatroids is in network design. Two flavours of network design problems are addressed in [10] using g-polymatroids — undirected pair-requirements and directed uniform requirements. One obtains min-max relations and algorithms for edge connectivity augmentation, even subject to degree bounds. In these applications, it is important that g-polymatroids can be defined by skew-submodular or intersecting-submodular functions. Total dual laminarity is the typical property used to show that such functions define g-polymatroids: it is therefore natural that we try to properly understand this property.

Given a set function p, consider the problem of k-colouring the ground set so that each set S gets at least $p(S)$ different colours. When p is supermodular, or even the maximum of two supermodular functions, this "supermodular colouring" problem can be attacked with g-polymatroids, and one can show that a colouring exists except if one of the obvious obstructions $p(S) > |S|$ or $p(S) > k$ holds for some S. This was proven by Schrijver [25], simplified by Tardos [29] and Schrijver [26], and a variant was proven by Király (as described in [3]). We will prove a more general version of this theorem, as an example of how TDL can be used in a proof.

The *n-permutahedron* is a classical polytope, defined as the convex hull of the $n!$ permutations of $(1, 2, \dots, n)$. For example, the g-polymatroid in the left part of Figure 1 on page 2 is essentially the 4-permutahedron. In 2005 Postnikov [21] defined *generalized permutahedra* as "deformations" of the permutahedron:

Definition 1.6. Let Π_n denote the set of all n-permutations. A *generalized permutahedron* is any polytope conv.hull$\{x_\pi \mid \pi \in \Pi_n\}$ such that the x_π satisfy, for all π and all neighbour transpositions $(i\ i+1)$, that $x_\pi - x_{\pi\circ(i\ i+1)}$ is a nonnegative multiple of $\chi_{\pi[i]} - \chi_{\pi[i+1]}$.

The focus of Postnikov's paper [21] is computing the volumes and integer volumes (Ehrhart theory) of generalized permutahedra. Note the similarity between Definition 1.6 and Tomizawa's theorem. In fact, the following theorem can be proven using Tomizawa's theorem as a starting point:

Theorem 1.7. *The class of generalized permutahedra is the same as the class of bounded base polyhedra.*

This result was also mentioned in [2, Thm. 2.1]. Postnikov et al. [22] proved that the vertex deformation used in Definition 1.6 can be rephrased in other equivalent ways. For example P is a base polytope if and only if its normal fan refines that of the permutahedron.

2 Total dual laminarity

As a general application of Edmonds' methods, two key steps in [9] were proving that every paramodular pair is TDL, and that the intersection of two TDL systems is totally dual integral.

Theorem 2.1 ([9]). *1. If the pair (p, b) is paramodular then it is TDL.*

2. If (p_1, b_1) and (p_2, b_2) are TDL pairs, then the linear system

$$\{x \in \mathbb{R}^n : \ p_i(S) \le x(S) \le b_i(S) \text{ for every } S \subseteq [n], i = 1, 2\}$$

is totally dual integral.

The core of the second statement is the fact that the incidence matrix of the union of two laminar families is totally unimodular. In this section we prove that only g-polymatroids can be described by TDL systems, we give short TDL-based proofs of several g-polymatroid properties, and we show that testing TDL is NP-hard.

2.1 All TDL Systems define Generalized Polymatroids

Theorem 2.2. *If (p, b) is totally dual laminar, then the polyhedron $Q(p, b)$ is a g-polymatroid. If in addition p and b are integral, then $Q(p, b)$ is an integral g-polymatroid.*

Proof. We assume $Q(p, b)$ is nonempty. Define i and a as in Equation (3) where $P = Q(p, b)$. Observe that $Q(p, b) = Q(i, a)$. We will prove the theorem by showing that (i, a) is paramodular.

An important special kind of laminar family is a *chain* family: a chain is a set family where for any two sets in the family, one contains the other.

Overview. The proof's heart has a combinatorial flavour similar to traditional dual uncrossing arguments: we gradually make an optimal dual solution in (2) more and more structured. First we show every optimal $y = (y^u, y^\ell)$ with $\mathtt{supp}(y^u) \cup \mathtt{supp}(y^\ell)$ laminar can be transformed into one such that $\mathtt{supp}(y^u)$ and $\mathtt{supp}(y^\ell)$ are two laminar families on *disjoint ground sets*. Then, we transform the laminar families into chain families on disjoint ground sets, so-called "dichain duals". Crucially, for every c, exactly one dichain dual is feasible, and so an optimal dual can be easily computed. By comparing it to other duals we get inequalities proving that (i, a) is paramodular. Now we give the details.

To begin with, we normalize the form of the program; we call the (p, b) formulation (1) and (2) the *old* primal and dual, whereas $\{x \mid \forall S \subset [n], i(S) \leq x(S) \leq a(S)\}$ is the *new* primal. The new primal LP in matrix form is $\{\max cx \mid x \in \mathbb{R}^n, i \leq \chi x \leq a\}$ and its dual is

$$\{\min y^u a - y^\ell i \mid y^u \geq \mathbf{0}, y^\ell \geq \mathbf{0}, (y^u - y^\ell)\chi = c\}. \tag{4}$$

Proposition 2.3. *For every c with finite optimum, the new dual (4) has an optimum $Y = (Y^u, Y^\ell)$ such that $\mathtt{supp}(Y^u) \cup \mathtt{supp}(Y^\ell)$ is a laminar family, i.e. (i, a) is also TDL.*

Proof. We know by the hypothesis of the theorem that the old dual has such an optimum $Y = (Y^u, Y^\ell)$ whose combined support is laminar. We show that Y is an optimal solution of the new dual. For this it suffices to observe that for every old primal constraint getting positive dual in Y, the new primal contains that same constraint. To see this for a packing constraint $x(S) \leq b(S)$ getting positive dual, obviously $a(S) \leq b(S)$, but also by complementary slackness an optimal primal x satisfies $x(S) = b(S)$, so $a(S) \geq b(S)$. The covering case is similar. □

From now on we only work with the new primal/dual, so we just call them the primal/dual.

Proposition 2.4. *For every c with finite optimum, the dual (4) has an optimum Y such that $\mathtt{supp}(Y^u)$ and $\mathtt{supp}(Y^\ell)$ are laminar families on* disjoint *ground sets.*

Proof. Let $Y = (Y^u, Y^\ell)$ represent the dual guaranteed by Proposition 2.3, so $\mathcal{L} = \mathtt{supp}(Y^u) \uplus \mathtt{supp}(Y^\ell)$ is laminar (possibly with repeats). Fix a tree representation of $\mathcal{L}$, meaning a forest of rooted trees on node set $\mathcal{L}$ so that each child is a subset of its parent, and the roots are disjoint; it is unique up to ordering of the repeats. Each set in $\mathcal{L}$ has a u *sign* if it came from $\mathtt{supp}(Y^u)$ and an ℓ *sign* if it came from $\mathtt{supp}(Y^\ell)$.

If every set in $\mathcal{L}$ has the same sign as its parent, we are done. Otherwise, take an inclusion-maximal parent-child pair $P \supseteq C$ whose signs differ. Suppose P has sign u and C has sign ℓ, the other case is similar. Let $\delta = \min\{Y^u_P, Y^\ell_C\}$, and define $\widetilde{Y}$ to be the same as Y except

$$\widetilde{Y}^u_P := Y^u_P - \delta; \qquad \widetilde{Y}^\ell_C := Y^\ell_C - \delta; \qquad \widetilde{Y}^u_{P\setminus C} := Y^u_{P\setminus C} + \delta.$$

We claim that $\widetilde{Y}$ is still an optimal dual solution whose support is laminar. First, the support is still laminar since $\mathcal{L} \cup \{P \setminus C\}$ is laminar and $\mathtt{supp}(\widetilde{Y})$ is this family minus C and/or P. Second, $\widetilde{Y}$ is feasible since $(\widetilde{Y}^u - \widetilde{Y}^\ell)\chi = (Y^u - Y^\ell)\chi + \delta(-\chi_P + \chi_{P\setminus C} + \chi_C) = c + \mathbf{0}$. To show $\widetilde{Y}$ is still optimal it is necessary and sufficient to show that the change in objective, which equals $\delta(-a(P) + i(C) + a(P \setminus C))$, is nonpositive. In other words we need $a(P) \geq a(P \setminus C) + i(C)$; to see this consider any x^* achieving the maximum in the definition of $a(P \setminus C)$, i.e. such that $a(P \setminus C) = x^*(P \setminus C) = \max_{x \in Q(p,b)} x(P \setminus C)$. Then $a(P) \geq x^*(P) = x^*(P \setminus C) + x^*(C) \geq a(P \setminus C) + i(C)$, by definition of $i(C)$.

To show this argument can terminate, suppose we chose the original Y to have minimal 1-norm $\|Y\|_1$ — we can define this Y with a linear program which ensures the infimum is achieved. Observe the transformation $Y \mapsto \widetilde{Y}$ decreases the 1-norm by δ. Consequently for this extremal Y, no parent-child pair has opposing signs, and this Y is what Proposition 2.4 asked for. ☐

Proposition 2.5 (Optimal dichain duals exist)**.** *For every c with finite optimum, the dual (4) has an optimum Y such that* $\mathtt{supp}(Y^u)$ *and* $\mathtt{supp}(Y^\ell)$ *are chain families on disjoint ground sets (a* dichain dual*).*

Proof. The argument is very similar to the previous proposition but simpler and so is just sketched. Start with the $Y = (Y^u, Y^\ell)$ guaranteed by Proposition 2.4. In the laminar family $\mathtt{supp}(Y^u)$ (Y^ℓ is analogous), if it is laminar but not a chain, it has a pair of disjoint sets; let S, T be a pair of such sets with maximal combined size. Then we increase $Y^u_{S\cup T}$ and decrease Y^u_S and Y^u_T until one of them becomes zero. This operation preserves laminarity of $\mathtt{supp}(Y^u)$, retains feasibility and optimality of Y (here we use that $a(S \cup T) \leq a(S) + a(T)$), decreases its 1-norm, and does not change the ground set of the laminar family $\mathtt{supp}(Y^u)$. ☐

Proposition 2.6 (Feasible dichain duals are unique)**.** *For every c, there is at most one dichain dual (Y^u, Y^ℓ) such that $(Y^u - Y^\ell)\chi = c$.*

Proof. In fact there is always *exactly* one such dual: i.e. $c_j = \{\sum_{S:j\in S} Y^u_S - Y^\ell_S\}_j$ is a bijection between dichain duals Y and real vectors $c \in \mathbb{R}^n$. The kth largest positive value in $\{c_j \mid j \in [n]\}$ corresponds to the kth inclusion-smallest set L_k in $\mathtt{supp}(Y^u)$, and the dual value $Y^u_{L_k}$ equals the difference between the kth largest and $(k+1)$th largest values in $\{0\} \cup \{c_j \mid j \in [n]\}$. We deal with negative values and Y^ℓ similarly. A short computation with telescoping sums confirms $(Y^u - Y^\ell)\chi = c$, and a standard proof by induction on k gives uniqueness. ☐

Combining the previous two propositions, we get the following.

Corollary 2.7. *For every c, any feasible dichain dual to (4) is optimal.*

Proof. This is immediate if the optimum is finite. Otherwise the optimal value is $-\infty$, and the dichain dual must also have value $-\infty$ by weak duality. □

Now we are in good shape to complete the proof. First we show a is submodular, i.e. for any $P, Q \subseteq [n]$ that $a(P \cup Q) + a(P \cap Q) \leq a(P) + a(Q)$. Let c assign value 2 to elements of $P \cap Q$, value 1 to elements of the symmetric difference of P and Q, and 0 to all other elements of $[n]$. Since $c = \chi_{P \cup Q} + \chi_{P \cap Q}$, one feasible dual Y is to set $Y^u_{P \cup Q} = Y^u_{P \cap Q} = 1$ and zero elsewhere; it is a dichain dual and hence by Corollary 2.7 the optimal LP value is $a(P \cup Q) + a(P \cap Q)$. On the other hand, another feasible dual is $Y^u_P = Y^u_Q = 1$ and zero elsewhere, hence its objective value $a(P) + a(Q)$ is at least the optimum $a(P \cup Q) + a(P \cap Q)$ and we are done.

The proof that i is supermodular is similar; and to show the cross-inequality $a(P) - i(Q) \geq a(P \setminus Q) - i(Q \setminus P)$ for all $P, Q \subset [n]$ we repeat the argument with $c = \chi_P - \chi_Q = \chi_{P \setminus Q} - \chi_{Q \setminus P}$, by comparing the feasible dual $Y^u_P = Y^\ell_Q = 1$ to the optimal dual $Y^u_{P \setminus Q} = Y^\ell_{Q \setminus P} = 1$ (and zeroes elsewhere). This completes the proof of the first part of Theorem 2.2.

The second part follows easily, since by Theorem 2.1, the system (1) is TDI for a TDL pair (p, b). □

As an aside, when a dichain dual for a given c is optimal for (2), it implies that the iterative greedy algorithm is correct for that c. This reproves the known fact that the iterative greedy algorithm works for g-polymatroids. We consider the case $c_1 > c_2 > \cdots > 0$ where c has finite maximum; the other cases are similar. Propositions 2.5 and 2.6 determine the unique dual optimum. Only one primal solution satisfies complementary slackness with all positive dual variables, namely the primal $\big(a([i]) - a([i-1])\big)_{i=1}^n$. So this primal, call it x^*, is the unique primal optimum. What solution x^g does the iterative greedy algorithm produce? We have $x^g_1 \geq x^*_1$ and $x^g_1 \leq a([1]) = x^*_1$ since the latter is a constraint of the system. So $x^g_1 = x^*_1$. Likewise $x^g_2 \geq x^*_2$, and $x^g_2 \leq a([2]) - a([1]) = x^*_2$ by combining the constraint $x([2]) \leq a([2])$ with the fact that x_1 is fixed at $a([1])$. By induction, $x^g = x^*$.

In contrast, that the iterative greedy algorithm works for some c does not imply that the dichain dual for that c is optimal. For example, take $c = (3, 1 + \epsilon, 1)$ with the right-hand polytope in Figure 1 on page 2. This is surprising, since if the iterative greedy algorithm works for *all* c, we have a g-polymatroid and it implies that the dichain duals are optimal in (2) for *all* c.

2.2 Intersecting Paramodularity

We mention one well-known theorem that follows easily from Theorem 2.2. Two sets S and T *conflict* if all of $S \cap T, S \setminus T, T \setminus S$ are nonempty; note that a set system is laminar iff it has no conflicting pair of sets.

Definition 2.8. A pair $(p : 2^{[n]} \to \mathbb{R} \cup \{-\infty\}, b : 2^{[n]} \to \mathbb{R} \cup \{+\infty\})$ is *intersecting paramodular* if the supermodular, submodular, and cross-inequalities hold for every pair of conflicting sets. That is, for any conflicting S and T, we require $b(S \cap T) + b(S \cup T) \leq b(S) + b(T)$, $p(S \cap T) + p(S \cup T) \geq p(S) + p(T)$, and $b(S \setminus T) - p(S \setminus T) \leq b(S) - p(T)$.

The values of $p(\varnothing)$ and $b(\varnothing)$ have no effect on whether (p, b) is intersecting paramodular.

Theorem 2.9. *When (p, b) is intersecting paramodular, $Q(p, b)$ is a g-polymatroid.*

The original proof of this theorem [13, Prop. 2.5] uses the "truncation" method.

Proof. By Theorem 2.2 it will suffice to show (p, b) is totally dual laminar. We may assume $Q(p, b) \neq \varnothing$. Fix any primal maximization objective c for which the primal is bounded. Along the lines of standard uncrossing arguments, let y be an optimal dual solution to (2), and moreover one for which $y \cdot \mu := \sum_S (y^u_S + y^\ell_S)(n - |S|)^2$ is minimal among all optima. We claim this y has laminar support. If not, there are two positive dual variables for two conflicting sets S, T. In the case that $y^u_S, y^u_T > 0$, consider decreasing y^u_S, y^u_T by $\epsilon := \min\{y^u_S, y^u_T\}$ and increasing $y^u_{S \cup T}, y^u_{S \cap T}$ by ϵ; this would maintain dual feasibility, maintain dual optimality since the submodular inequality holds for b on S and T, and strictly decrease $y \cdot \mu$, a contradiction. The other two cases are similar, establishing that y has laminar support as needed. ☐

2.3 Intersections with Boxes and Planks

A classical property of g-polymatroids that we will use is the following.

Theorem 2.10 ([9])**.** *The family of g-polymatroids is closed under intersecting with* planks $\{x \mid \ell_0 \leq \|x\|_1 \leq u_0\}$ *and* boxes $\{x \mid \ell \leq x \leq u\}$.

Proof. In the TDL framework, the crux is the following:

Observation 2.11. *If L is laminar on ground set $[n]$, then so is $L \cup \{[n]\}$, and $L \cup \{\{i\}\}$ for any $i \in [n]$.*

Consider taking a g-polymatroid P and adding a box upper bound constraint; the other cases are similar. Let $Q(p, b) = P$ be a TDL formulation and add the constraint $x_i \leq u_i$. Whereas the dual of (1) is (2), adding $x_i \leq u_i$ to the primal gives the dual

$$\{\min y^u b - y^\ell p + y^\nu u_i \mid y \geq \mathbf{0}, (y^u - y^\ell)\chi + y^\nu \chi_i = c\}. \tag{5}$$

The new program is TDL for the following reason. Take any dual optimum (Y^u, Y^ℓ, Y^ν) for (5). Let (Z^u, Z^ℓ) be a dual optimum for the original dual (2) under cost vector $c' = c - Y^\nu \chi_i$, such that $\mathtt{supp}(Z) = L$ is laminar. Then (Z^u, Z^ℓ, Y^ν) is an optimal dual for (5), and it has laminar support by Observation 2.11. ☐

2.4 Hardness of Total Dual Laminarity

Theorem 2.12. *Deciding whether a given system is TDL is* NP*-hard.*

Proof. We reduce the 3-dimensional perfect matching problem to it, which is known to be NP-complete. Let $\mathcal{H} = (V_1, V_2, V_3; \mathcal{E})$ be an instance of the 3-dimensional perfect matching problem, that is, a 3-uniform hypergraph on vertex set $V_1 \cup V_2 \cup V_3$ (where V_1, V_2 and V_3 are disjoint and equal in size) and edge set $\mathcal{E} \subseteq V_1 \times V_2 \times V_3$, where the goal is to find a matching $M \subseteq \mathcal{E}$ which covers all vertices. For convenience we assume that the edges cover V_3. We construct the following linear system consisting only of homogeneous equalities.

$$\{x \in \mathbb{R}^{V_1 \cup V_2 \cup V_3} \mid \; x(e) = 0 \; \forall e \in \mathcal{E}, \; x(v) = 0 \; \forall v \in V_1 \cup V_2\}.$$

The dual system is

$$\begin{aligned}\{y \in \mathbb{R}^{\mathcal{E} \cup V_1 \cup V_2} \mid & \\ \sum_{e: v \in e} y_e &= c_v \quad \forall v \in V_3, \\ y_v + \sum_{e: v \in e} y_e &= c_v \quad \forall v \in V_1 \cup V_2\}.\end{aligned}$$

We claim that this system is TDL if and only if $\mathcal{H}$ has a perfect matching. Since V_3 is covered, a dual solution always exists, and all are optimal, thus the system is TDL if and only if for every objective function c there is a dual solution $y \in \mathbb{R}^{\mathcal{E} \cup V_1 \cup V_2}$ for which $\mathtt{supp}(y)$ is laminar.

Suppose that the system is TDL, and take such a y for $c = \mathbf{1}$. Now every vertex in V_3 has to be covered with an edge e with positive dual variable y_e, and these have to be disjoint. In other words, $\mathtt{supp}(y)$ has to contain a perfect matching.

For the other direction, suppose that M is a perfect matching in $\mathcal{H}$, and let c be an objective function. Let us define y by

$$\begin{aligned} y_e &:= \begin{cases} c_{v_3} & \text{if } e = \{v_1, v_2, v_3\} \in M, \\ 0 & \text{if } e \notin M, \end{cases} \\ y_v &:= c_v - y_e \quad \text{if } v \in e \in M, v \in V_1 \cup V_2. \end{aligned}$$

The support of y is laminar and y is an optimal dual solution, so we are done. □

3 Decomposition of Generalized Polymatroids

Recall that in dimension n, a base polyhedron is contained within a hyperplane and thus has dimension at most $(n-1)$. If this holds with equality, we call the base polyhedron *max-dimensional*. The following decomposition theorem is analogous to the decomposition into connected components of a matroid, or more generally of a submodular system, see [15].

Theorem 3.1. *Every nonempty g-polymatroid Q is the direct product of at most one full-dimensional g-polymatroid and some (possibly zero) max-dimensional base-polyhedra.*

Equivalently, every non-max-dimensional base polyhedron is the direct product of several max-dimensional base polyhedra.

Proof. Let (p, b) be a paramodular pair which defines Q. First let us prove that the affine hull of Q is of the form $\{x \in \mathbb{R}^n : x(A_i) = a_i, i \in [t]\}$ for some subpartition $\mathcal{A} = \{A_1, A_2, \dots A_t\}$ of $[n]$. We know that the affine hull is the intersection of the implicit equalities (from the system). An equality $x(S) = b(S)$ is implicit if and only if $p(S) = b(S)$ if and only if the equality $x(S) = p(S)$ is implicit. Let us call such a set *fixed-sum*.

If S and T are fixed-sum, then so are $S \cap T$ and $S \cup T$:

$$\begin{aligned} b(S \cap T) + b(S \cup T) \le b(S) + b(T) = p(S) + p(T) \\ \le p(S \cap T) + p(S \cup T) \le b(S \cap T) + b(S \cup T). \end{aligned}$$

Also, if S and T are fixed-sum, then $S \setminus T$ and $T \setminus S$ are also fixed-sum:

$$\begin{aligned} b(S \setminus T) - p(T \setminus S) \le b(S) - p(T) = p(S) - b(T) \\ \le p(S \setminus T) - b(T \setminus S) \le b(S \setminus T) - p(T \setminus S). \end{aligned}$$

It follows that the inclusion-minimal fixed-sum sets form a subpartition, and that every other fixed-sum set is a disjoint union of them. So they form the desired subpartition $\mathcal{A}$.

The empty set is trivially fixed-sum, and if no other set is fixed-sum, then Q is full-dimensional and we are done. If the only fixed-sum sets are the empty set and $[n]$, then Q is a max-dimensional base polyhedron and we are done again. Otherwise, take a fixed-sum set A other than $[n]$ and $\varnothing$. We claim that $Q = Q_1 \times Q_2$, where Q_1 is a base polyhedron on A and Q_2 is a g-polymatroid on $[n] \setminus A$, then we are done by induction. The following lemma implies this claim.

Lemma 3.2. *For paramodular (p, b), if $p(T) = b(T)$ (i.e. T is fixed-sum), then for any $X \subseteq [n]$ we have $b(X) = b(X \cap T) + b(X \setminus T)$ and $p(X) = p(X \cap T) + p(X \setminus T)$.*

Proof. We derive four inequalities from the cross-inequality and submodularity:

$$\begin{aligned} b(X \cap T) + b(X \setminus T) &\ge b(X) + b(\varnothing) \\ b(X \cup T) - p(T \setminus X) &\ge b(X) - p(\varnothing) \\ b(X) + b(T) &\ge b(X \cap T) + b(X \cup T) \\ b(X) - p(T) &\ge b(X \setminus T) - p(T \setminus X). \end{aligned}$$

The sum of the four inequalities has the same left- and right-hand side, once we use the fact that $b(T) = p(T)$ and $b(\varnothing) = p(\varnothing) = 0$. So all four inequalities hold with equality. The first one gives the first half of the lemma. By switching the role of p and b, we get the second half of the lemma. □

This completes the proof of Theorem 3.1. □

4 Recognizing Generalized Polymatroids

In this section we give a polynomial-time algorithm that decides whether a given LP of the form (1) describes a g-polymatroid. Here the inequalities where $b(S) = +\infty$ or $p(S) = -\infty$ are not part of the input.

Theorem 4.1. *There is a polynomial-time algorithm, which on input (A, b), determines whether the polyhedron $\{x \mid Ax \leq b\}$ is a g-polymatroid.*

First we deal with the case that the polyhedron is full-dimensional, in which case we characterize the linear systems that define g-polymatroids; afterwards, we show how to reduce the general case to the full-dimensional one with the help of Theorem 3.1.

We can always make the following assumption:

Assumption 4.2. The input polyhedron is *minimally described* in the sense that deleting any inequality would yield a strictly larger polyhedron.

This is without loss of generality because we can convert an arbitrary description to a minimal one by removing redundant inequalities one by one. Cheching redundancy of an inequality can be done in polynomial time using linear programming.

4.1 The Full-Dimensional Case

For full-dimensional polyhedra, the minimal description is known to be unique up to scaling inequalities by a positive scalar; also, every inequality in the minimal description defines a facet. Moreover, by definition, a g-polymatroid's facet-defining inequalities are of the form $x(S) \geq \beta$ or $x(S) \leq \beta$ for some S and β. So by scaling we assume all input inequalities are represented by the following families $\mathcal{B}$ and $\mathcal{P}$.

Definition 4.3. Let $\mathcal{B}$ be the family of all S where $x(S) \leq b(S)$ is part of the input (i.e. $b(S) \neq +\infty$). Similarly let $\mathcal{P}$ be the family of all S where $x(S) \geq p(S)$ is part of the input.

Our proof method will use the functions $i(S)$ and $a(S)$ described by (3), where $P = Q(p, b)$ is the input polyhedron. Note that for any particular set S, $i(S)$ and $a(S)$ can be computed in polynomial time. Moreover, note by minimality that $i(S) = p(S)$ for all $S \in \mathcal{P}$ and similarly for $\mathcal{B}$. The core of our approach is the following new characterization:

Theorem 4.4. *Suppose that for a pair (p, b), the polyhedron $Q(p, b)$ is full-dimensional and minimally described. Then $Q(p, b)$ is a g-polymatroid if and only if*

(i) for every $S, T \in \mathcal{B}$, $a(S \cup T) + a(S \cap T) \leq b(S) + b(T)$ holds,

(ii) for every $S, T \in \mathcal{P}$, $i(S \cup T) + i(S \cap T) \geq p(S) + p(T)$ holds, and

(iii) for every $S \in \mathcal{B}$ and $T \in \mathcal{P}$, $a(S \setminus T) - i(T \setminus S) \leq b(S) - p(T)$ holds.

The theorem yields our polynomial-time algorithm (the full-dimensional special case of Theorem 4.1): simply iterate through every pair of sets in the input, and check these conditions.

Proof. The "only if" direction is the easy one. If $Q(p,b)$ is a g-polymatroid, then (i,a) is paramodular and $Q(p,b) = Q(i,a)$. Since $a(X) \leq b(X)$ for all sets X, and a is submodular, we have $a(S \cup T) + a(S \cap T) \leq a(S) + a(T) \leq b(S) + b(T)$. The other cases are similar.

To prove the "if" part, we will show that (p,b) is TDL, that is, for every objective function $c \in \mathbb{R}^V$ for which a dual optimal solution exists, there is a laminar one. Using Theorem 2.2 it follows that $Q(p,b)$ is a g-polymatroid.

Let $M_{\mathcal{B}}$ and $M_{\mathcal{P}}$ be the matrices whose rows are indexed by $\mathcal{B}$ and $\mathcal{P}$ respectively, and where the rows are the characteristic vectors of their indices. Let M denote the matrix $\left(\begin{smallmatrix} M_{\mathcal{B}} \\ -M_{\mathcal{P}} \end{smallmatrix}\right)$.

For every set $S \subseteq [n]$ with $a(S) = \max_{x \in Q(p,b)} S(x)$ finite, let $(\beta^S, \pi^S) \in \mathbb{R}_+^{\mathcal{B} \cup \mathcal{P}}$ be an optimal dual, i.e. one that satisfies

$$\begin{aligned} \chi_S &= (\beta^S, \pi^S) M & (6) \\ a(S) &= (\beta^S, \pi^S)(b, -p). & (7) \end{aligned}$$

Likewise when $i(S) = \min_{x \in Q(p,b)} S(x)$ is finite, let $(\beta^{-S}, \pi^{-S}) \in \mathbb{R}_+^{\mathcal{B} \cup \mathcal{P}}$ be an optimal dual,

$$\begin{aligned} -\chi_S &= (\beta^{-S}, \pi^{-S}) M & (8) \\ -i(S) &= (\beta^{-S}, \pi^{-S})(b, -p). & (9) \end{aligned}$$

For certain sets S and T we define a vector in $\mathbb{R}^{\mathcal{B} \cup \mathcal{P}}$, which will be used for modifying the dual. Let e_S denote the vector with 1 in the S component and 0 elsewhere — it lies in $\mathbb{R}^{\mathcal{B}}$ or $\mathbb{R}^{\mathcal{P}}$ depending on context. Then,

- if $S, T \in \mathcal{B}$ conflict and $a(S)$ and $a(T)$ are bounded, define $u(S,T)$ to be
$$u(S,T) = -(e_S, \mathbf{0}) - (e_T, \mathbf{0}) + (\beta^{S \cup T}, \pi^{S \cup T}) + (\beta^{S \cap T}, \pi^{S \cap T}); \qquad (10)$$
- if $S, T \in \mathcal{P}$ conflict and $i(S)$ and $i(T)$ are bounded, define $v(S,T)$ to be
$$v(S,T) = -(\mathbf{0}, e_S) - (\mathbf{0}, e_T) + (\beta^{-S \cup T}, \pi^{-S \cup T}) + (\beta^{-S \cap T}, \pi^{-S \cap T}); \qquad (11)$$
- if $S \in \mathcal{B}$ and $T \in \mathcal{P}$ conflict and $a(S)$ and $i(T)$ are bounded, define $w(S,T)$ to be
$$w(S,T) = -(e_S, \mathbf{0}) - (\mathbf{0}, e_T) + (\beta^{S \setminus T}, \pi^{S \setminus T}) + (\beta^{-T \setminus S}, \pi^{-T \setminus S}). \qquad (12)$$

Proposition 4.5. *For the vectors defined above, the following properties hold:*

(a) The vectors $u(S,T)$, $v(S,T)$, $w(S,T)$ are always nonzero.

(b) $u(S,T)M = v(S,T)M = w(S,T)M = \mathbf{0}$.

(c) $u(S,T)$, $v(S,T)$ and $w(S,T)$ are weakly improving directions for the objective function $(b,-p)$.

Proof. *(a)* If $u(S,T)$ were $\mathbf{0}$, then $\mathtt{supp}((\beta^{S\cup T},\pi^{S\cup T})+(\beta^{S\cap T},\pi^{S\cap T}))=\{S,T\}$. But, using the fact that S and T conflict, it is easy to see that no dual can meet condition (6) in the definition of $(\beta^{S\cap T},\pi^{S\cap T})$ and also have support that is a subset of $\{S,T\}$. The arguments for $v(S,T)$ and $w(S,T)$ are similar.

(b) $u(S,T)M=-\chi_S-\chi_T+\chi_{S\cup T}+\chi_{S\cap T}=\mathbf{0}$, and similarly for the other cases.

(c) $u(S,T)(b,-p)=-b(S)-b(T)+a(S\cup T)+a(S\cap T)\leq 0$, this was condition (i). The other cases follow likewise from conditions (ii) and (iii). □

Let C be the cone generated by these vectors,

$$C:=\text{cone}(\{u(S,T):S,T\in\mathcal{B}\text{ conflict}\}\cup\{v(S,T):S,T\in\mathcal{P}\text{ conflict}\}\\ \cup\{w(S,T):S\in\mathcal{B},T\in\mathcal{P}\text{ conflict}\}).$$

Proposition 4.6. *The cone C is pointed, i.e. it does not contain any line.*

Proof. For some number N, let $(N+\mu)$ be the vector whose value in the coordinate indexed by each set S is $N+(n-|S|)^2$. We claim that for N sufficiently large, $(N+\mu)$ has positive scalar product with all the generators of C, which will complete the proof. To see this, one part is to observe that $\mathbf{1}\cdot(\beta^X,\pi^X)\geq 1$ for any nonempty X, with equality iff $(\beta^X,\pi^X)=(e_X,\mathbf{0})$; and similarly for $-X$. It follows that $\mathbf{1}\cdot u(S,T)$ is nonnegative, with equality only when $(\beta^{S\cup T},\pi^{S\cup T})=(e_{S\cup T},\mathbf{0})$ and $(\beta^{S\cap T},\pi^{S\cap T})=(e_{S\cap T},\mathbf{0})$. Furthermore in this case, $(N+\mu)\cdot u(S,T)=(n-|S\cup T|)^2+(n-|S\cap T|)^2-(n-|S|)^2-(n-|T|)^2>0$, since S and T conflict. The other details are similar. □

Proposition 4.7. *If for a dual solution y the affine cone $y+C$ intersects the dual polyhedron only in y, then $\mathtt{supp}(y)$ is laminar.*

Proof. Write $y=(y^u,y^\ell)$. Suppose in contradiction of the claim that there are two conflicting sets $S,T\in\mathcal{B}$, for which y^u_S and y^u_T are positive; the other cases are similar. Then for sufficiently small $\epsilon>0$, $y':=y+\epsilon u(S,T)$ lies in $y+C$ and has $y'\geq 0$. Moreover, y' is dual feasible because of part *(b)* of Proposition 4.5, and $y'\neq y$ because of part *(a)*. This contradicts the assumption of the proposition. □

Due to the above proposition it is enough to give an optimal dual solution y for which the intersection of $y+C$ and the dual polyhedron is $\{y\}$. The existence of such a vector follows from the next two propositions.

Proposition 4.8. *If P is a bounded polytope and C is a pointed cone, then there exists a vector $y\in P$ such that $(y+C)\cap P=\{y\}$.*

Proof. Since C is pointed, there is a vector c with which every vector in C has positive scalar product. Let y be maximal in P for the objective c. Then $(y+C)\cap P=\{y\}$. □

Proposition 4.9. *If a linear program with no all-zero rows defines a full-dimensional polyhedron, then the optimal face of the dual is bounded.*

Proof. Write $Ax \leq b$ for the linear program. Suppose otherwise that the optimal dual face contains a ray. This implies that there is a dual combination $y \geq 0$ of primal inequalities, $y \neq 0$, such that $yA = 0$ and (by optimality) $yb = 0$. Consequently the negative of some constraint can be obtained as a nonnegative combination of other constraints, so this constraint always holds with equality, contradicting full-dimensionality (using that the constraint is not all-zero). □

The propositions combine as follows: since $Q(p, b)$ is full-dimensional, Proposition 4.9 implies the optimal face of its dual is bounded. Apply Proposition 4.8 to the optimal face, obtaining an optimal y such that the only optimal point of $y + C$ is y. Further, by part *(c)* of Proposition 4.5, any feasible point of $y + C$ is optimal, so y is the only *feasible* point of $y + C$. So Proposition 4.7 applies and the proof of Theorem 4.4 is complete. □

This implies a test for max-dimensional base polyhedra, which will be useful later.

Corollary 4.10. *Let $P = Q(p, b) \cap \{x \mid x([n]) = c\}$ be of dimension $n - 1$. Then we can test in polynomial time whether P is a base polyhedron.*

Proof. We know that P is a base polyhedron if and only if, by projecting away some variable x_n, we get a g-polymatroid in $n - 1$ dimensions. Notice that this projection is given explicitly by $Q(p', b') \in \mathbb{R}^{n-1}$ where for all $S \subseteq [n - 1]$,

$$p'(S) = \max\{p(S), c - b([n] \setminus S)\} \qquad \text{and} \qquad b'(S) = \min\{b(S), c - p([n] \setminus S)\}.$$

We can test whether this is an $(n-1)$-dimensional g-polymatroid by Theorem 4.4. □

The proof of Theorem 4.4 implies the following.

Corollary 4.11. *If $Q(p, b)$ is a full-dimensional g-polymatroid, then (p, b) is TDL.*

4.2 The General Case

The proof method of Theorem 4.4 does not work directly in the non-full-dimensional case, because the system is not necessarily TDL, as the following example shows.

Example 4.12. *Consider the LP with 6 constraints $\{x_i + x_j \geq 1, x_i + x_j \leq 1\}_{i,j \in [3], i \neq j}$. It defines a g-polymatroid (the single point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$), but it is not totally dual laminar.*

We use the decomposition from Theorem 3.1 to get around this obstacle.

of Theorem 4.1. It is useful to first check whether the affine hull has the correct form.

Proposition 4.13. *For a g-polymatroid, the affine hull is of the form $\{x \mid \forall i, x(A_i) = c_i\}$ for some subpartition $\mathcal{A} = \{A_i\}_i$ of $[n]$.*

Proof. This follows from the proof of Theorem 3.1, then noting that the affine hull of a full-dimensional g-polymatroid is all of its ambient space, and the analogue for max-dimensional base polyhedra. □

Our algorithm begins by checking whether the polyhedron's affine hull has the form in Proposition 4.13. Notice that an inequality $a_i x \leq b_i$ is an implicit equality if the minimum of $a_i x$ is b_i, and in this way we can compute a system $A^= x = b^=$ of linear equalities defining the affine hull.

Proposition 4.14. *In polynomial time we can check whether $P = \{x \mid A^= x = b^=\}$ is of the form $\{x \mid \forall i, x(A_i) = c_i\}$ for some subpartition $\mathcal{A} = \{A_i\}_i$ of $[n]$, and find $\mathcal{A}, c$ if so.*

Proof. We may assume that P has this form, and concentrate on the problem of finding $\mathcal{A}, c$. This is because we can run such an algorithm on any P, and then merely check that the output of the algorithm (if it does not crash) satisfies $\{x \mid A^= x = b^=\} = P$, which is a matter of seeing if each equality defining one system is implied by the other system, which can be done using a subroutine to compute matrix ranks.

We start identifying parts of the subpartition. For $I \subseteq [n]$ let P_I be the projection of P on to the variables $\{x_i\}_{i \in I}$. We can check in polynomial time whether P_I has full dimension $|I|$, by testing whether there is any vector y such that $yA^=$ is zero on all coordinates of $[n] \setminus I$, and nonzero on at least one coordinate of I.

Observe that $\dim(P_{A_i}) < |A_i|$, and moreover that $\dim(P_I) < |I|$ if and only if I contains some A_i. To begin with, if $\dim(P) = n$ then $P = \mathbb{R}^n$ and the algorithm returns "yes," with $\mathcal{A} = c = \varnothing$. Otherwise, initialize $I = [n]$, then for each element $j \in I$ in turn, delete j from I unless it would cause the new I to satisfy $\dim(P_I) = |I|$. We may set A_1 equal to this final I. Similarly, if $\dim(P_{[n] \setminus A_1}) = n - |A_1|$ then we are done, otherwise we let A_2 be an inclusion-minimal subset of $[n] \setminus A_1$ with $\dim(P_{A_2}) < |A_2|$. Iterating this gives $\mathcal{A}$, then computing c is easy. $\square$

Now that we have the subpartition we want to check whether Q is a direct product of some polyhedra on the sets in $\mathcal{A}$ and on $[n] \setminus \cup \mathcal{A}$. Using the following lemma we can compute the linear systems describing these polyhedra if they exist. We denote the ith row of a matrix M by m_i and the ith coordinate of a vector v by v_i.

Lemma 4.15. *If a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is a direct product of two polyhedra $P = P_1 \times P_2$ where $P_1 \subseteq \mathbb{R}^I$ and $P_2 \subseteq \mathbb{R}^{[n] \setminus I}$, then P_1 is described by the system $\{x \in \mathbb{R}^I \mid A'x \leq b'\}$ and P_2 by the system $\{x \in \mathbb{R}^{[n] \setminus I} \mid A''x \leq b''\}$, where A' and A'' are the submatrices of A restricted to I and $[n] \setminus I$ respectively and the right hand sides are $b'_i := \max_{x \in P} a'_i x$ and $b''_i := \max_{x \in P} a''_i x$.*

Proof. Let x_I and $x_{[n] \setminus I}$ denote the restrictions of x to I and $[n] \setminus I$ respectively. Let $P' := \{x \mid A'x_I \leq b',\ A''x_{[n] \setminus I} \leq b''\}$. It is clear that $P \subseteq P'$ since P' consists of inequalities that are valid for P. For the other direction, it is enough to show each $a_i \leq b_i$ is valid for P'. Let x^1 and x^2 maximize a'_i and a''_i respectively in P, then the vector $(x^1_I, x^2_{[n] \setminus I}) \in P$ maximizes both a'_i and a''_i (it is in P because P is a direct product). Thus

$$b'_i + b''_i = a'_i(x^1_I, x^2_{[n] \setminus I}) + a''_i(x^1_I, x^2_{[n] \setminus I}) = a_i(x^1_I, x^2_{[n] \setminus I}) \leq b_i.$$

which shows $a_i x \leq b_i$ is implied by the two inequalities $a'_i x_I \leq b'_i$ and $a''_i x_{[n] \setminus I} \leq b''_i$ that define P'. $\square$

With these tools, our complete algorithm for recognizing g-polymatroids goes as follows. First, check whether the affine hull could be the affine hull of a g-polymatroid, using Proposition 4.14, and compute the subpartition $\mathcal{A}$. Next we check whether Q is the direct product of some polyhedra on the sets A_i and on $[n] \setminus \cup\mathcal{A}$: using Lemma 4.15 we compute the possible linear descriptions of the factors Q_i and then check whether their direct product is Q. We then seek to use Theorem 4.4 (resp. Corollary 4.10) to check whether Q_i is a g-polymatroid (resp. base polyhedron). $\square$

4.3 Recognizing Integral Generalized Polymatroids

We can also decide whether a given linear system of the form (1) describes an integer g-polymatroid. Again, there is a difference between the full-dimensional case and the non-full-dimensional case. Suppose p and b are integral. If $Q(p, b)$ is a full-dimensional g-polymatroid, then it is an integral one, since the proof of Theorem 4.4 gave that the system is TDL, thus TDI. But $Q(p, b)$ may be a non-integral g-polymatroid when it is non-full-dimensional, see the example at the start of Section 4.2.

Nonetheless, we now describe an algorithm to determine whether an arbitrary polyhedron is an integral g-polymatroid. Assume without loss of generality that the system is given by a minimal description, and as in the proof of Theorem 4.1 we may assume the description is as in (1). Note p and b must be integral in order for $Q(p, b)$ to be integral. In the full-dimensional case we are done by the above remark. In the case that $Q(p, b)$ is a max-dimensional base polyhedron with $x([n]) = c$, it is additionally necessary that c is integral, but also sufficient by considering the correspondence between base polyhedra and g-polymatroids. Finally, in the general case, by Theorem 3.1 and Lemma 4.15 we can compute the description of some full-dimensional g-polymatroids and base polyhedra, whose direct product is our g-polymatroid, and with the above method we can check whether these are integer polyhedra. Since the direct product of several g-polymatroids is integral if and only if each individual one is integral, this answers whether our g-polymatroid is integral.

Note that we change the system during the algorithm, so we may ask whether there is a necessary and sufficient condition in terms of p and b. The answer is positive:

Theorem 4.16. *Suppose that $Q(p, b)$ is a g-polymatroid, and that it is minimally described. Then $Q(p, b)$ is an integer g-polymatroid if and only if p and b are integral and on every fixed-sum set, the sum is integer.*

Proof. The conditions are clearly necessary, because of minimality. For sufficiency suppose that p and b are integral and on every fixed-sum set the sum is integer. It is enough to show that the full dimensional g-polymatroid resp. max dimensional base polyhedra according to Theorem 3.1 have integral describing systems, because then by the above argument they are integer polyhedra and so is $Q(p, b)$. We use the following proposition.

Proposition 4.17. *Let Q be a polyhedron for which $Q = Q_1 \times Q_2$ where $Q_1 \subseteq \mathbb{R}^I$ and $Q_2 \subseteq \mathbb{R}^{[n]\setminus I}$. Suppose that $ax \leq b$ is an inequality in a system of Q which is not redundant and let $a'x \leq b'$ and $a''x \leq b''$ be the inequalities for Q_1 resp. Q_2 according to Lemma 4.15. Then one of them is an implicit equality.*

Proof. Let $\dim(Q) = d$ and $\dim(Q_j) = d_j$ for $j = 1, 2$. Because $ax \leq b$ is not redundant, the face $F := \{x \in Q \mid ax = b\}$ has dimension at least $d - 1$. Define the face $F_1 := \{x \in Q_1 \mid a'x = b'\}$ of Q_1 and similarly define $F_2 := \{x \in Q_2 \mid a''x = b''\}$. Then $F = F_1 \times F_2$ and $\dim(F) = \dim(F_1) + \dim(F_2)$. While $\dim(F_j) \leq d_j$, this cannot hold strictly for both $j = 1$ and $j = 2$ since $\dim(F) > d - 2$. The j with $\dim(F_j) = d_j$ has $F_j = Q_j$ and yields the claimed implicit equality. □

Since every implicit equality is integer in the system with right hand p and b, by the above proposition, the systems that we get using Lemma 4.15 have also integer right hand side and it remains true that implicit equalities are integer. By iterating this, we get describing systems with integer right hand side for the terms in the decomposition according to Theorem 3.1. This completes the proof of Theorem 4.16. □

4.4 Oracle Model

Since we came up with a polynomial-time algorithm to recognize g-polymatroids when they are presented explicitly, it is also interesting to consider whether the same could be accomplished when the input polyhedron is given in an implicit form. Say that a *linear optimization oracle* for a polyhedron P takes a cost-function c as input, and returns a point on P which maximizes $c \cdot x$. Then, the following information-theoretic argument shows that we cannot recognize g-polymatroids with any number of queries polynomial in n. Recall the permutahedron

$$\Pi := \{x \in \mathbb{R}^n \mid x([n]) = \tbinom{n+1}{2}; \forall S \subset [n], x(S) \geq \tbinom{|S|+1}{2}\}$$

which is a "generic" max-dimensional base polytope, whose vertices are the permutations of $[n]$. One may show that, when $n \equiv 2 \pmod 4$, if we delete any one constraint for some S with $|S| = n/2$, the modified polyhedron, call it Π_S, is no longer a g-polymatroid. Furthermore, one may show that if a query can distinguish Π from Π_S, then that query cannot distinguish Π from $\Pi_{S'}$, where S' is any other $(n/2)$-subset of $[n]$. Therefore, no deterministic algorithm can recognize g-polymatroids with fewer than $\binom{n}{n/2} = \Omega(2^n/\sqrt{n})$ queries, and likewise any randomized algorithm that is correct 2/3 of the time on all inputs needs $\Omega(2^n/\sqrt{n})$ queries.

5 Intersection Integrality

Theorem 5.1. *Let P be a polyhedron whose intersection with each integral g-polymatroid is integral. Then P is an integral g-polymatroid.*

Proof. Suppose that the nonempty polyhedron P is not an integer g-polymatroid. We want to give an integral g-polymatroid Q for which $P \cap Q$ is not integral. We can assume that P is an integral polyhedron since if not, then $Q_1 = \mathbb{R}^n$ will do.

Assume that P is bounded and integer. Then Theorem 1.5 implies that there is an edge of P whose direction v is not in $E := \{\chi_i : i \in [n]\} \cup \{-\chi_i : i \in [n]\} \cup \{\chi_i - \chi_j : i, j \in [n]\}$. Let z be an integer point on this edge. The cube $z + [-1, 1]^n$ is a g-polymatroid, thus we can assume that its intersection with P is integer. This implies that v can be chosen $\{0,1,-1\}^n$ and $z+v$ is in P. Since $v \notin E$, there are two coordinates of v which are the same, both 1 or -1, we can assume that $v_1 = v_2 = 1$. Then the g-polymatroid Q_2 defined by the paramodular pair

$$p(S) := \begin{cases} z_1 + z_2 + 1 & \text{if } S = \{1,2\}, \\ -\infty & \text{otherwise,} \end{cases}$$

$$b(S) := \begin{cases} z_1 + z_2 + 1 & \text{if } S = \{1,2\}, \\ \infty & \text{otherwise,} \end{cases}$$

is the affine hyperplane $z + \{x \in \mathbb{R}^n : \; x_1 + x_2 = 1\}$ which intersects the edge $z + tv$ in a noninteger vector $z + \frac{1}{2}v$. Thus Q_2 intersects P in a noninteger polyhedron, too.

Assume now that P is an unbounded integer polyhedron. By Theorem 1.5, there is a vector z such that the tangent cone of P at z is not generated by vectors in the set E. Since P is integral, we can choose z to be an integral vector. Let C be the cube $z + [-1,1]^n$. Then $P \cap C$ is a bounded polyhedron which is — again by Theorem 1.5 — not a g-polymatroid, since the tangent cone at z did not change. Thus we can use the bounded case, which implies that there is a polymatroid Q_3 for which $P \cap C \cap Q_3$ is non-integer. Since the intersection of an integral g-polymatroid with an integral box is again an integral g-polymatroid [13], $C \cap Q_3$ is an integral g-polymatroid which intersects P in a non-integer polyhedron. □

The pseudo-recursive characterization in Theorem 5.1 can be refined to ones less dependent on external definitions:

Corollary 5.2. *The polyhedron $P \subseteq \mathbb{R}^n$ is an integral g-polymatroid if and only if it has integral intersection with each polyhedron Q of the following form: Q has some fixed integral coordinates $\{c_i\}_{i \in F}$, optionally two distinct coordinates $j,k \notin F$ with fixed integral sum c, and the remaining coordinates free, i.e.*

$$\begin{aligned} &Q = \{x \in \mathbb{R}^n \mid x_i = c_i, \forall i \in F; \;\; x_j + x_k = c\} \\ &\quad \textit{or} \\ &Q = \{x \in \mathbb{R}^n \mid x_i = c_i, \forall i \in F\}. \end{aligned} \tag{13}$$

Proof. To prove the easy $\Rightarrow$ direction of the proof, it is enough to verify that each such Q is an integral g-polymatroid. This follows from standard constructions [12, Thm. 2.8]: Q is a direct sum of copies of $\mathbb{R}$, integer singleton sets, and possibly the plank $x_j + x_k = c$.

So now we focus on the $\Leftarrow$ direction: given a polyhedron P which is not an integral g-polymatroid, find an integral g-polymatroid Q of the desired form such that $P \cap Q$ is non-integral. According to the proof of Theorem 5.1, there is an integer g-polymatroid Q — either $\mathbb{R}^n$, or an integer box, or the intersection of an integer box with the integer plank $\{x \mid x_j + x_k = c\}$ — so that $P \cap Q$ has a non-integer vertex z. In the third case, direct computation shows that Q is either an $(n-2)$-dimensional box with two fixed integer coordinates, or the direct product of an $(n-2)$-dimensional box with a line segment of the form $\{x \mid x_i + x_j = c, \ell \leq x_i \leq u\}$.

Next, let Q' be the minimal face of Q containing z, and let Q'' be the affine hull of Q'. Now z is a vertex of $P \cap Q'$ since $Q' \subseteq Q$. Also, z is a vertex of $P \cap Q''$ since Q' and Q'' are identical in a neighbourhood of z (by our choice of Q').

We claim Q'' is the desired integral g-polymatroid. This is accomplished by the straightforward verification that no matter which of the three cases we are in, and no matter which face of Q is Q', we can describe Q'' in the desired form. This completes the proof. ☐

Corollary 5.3. *The polyhedron $P \subseteq \mathbb{R}^n$ is an integral g-polymatroid if and only if, for every Q which is an integer translate of a matroid independent set polytope, $P \cap Q$ is integral.*

Proof. Let Q_0 be the polyhedron guaranteed by Corollary 5.2. The proof of Corollary 5.2 guarantees that $P \cap Q_0$ has a non-integer vertex z. We consider two cases depending on which of the two equations in (13) defines Q_0. In the first case, Q_0 has a constraint $x_j + x_k = c$. The second case will turn out to be just a simpler version of the first, so we omit its proof.

Let Q'_0 be obtained from Q_0 by replacing the equality constraint $x_j + x_k = c$ by the inequality $x_j + x_k \leq c$. We claim z is still a vertex of $Q'_0 \cap P$, which is evident since any expression as z as a strictly convex sum of two points in $Q'_0 \cap P$ would have both of these two points satisfying $x_j + x_k = c$, contradicting that z is a vertex of $Q_0 \cap P$. Then, let Q''_0 be obtained from Q'_0 as $Q''_0 := \{x \in Q'_0 \mid \lfloor z \rfloor \leq x \leq \lceil z \rceil\}$ (here floor and ceiling act component-wise). Since $z \in Q''_0 \subseteq Q'_0$, we still have that z is a vertex of $P \cap Q''_0$. Moreover, $Q''_0 - \lfloor z \rfloor$ is easily verified to be the independent set polytope of a matroid on $[n]$ where elements $\{i \mid z_i \text{ integer}\}$ are loops, elements j and k are parallel, and all other elements are co-loops. So $Q = Q''_0$ proves the corollary. ☐

Corollary 5.3 directly implies the following description of g-polymatroids as an axiomatic generalization of matroids:

Corollary 5.4. *Let $\mathcal{C}$ be an inclusion-maximal class of polyhedra such that (i) $\mathcal{C}$ includes all matroid independent set polytopes, (ii) $\mathcal{C}$ is closed under integer translation, and (iii) the intersection of any two polyhedra in $\mathcal{C}$ is integral. Then $\mathcal{C}$ equals the class of all integral g-polymatroids.*

6 Truncation-paramodularity

In this section, we introduce *truncation paramodularity*, a new notion implying total dual laminarity. As illustrated by the diagram in the introduction, truncation paramodularity is implied by the notion of *near paramodularity* from [11, 12].

Definition 6.1 (separation, near paramodularity [11, 12])**.** We call a set S *b-separable from below* if there is a non-trivial partition $\{S_i : i \in [t]\}$ of S for which $\sum b(S_i) \leq b(S)$. Similarly, S is *p-separable from above* if there is a non-trivial partition $\{S_i : i \in [t]\}$ of S for which $\sum p(S_i) \geq p(S)$. We omit "from above/below" when the context is clear.

The pair (p, b) is *near paramodular* if it satisfies the following:

(i) b satisfies the submodular inequality for non-b-separable conflicting sets,

(ii) p satisfies the supermodular inequality for non-p-separable conflicting sets,

(iii) the cross-inequality $b(S) - p(T) \geq b(S \setminus T) - p(T \setminus S)$ holds for every conflicting non-b-separable S and non-p-separable T.

It is clear from the definition that an intersecting paramodular pair is also near paramodular. Next, we introduce the weaker notion of *truncation paramodularity.*

Definition 6.2 (truncation, truncation paramodularity)**.** The *upper truncation* of a set function $p : 2^{[n]} \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$p^{\wedge}(S) = \max\{\sum_{Z \in \mathcal{F}} p(Z) \mid \mathcal{F} \text{ is a partition of } S\},$$

where the trivial partition $\{S\}$ is also allowed. Similarly, the *lower truncation* of a set function $b : 2^{[n]} \to \mathbb{R} \cup \{+\infty\}$ is

$$b^{\vee}(S) = \min\{\sum_{Z \in \mathcal{F}} b(Z) \mid \mathcal{F} \text{ is a partition of } S\}.$$

The pair (p, b) is *truncation paramodular* when $(p^{\wedge}, b^{\vee})$ is near paramodular.

Proposition 6.3. *Near paramodularity implies truncation paramodularity.*

Proof. There are two useful observations to make here (along with analogues for p): (i), that b-separability is identical to $b^{\vee}$-separability; (ii), that every non-b-separable set S has $b^{\vee}(S) = b(S)$. This gives an alternate definition of truncation paramodularity, that every conflicting pair of non-separable sets should satisfy paramodular inequalities like $p^{\wedge}(S \cup T) + p^{\wedge}(S \cap T) \geq p(S) + p(T)$ and analogues. Using that definition along with $b^{\vee} \leq b$ and $p^{\wedge} \geq p$, the result follows. ☐

We now show the new notion is still strong enough to imply total dual laminarity:

Theorem 6.4. *If the pair (p, b) is truncation-paramodular, then it is TDL.*

Proof. We have to show that for any integral objective function c, there is a laminar optimal dual solution. We can assume that there is an integral optimal dual solution since if y is an arbitrary rational optimal dual solution, and N is the lowest common denominator of y, then for the objective function Nc, Ny is an integral optimal dual solution and the set of possible support systems did not change. Let K be an integer such that there exists an integral optimal dual solution y satisfying $y^{\ell}\chi \leq K\mathbf{1}$ and $y^{u}\chi \leq K\mathbf{1}$. Let us call y *small* if it satisfies these conditions.

Let us order the subsets of $[n]$ in such a way that if $X \subset Y$ then X comes first, that is, we take a linear extension of the poset $(2^{[n]}, \subseteq)$. Let $y = (y^{\ell}, y^{u})$ be the small integral optimal dual solution for which y^{ℓ} is lexicographically maximal in the above order, and with respect to this, y^{u} is lexicographically maximal.

We claim that no set in $\mathtt{supp}(y^{\ell})$ is p-separable and no set in $\mathtt{supp}(y^{u})$ is b-separable. Suppose indirectly that for a partition $\{X_i : i \in [t]\}$ of $X \in \mathtt{supp}(y^{\ell})$, $\sum p(X_i) \geq p(X)$

holds. Then by decreasing y^ℓ on X by one and increasing it on each X_i by one, we get a small integral optimal dual solution for which the first part is lexicographically larger than y^ℓ, a contradiction. The other part is similar.

Now we claim that $\mathtt{supp}(y^\ell) \cup \mathtt{supp}(y^u)$ is laminar. Suppose first that there are conflicting sets X, Y in $\mathtt{supp}(y^\ell)$. Since X and Y are not p-separable, inequality $p^\wedge(X \cap Y) + p^\wedge(X \cup Y) \geq p(X) + p(Y)$ holds, with partitions $\mathcal{F}^\cap$ and $\mathcal{F}^\cup$ giving the upper truncation values. Thus if we decrease y^ℓ on X and Y by 1 and increase it on the elements of $\mathcal{F}^\cap$ and $\mathcal{F}^\cup$ by 1, we get again a small integral optimal dual solution for which the first part is lexicographically larger than y^ℓ, a contradiction. We can prove similarly that $\mathtt{supp}(y^u)$ is laminar. Now suppose that for $X \in \mathtt{supp}(y^u)$ and $Y \in \mathtt{supp}(y^\ell)$, X and Y are conflicting. Since X is not b-separable and Y is not p-separable, inequality $b(X) - p(Y) \geq b^\vee(X \setminus Y) - p^\wedge(Y \setminus X)$ holds, with partitions $\mathcal{F}^1$ and $\mathcal{F}^2$ giving the truncation values. Thus if we decrease y^u on X and y^ℓ on Y by 1 and increase y^u on the elements of $\mathcal{F}^1$ and y^ℓ on the elements of $\mathcal{F}^2$ by 1, we get again a small integral optimal dual solution which is lexicographically larger than y, a contradiction. This proves total dual laminarity. ☐

6.1 An application: the supermodular coloring theorem

The following colouring theorem is an extension of Schrijver's supermodular coloring theorem [25], and of the skew-supermodular colouring theorem in [3]. Our proof is a descendant of Schrijver's proof [26, §49.11c]. We show that it is a consequence of Theorem 6.4.

Theorem 6.5. *Let k be a positive integer and let f_1 and f_2 be nonnegative integer-valued set functions on ground set $[n]$, which satisfy the following properties:*

(i) $\max\{f_1(S), f_2(S)\} \leq \min\{k, |S|\}$ *for each* $S \subseteq [n]$,

(ii) for $i \in \{1, 2\}$ *and for every conflicting* $S, T \subset [n]$, *there exist* $U \subseteq S \cup T$ *and* $I \subseteq S \cap T$ *such that* $f_i(U) + f_i(I) \geq f_i(S) + f_i(T)$.

Then $[n]$ can be coloured with k colours so that every set $S \subseteq [n]$ contains at least $\max\{f_1(S), f_2(S)\}$ *colours. Moreover there is such a colouring where each colour is used $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ times.*

Proof. We can assume w.l.o.g. that f_1 and f_2 have value 1 on every singleton. We use induction on k; the claim is evident for $k = 1$. For the inductive step, we want to define the k-th color class C so that $f_i'(S) := \max\{f_i(S), \max_{X \subseteq C} f_i(S \cup X) - 1\}$ $(i = 1, 2)$ fulfill the criteria on ground set $[n] \setminus C$ with $k - 1$ colors. Equivalently, C has to satisfy $p_i(S) \leq |C \cap S| \leq b_i(S)$ $(i = 1, 2)$ for every set $S \subseteq [n]$, where

$$\begin{aligned} p_i(S) &:= \begin{cases} 1 & \text{if } S \text{ is minimal such that } f_i(S) = k, \\ -\infty & \text{otherwise}, \end{cases} \\ b_i(S) &:= |S| - f_i(S) + 1. \end{aligned}$$

In other words $\chi_C \in Q(p_1, b_1) \cap Q(p_2, b_2)$. In addition, we also require that $\lfloor n/k \rfloor \leq |C| \leq \lceil n/k \rceil$.

We claim that (p_i, b_i) is a truncation-paramodular pair for $i = 1, 2$. First, p_i clearly satisfies (ii) of Definition 6.2, since the minimal sets on which f_i is k are disjoint.

For some i, let S and T be conflicting and not b_i-separable. There exist $U \subseteq S \cup T$ and $I \subseteq S \cap T$ such that $f_i(U) + f_i(I) \geq f_i(S) + f_i(T)$. Using that b_i is 1 on each singleton, we have

$$b_i^\vee(S \cup T) \leq b_i(U) + |(S \cup T) \setminus U| = |S \cup T| - f_i(U) + 1 \text{ and}$$
$$b_i^\vee(S \cap T) \leq b_i(I) + |(S \cap T) \setminus I| = |S \cap T| - f_i(I) + 1,$$

hence

$$\begin{aligned} b_i(S) + b_i(T) &= |S| + |T| - f_i(S) - f_i(T) + 2 \\ &\geq |S \cup T| + |S \cap T| - f_i(U) - f_i(I) + 2 \geq b_i^\vee(S \cup T) + b_i^\vee(S \cap T). \end{aligned}$$

Finally we show that (iii) of Definition 6.2 is trivially satisfied because there are no conflicting sets S and T with that property. Let S be a minimal set such that $f_i(S) = k$, and let T be a conflicting set; we claim that T is b_i-separable. Indeed, we know that there are sets $U \subseteq S \cup T$ and $I \subseteq S \cap T$ such that $f_i(U) + f_i(I) \geq f_i(S) + f_i(T)$. We have $f_i(U) \leq k = f_i(S)$, hence $f_i(I) \geq f_i(T)$. This gives $b_i(T) \geq b_i(I) + |T \setminus I|$, so the partition $\{I, \{v : v \in T \setminus I\}\}$ shows that T is b_i-separable.

Since (p_i, b_i) is truncation-paramodular, $Q(p_i, b_i)$ is an integer g-polymatroid ($i = 1, 2$). Thus the common intersection with a plank $Q(p_1, b_1) \cap Q(p_2, b_2) \cap \{x : \lfloor n/k \rfloor \leq \mathbf{1}x \leq \lceil n/k \rceil\}$ is integral. It is also non-empty, because the vector $\frac{1}{k}\mathbf{1}$ is an element. We can choose an arbitrary set C whose characteristic vector is in the polyhedron, and get the remaining $k - 1$ colour classes by induction. □

Remark. If f is a skew-supermodular function, then we can construct a function f' by $f'(S) = 0$ if $f(S) \leq 0$ or there is a set $T \subsetneq S$ such that $f(T) \geq f(S)$, and $f'(S) = f(S)$ otherwise. The set function f' satisfies the properties of Theorem 6.5, and a feasible colouring for f' is also feasible for f. Thus Theorem 6.5 is a generalization of the skew-supermodular colouring theorem in [3].

6.2 Checking truncation-paramodularity in polynomial time

Can truncation-paramodularity of a pair (p, b) be checked in polynomial time if the input consists of the finite values of the two functions? The naive approach does not work — indeed, separability testing and computing $p^\vee / b^\wedge$ are NP-hard, by reduction from 3-dimensional matching. Nonetheless, in contrast to the hardness of checking total dual laminarity,

Theorem 6.6. *Let $p : 2^{[n]} \to \mathbb{Z} \cup \{-\infty\}$ and $b : 2^{[n]} \to \mathbb{Z} \cup \{+\infty\}$ be set functions, given by an explicit enumeration of their finite values. We can decide in polynomial time if (p, b) is a truncation-paramodular pair.*

Proof. Let $\mathcal{B}$ (resp. $\mathcal{P}$) be the family of all sets where b (resp. p) is finite. We first show an algorithm that decides if $b^\vee$ satisfies the submodular inequality for conflicting sets which are non-b-separable, and at the same time identifies all non-b-separable sets in $\mathcal{B}$.

We enumerate all sets in $\mathcal{B}$ and all conflicting pairs $S, T \in \mathcal{B}$ in one series $A_1, A_2, \dots, A_k$ in an order of increasing size, where the size of a pair is the size of the union. We consider the sets in this order. Suppose that for a given index t we have already identified all non-b-separable sets with index smaller than t, and we have established that the submodular inequality for $b^\vee$ holds for all conflicting non-b-separable pairs of index smaller than t.

Suppose first that A_t is a set $S \in \mathcal{B}$.

Proposition 6.7. *For any $T \subsetneq S$, $b^\vee(T) = \max\{x(T) \mid x(Z) \le b(Z)\ \forall Z \subseteq T\}$.*

Proof. Let $\gamma = \max\{x(T) \mid x(Z) \le b(Z)\ \forall Z \subseteq T\}$. At this point of the algorithm we know that the set function $b^\vee \mid_{Z:Z\subseteq T}$ is submodular on conflicting non-b-separable pairs. Therefore the LP $\max\{x(T) \mid x(Z) \le b(Z)\ \forall Z \subseteq T\}$ has a laminar dual optimal solution y, which satisfies $yb = \gamma$ and $y\chi = \chi_T$. By laminarity, the inclusionwise maximal elements of $\mathtt{supp}(y)$ form a partition $\mathcal{F}$ of T.

We claim that $\sum_{Z\in\mathcal{F}} b(Z) = \gamma$. Indeed, let $\epsilon = \min\{y_Z : Z \in \mathcal{F}\}$. If $\sum_{Z\in\mathcal{F}} b(Z) > \gamma$, then we can construct a dual solution y' of objective value smaller than γ by

$$y'_Z = \begin{cases} \frac{y_Z-\epsilon}{1-\epsilon} & \text{if } Z \in \mathcal{F}, \\ \frac{y_Z}{1-\epsilon} & \text{if } Z \notin \mathcal{F}. \end{cases}$$

This would contradict the optimality of y, thus $b^\vee(T) = \gamma$. ☐

Due to the proposition we can test in polynomial time whether S is b-separable: we can compute $b^\vee(S \setminus T)$ for every $T \in \mathcal{B}$ which is a subset of S. Then S is non-b-separable if and only if $b^\vee(S \setminus T) + b(T) > b(S)$ for any such T.

Suppose now that A_t is a conflicting pair $S, T \in \mathcal{B}$. We have already checked if both are b-separable; let us assume that they are. A proof similar to the proof of the above proposition shows that we can compute $b^\vee(U)$ for any $U \subsetneq S \cup T$. Thus we can compute $b^\vee(S \cap T)$, and we can also determine $b^\vee(S \cup T)$ by computing $b^\vee((S \cup T) \setminus U)$ for every $U \in \mathcal{B}$ which is a subset of $S \cup T$. Therefore we can decide whether $b(S) + b(T) \ge b^\vee(S \cap T) + b^\vee(S \cup T)$ holds. This concludes the description of the first algorithm.

An analogous algorithm can be used to decide if $p^\wedge$ satisfies the supermodular inequality for conflicting sets which are not separable from above with respect to p, and to identify all separable sets in $\mathcal{P}$.

It remains to check whether the cross-inequality for $p^\wedge$ and $b^\vee$ holds for conflicting non-b-separable pairs. Since we have already identified the non-separable sets, and we can compute $p^\wedge$ and $b^\vee$ on any set by linear programming, this can be done in polynomial time. ☐

Testing near paramodularity is likewise in P; the essential difference is that with near parmodularity, we do not need to keep track of the values of $p^\vee/b^\wedge$.

7 Minkowski Sum Characterization

In this section we use several results proven for generalized permutahedra, and the fact that this class is equivalent to bounded base polyhedra (Theorem 1.7). The following was shown in Proposition 2.4 in the arXiv version of [1]:

Theorem 7.1. *If P is a bounded base polyhedron, then there exist nonnegative real coefficients $\{\lambda_I^-, \lambda_I^+\}_{\varnothing \neq I \subseteq [n]}$ such that*

$$P + \sum_I \lambda_I^- \triangle_I = \sum_I \lambda_I^+ \triangle_I \tag{14}$$

where + means the Minkowski sum. Moreover, each bounded base polyhedron has such a representation where either λ_I^+ or λ_I^- is zero for all I, and this constrained representation is unique.

We observe that the following converse holds:

Observation 7.2. Only *bounded base polyhedra can satisfy the representation in the statement of Theorem 7.1.*

Proof. As mentioned earlier, [22, Prop. 3.2] showed that P is a generalized permutahedron if and only if its normal fan refines the normal fan of the permutahedron. So we need only show that when the condition (14) holds, P satisfies this refinement condition. Unwrapping the definitions of refinement, normal fans, and permutahedra, we must show that for all n-permutations π, all objective functions c with $c_{\pi[1]} \geq c_{\pi[2]} \geq \cdots \geq c_{\pi[n]}$ have a common maximizer x_π in P.

For any π, all its corresponding c have a common maximizer for the right-hand side of (14), namely $\sum_I \lambda_I^+ \chi_{\mathrm{argmin}\{\pi[i]:i\in I\}}$. The summand $\sum_I \lambda_I^- \triangle_I$ has a similar maximizer. Then the equality of Minkowski sums (14) implies that P has a common maximizer $x_\pi = \sum_I (\lambda_I^+ - \lambda_I^-)\chi_{\mathrm{argmin}\{\pi[i]:i\in I\}}$ for all such c, as needed. □

Then Theorem 1.3 follows from Theorem 7.1, the observation above, and Theorem 1.4.

8 Open Questions

We showed that checking total dual laminarity is NP-hard. We can show that the problem lies in $\mathsf{coNP}^{\mathsf{NP}} = \Pi_2^{\mathsf{P}}$: first we check for g-polymatroidality, and then for every combinatorial order type of objective (i.e., every cone in the normal fan of generic g-polymatroids) we check boundedness for those objectives, and if so, whether a dual laminar optimum exists. This still leaves a gap: is testing TDL Π_2^{P}-complete, NP-complete, or something else?

It would be interesting to come up with characterizations of other well-known general classes of polyhedra. One example would be a clean characterization of polyhedra that can be obtained as the intersection of two g-polymatroids, or a characterization of those set-families which equal the common independent sets of two matroids. Another example would be to characterize lattice polyhedra.

There is some interesting recent work on what constitutes an obstacle to submodularity [28], which seems to be relevant. We would be interested in progress on the following *submodular extension* problem: given n and a collection of pairs $\{(S_i \subseteq [n], v_i)\}_i$, determine whether there is any submodular f so that $f(S_i) = v_i$ for all i. It can be solved in exponential time by linear programming on all 2^n values of f, like in [28]. Is there a polynomial-time algorithm, or is this problem NP-hard?

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