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Degree bounded forest covering

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Abstract

We prove that for an undirected graph with arboricity at most $k + \epsilon$, its edges can be decomposed into k forests and a subgraph with maximum degree $\lceil \frac{k\epsilon+1}{1-\epsilon} \rceil$. The problem is solved by a linear programming based approach: we first prove that there exists a fractional solution to the problem, and then use a result on the degree bounded matroid problem by Király, Lau and Singh [5] to get an integral solution.

1 Introduction

Let $G = (V, E)$ be an undirected graph without loops. The set of edges induced by a node set $X \subseteq V$ is denoted by $E[X]$. The *arboricity* of G is defined as

$$\max_{X\subseteq V, |X|\geq 2} \frac{|E[X]|}{|X| - 1} \; .$$

A well-known result of Nash-Williams [8] states that a graph G can be covered by k forests if and only if its arboricity is at most k. If G has arboricity $k + \epsilon$ for some $0 < \epsilon < 1$, then this implies that it can be covered by $k + 1$ forests, but not by k forests. It is natural to ask whether, if ϵ is small, then G can "almost" be covered by k forests in some sense. Recently, Montassier et al. [6] proposed a conjecture of that flavor, where "almost" means that the remaining edges form a forest of low maximum degree.

Conjecture 1.1 ([6])**.** *If the arboricity of G is at most $k+\epsilon$ for some $0 < \epsilon < 1$, then G decomposes into $k+1$ forests, one of which has maximum degree at most $\left\lceil \frac{(k+1)\epsilon}{1-\epsilon} \right\rceil$.*

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This conjecture is best possible as shown by examples in [6]. Partial results are obtained by combinatorial method [6, 4] and by topological method [3], and related results are known for planar graphs [1, 2]. In this paper we are interested in a weaker form of the conjecture, where the bounded degree subgraph is not required to be a forest.

Conjecture 1.2. *If the arboricity of G is at most $k+\epsilon$ for some $0<\epsilon<1$, then G contains k forests such that the edges not covered by any of them form a subgraph of maximum degree at most $\left\lceil \frac{(k+1)\epsilon}{1-\epsilon} \right\rceil$.*

This weaker conjecture is also of interest by itself, and it has applications in bounding the game chromatic number [7]. Partial results towards this weaker conjecture are obtained in [7, 1, 6]. Recently, for $\epsilon \geq \frac{1}{2}$, Conjecture 1.2 was shown to be true by Kim et al. [4], but the case $\epsilon < \frac{1}{2}$ remains open (there are some special values for which it is known, see [4]). Our main result is the following theorem which almost proves Conjecture 1.2.

Theorem 1.3. *Let G be a graph with arboricity at most $k+\epsilon$, where k is a positive integer and $0<\epsilon\leq\frac{1}{2}$. Then G contains k forests such that the edges not covered by any of them form a subgraph of maximum degree at most $\left\lceil \frac{(k+1)\epsilon}{1-\epsilon} \right\rceil + 1 = \left\lceil \frac{k\epsilon+1}{1-\epsilon} \right\rceil$.*

Unlike previous approaches, we use a linear programming based approach to tackle this problem. We first prove a fractional version of Conjecture 1.2 (see Theorem 2.1), and then show that Theorem 1.3 follows from a result of the degree bounded matroid problem [5]. A consequence of this approach is that the forests satisfying Theorem 1.3 can be constructed in polynomial time.

2 Relation to Degree Bounded Matroids

In the degree lower-bounded matroid independent set problem, we are given a matroid $M=(V,\mathcal{I})$, a hypergraph $H=(V,E)$, and lower bounds $f(e)$ for each hyperedge $e\in E(H)$. The task is to find an independent set I with $|I\cap e|\geq f(e)$ for each hyperedge $e\in E(H)$. The forest covering problem can be reduced to a degree lower-bounded independent set problem: It is a well-known consequence of the matroid union theorem that for any graph G and positive integer k there is a matroid M_k with ground set E whose independent sets are the edge sets that can be covered by k forests. Given an undirected graph $G=(V,E)$ and the forest covering problem with parameter k and Δ where Δ is the target maximum degree of the remaining graph, we set the matroid to be M_k and define the hypergraph H with $V(H)=E(G)$ and $E(H)=\{\delta(v): v\in V(G)\}$ where $\delta(v)$ is the set of edges with exactly one endpoint in v, and set the lower bound for each hyperedge to be $d_G(v)-\Delta$ where $d_G(v)=|\delta(v)|$ is the degree of v in G. Then it can be seen that the degree bounded matroid problem in this setting is equivalent to the forest covering problem.

The result in [5] states that if there is a feasible solution to a linear programming relaxation of the degree bounded matroid problem, then there is an integral solution to the problem which violates the degree constraints by at most one[1]. The corresponding linear programming relaxation for the forest covering problem with parameter k is the following, where the objective is to minimize the maximum degree of the remaining graph. In the following let $d(v)$ denote the degree of node v in G, and for $x \in \mathbb{R}^E$ let $d_x(v) = \sum_{uv \in E} x_{uv}$.

$$\begin{aligned}
\min \quad & \Delta && && (1)\\
\text{s.t.} \quad & x(E[X]) \leq k(|X| - 1) && \text{for every } \emptyset \neq X \subseteq V && (2)\\
& 0 \leq x_e \leq 1 && \text{for every } e \in E && (3)\\
& d_x(v) \geq d_G(v) - \Delta && \text{for every } v \in V && (4)
\end{aligned}$$

The associated matroid polyhedron of M_k is described by (2) and (3). The requirement that $d_G(v) - d_x(v) \leq \frac{(k+1)\epsilon}{1-\epsilon}$ for every $v \in V$ can be written as a degree lower bound for x by setting $\Delta = \frac{(k+1)\epsilon}{1-\epsilon}$:

$$d_x(v) \geq d_G(v) - \frac{(k+1)\epsilon}{1-\epsilon} \qquad \text{for every } v \in V. \tag{5}$$

The result in [5] states that if the system (2),(3),(5) has a solution, then the matroid has an independent set F which almost satisfies the degree bounds:

$$d_F(v) \geq d_G(v) - \left\lceil \frac{(k+1)\epsilon}{1-\epsilon} \right\rceil - 1 \qquad \text{for every } v \in V. \tag{6}$$

This would imply Theorem 1.3 if the system (2),(3),(5) was always feasible when the graph has arboricity at most $k+\epsilon$. We prove that this fractional version of Conjecture 1.2 is true.

Theorem 2.1. *Let G be a graph with arboricity at most $k + \epsilon$, where k is a positive integer and $0 < \epsilon \leq \frac{1}{2}$. Then the system (2),(3),(5) has a feasible solution.*

We remark that this fractional version is also true if $\frac{1}{2} \leq \epsilon < 1$, and it is in fact easier to prove. However, this is less interesting because in this case Conjecture 1.2 itself has been proved in [4], so we only sketch the proof at the end of Section 3.

An additional consequence of the method in [5] is that if we are given a cost function $c : E \to \mathbb{R}_+$, and the minimum cost of a solution of (2),(3),(5) is z_{LP}, then there are k forests with total cost at most z_{LP} that satisfy the condition of Theorem 1.3, and these can be found in polynomial time.

3 Proof of the Fractional Conjecture

Instead of trying to describe an optimal solution to the linear program described by (2),(3),(5), we will give an upper bound for the objective value of the dual linear

[1]More precisely the result in [5] applies to the degree bounded matroid basis problem where the returned solution is required to be a basis of the matroid, but it is easy to reduce the degree bounded matroid independent set problem to that problem by adding dummy variables and we omit the details here.

program of (1)-(4) (when the arboricity of the graph is at most $k+\epsilon$), which is the following.

$$\begin{aligned}
\max \quad & \sum_{v\in V} d_G(v)\pi_v - \sum_{\emptyset\neq X\subseteq V} k(|X|-1)\mu_X - \sum_{e\in E}\rho_e \\
\text{s.t.} \quad & \pi_u + \pi_v - \sum_{Z:uv\in E[Z]} \mu_Z - \rho_{uv} \leq 0 \qquad \text{for every } uv\in E \\
& \sum_{v\in V}\pi_v \leq 1 \\
& \pi \geq 0 \\
& \mu \geq 0 \\
& \rho \geq 0
\end{aligned}$$

In an optimal dual solution we have $\rho_{uv} = \max\{\pi_u+\pi_v-\sum_{Z:uv\in E[Z]}, 0\}$. By writing $\sum_{v\in V} d_G(v)\pi_v = \sum_{uv\in E}(\pi_u+\pi_v)$ and eliminating the variables ρ, we get a simpler equivalent form.

$$\max \quad \sum_{uv\in E} \min\left\{\pi_u+\pi_v, \sum_{Z:uv\in E[Z]}\mu_Z\right\} - \sum_{\emptyset\neq X\subseteq V} k(|X|-1)\mu_X \tag{7}$$

$$\text{s.t.} \quad \sum_{v\in V}\pi_v \leq 1 \tag{8}$$

$$\pi \geq 0 \tag{9}$$

$$\mu \geq 0 \tag{10}$$

Let (π,μ) be an optimal dual solution. By duality, the following is equivalent to Theorem 2.1.

Theorem 3.1. *Let G be a graph with arboricity at most $k+\epsilon$, where k is a positive integer and $0<\epsilon\leq\frac{1}{2}$. Then*

$$\sum_{uv\in E} \min\left\{\pi_u+\pi_v, \sum_{Z:uv\in E[Z]}\mu_Z\right\} - \sum_{\emptyset\neq X\subseteq V} k(|X|-1)\mu_X \leq \frac{(k+1)\epsilon}{1-\epsilon}. \tag{11}$$

We will prove Theorem 3.1 in the rest of this section. Let $\mathcal{L} = \{\emptyset\neq X\subseteq V : \mu_X > 0\}$. By a standard uncrossing technique, we can simplify the optimal solution (π,μ) so that $\mathcal{L}$ is laminar, i.e. if X and Y in $\mathcal{L}$ are not disjoint, then $X\subseteq Y$ or $Y\subseteq X$.

Claim 3.2. *We may assume that $\mathcal{L}$ is laminar.*

Proof. Suppose that X and Y in $\mathcal{L}$ are not disjoint. It is easy to verify that if we decrease μ_X and μ_Y by $\min\{\mu_X,\mu_Y\}$, and increase $\mu_{X\cap Y}$ and $\mu_{X\cup Y}$ by $\min\{\mu_X,\mu_Y\}$, then we obtain a feasible dual solution whose objective value is at least as large, since $\min\{\pi_u+\pi_v, \sum_{Z:uv\in E[Z]}\mu_Z\}$ would not decrease for any $uv\in E$ and the second summation remains the same. □

The overall plan of the proof is as follows. We give an upper bound for the first term on the left hand side of (11) in form of definite integrals in Claim 3.3, and give lower bounds of the same form for the second term on the left hand side and also for the right hand side of (11). We then show in Lemma 3.4 that the required inequality holds for the integrands for any value of the variable, by using the assumption that the graph is of arboricity at most $k+\epsilon$.

Let us introduce some notation. For $X \in \mathcal{L}$, let $\alpha_X = \sum_{Z \supseteq X} \mu_Z$. Let $\alpha = \max\{\alpha_X : X \in \mathcal{L}\}$. For any $0 \le t \le \alpha$, let

$$\mathcal{L}_t = \{X \in \mathcal{L} : \alpha_X \ge t,\ \alpha_Y < t\ \forall Y \supsetneq X\}.$$

Note that the sets in $\mathcal{L}_t$ are disjoint because $\mathcal{L}$ is laminar. For any $0 \le t \le \alpha$ and $X \in \mathcal{L}_t$, let $X_t = \{v \in X : \ \pi_v \ge t\}$. Finally, given two node sets X and Y, let $d(X,Y)$ denote the number of edges with at least one endnode in both X and Y.

The first step of the proof is to give an upper bound for the first term of (11) that will turn out to be easier to estimate.

Claim 3.3.

$$\sum_{uv \in E} \min\left\{\pi_u + \pi_v, \sum_{Z : uv \in E[Z]} \mu_Z\right\}$$
$$\le \int_0^\alpha \sum_{X \in \mathcal{L}_t} \left(\frac{1}{1-\epsilon}|E[X_t]| + d(X_t, X \setminus X_t) + \frac{1-2\epsilon}{1-\epsilon}|E[X \setminus X_t]|\right) dt.$$

Proof. The integral on the right hand side is in fact a finite sum, so it is well-defined. To prove the inequality, we show that the contribution of each edge to the right hand side is at least its contribution to the left side. Let $e = uv \in E$ be an arbitrary edge, and let us assume $\pi_u \ge \pi_v$. Let X be the smallest set in $\mathcal{L}$ that contains both u and v; thus $\sum_{Z:uv\in E[Z]} \mu_Z = \alpha_X$. For any $t \in [0, \alpha_X]$, there is exactly one set $Z \in \mathcal{L}_t$ with $u, v \in Z$ since $\mathcal{L}$ is laminar and thus the sets in $\mathcal{L}_t$ are disjoint. We distinguish three cases.

1. $\pi_u \ge \alpha_X$. In this case the contribution of e to the left hand side is equal to α_X, and we will show that its contribution to the right hand side is at least α_X. When $t \in [0, \min\{\alpha_X, \pi_v\}]$, edge e is counted with weight $\frac{1}{1-\epsilon}$ in the right hand side because both u and v are in Z_t. If $\pi_v \ge \alpha_X$ then we are done. Otherwise e is counted with weight 1 when $t \in [\pi_v, \alpha_X]$ because $u \in Z_t$ but $v \notin Z_t$. Therefore the total contribution of e is at least α_X.

2. $\pi_u < \alpha_X \le \pi_u + \pi_v$. In this case the contribution of e to the left hand side is equal to α_X. In the right hand side, the edge e is counted with weight $\frac{1}{1-\epsilon}$ if $t \in [0, \pi_v]$ when both $u, v \in Z_t$, with weight 1 if $t \in [\pi_v, \pi_u]$ when $u \in Z_t$ and $v \notin Z_t$, and with weight $\frac{1-2\epsilon}{1-\epsilon}$ if $t \in [\pi_u, \alpha_X]$ when both $u, v \notin Z_t$. Thus the total contribution of e to the right hand side is equal to
$$\frac{1}{1-\epsilon}\pi_v + (\pi_u - \pi_v) + \frac{1-2\epsilon}{1-\epsilon}(\alpha_X - \pi_u) = \frac{1-2\epsilon}{1-\epsilon}\alpha_X + \frac{\epsilon}{1-\epsilon}\pi_u + \frac{\epsilon}{1-\epsilon}\pi_v.$$
Since $\pi_u + \pi_v \ge \alpha_X$ by assumption, this is at least α_X as desired.

3. $\pi_u + \pi_v \le \alpha_X$. In this case the contribution of e to the left hand side is equal to $\pi_u + \pi_v$. The contribution of e to the right hand side is the same as above: $\frac{1}{1-\epsilon}$ if $t \in [0, \pi_v]$, 1 if $t \in [\pi_v, \pi_u]$, and $\frac{1-2\epsilon}{1-\epsilon}$ if $t \in [\pi_u, \alpha_X]$, and thus the total contribution is equal to

$$\frac{1}{1-\epsilon}\pi_v + (\pi_u - \pi_v) + \frac{1-2\epsilon}{1-\epsilon}(\alpha_X - \pi_u).$$

   Since $\alpha_X - \pi_u \ge \pi_v$ and $\frac{1}{1-\epsilon} + \frac{1-2\epsilon}{1-\epsilon} = 2$, the contribution of e to the right hand side is at least $2\pi_v + (\pi_u - \pi_v) = \pi_u + \pi_v$ as desired (note that here we use the assumption that $\epsilon \le \frac{1}{2}$).

□

We reformulate the second term on the left side of (11) as an integral on the interval $[0, \alpha]$:

$$\sum_{X \subseteq V} k(|X|-1)\mu_X = \int_0^\alpha \sum_{X \in \mathcal{L}_t} k(|X|-1)\, dt.$$

The next step is to lower bound the constant on the right hand side of (11) by an integral with the same limits. Let us use the notation $\pi(X) = \sum_{v \in X} \pi_v$. By (8) we have

$$1 \ge \pi(V) \ge \sum_{v \in V} \min\{\pi_v, \sum_{Z: v \in Z} \mu_Z\} = \int_0^\alpha \sum_{X \in \mathcal{L}_t} |X_t|\, dt,$$

where the equality follows because the contribution of v to the right hand side is equal to $\min\{\pi_v, \sum_{Z:v\in Z} \mu_Z\}$. Thus

$$\frac{(k+1)\epsilon}{1-\epsilon} \ge \int_0^\alpha \sum_{X \in \mathcal{L}_t} \frac{(k+1)\epsilon}{1-\epsilon} |X_t|\, dt.$$

After these formulations, to prove Theorem 3.1, it suffices to show that

$$\int_0^\alpha \sum_{X \in \mathcal{L}_t} \left(\frac{1}{1-\epsilon}|E[X_t]| + d(X_t, X \setminus X_t) + \frac{1-2\epsilon}{1-\epsilon}|E[X \setminus X_t]| \right) dt$$
$$\le \int_0^\alpha \sum_{X \in \mathcal{L}_t} \left(\frac{(k+1)\epsilon}{1-\epsilon}|X_t| + k(|X|-1) \right) dt. \quad (12)$$

We show that the inequality holds for the integrands for any value of t between 0 and α, so it holds for the integrals as well. The assumption that G is of arboricity at most $k+\epsilon$ is only used in the following lemma.

Lemma 3.4. *For any $0 \le t \le \alpha$ and $X \in \mathcal{L}_t$, the following inequality holds:*

$$\frac{1}{1-\epsilon}|E[X_t]| + d(X_t, X \setminus X_t) + \frac{1-2\epsilon}{1-\epsilon}|E[X \setminus X_t]| \le k(|X|-1) + \frac{(k+1)\epsilon}{1-\epsilon}|X_t|.$$

Proof. The idea is to identify the high degree structures Y in $X \setminus X_t$, and then use the arboricity to bound $|E(X_t \cup Y)|$, while the number of remaining edges can be bounded by $k|X \setminus (Y \cup X_t)|$. Let $C_1, \dots, C_l$ be the components of $G[X \setminus X_t]$, and let Y be the union of the components where the average degree in G of the nodes is at least $k+1$, i.e.

$$Y = \bigcup \{C_i : 2|E[C_i]| + d(C_i, X_t) \geq (k+1)|C_i|\}.$$

Claim 3.5. *The following two inequalities hold for this set Y:*

$$2|E[Y]| + d(Y, X_t) \geq (k+1)|Y|, \tag{13}$$

$$d(X \setminus (Y \cup X_t), X) \leq k|X \setminus (Y \cup X_t)|. \tag{14}$$

Proof. Inequality (13) follows easily from the definition, since it holds for all components of $G[Y]$. To show inequality (14), observe that if $C_i \cap Y = \emptyset$, then $2|E[C_i]| + d(C_i, X_t) \leq k|C_i| + (|C_i| - 1)$. This implies, using that $|E[C_i]| \geq |C_i| - 1$ because of its connectedness, that $|E[C_i]| + d(C_i, X_t) \leq k|C_i|$. By summing over all components not in Y, we obtain that

$$d(X \setminus (Y \cup X_t), X) = \sum_{i: C_i \cap Y = \emptyset} (|E[C_i]| + d(C_i, X_t)) \leq k|X \setminus (Y \cup X_t)|.$$

□

First let us analyze the case when $X_t \cup Y = \emptyset$. Since all components have average degree less than $k+1$, we have $|E[X]| \leq \frac{k+1}{2}|X| - \frac{1}{2}$. A simple case analysis shows (using the fact that G has no loops) that this implies $|E[X]| \leq k(|X| - 1)$, so the Lemma is true in this case.

We may thus assume that $X_t \cup Y \neq \emptyset$. Since the arboricity of G is at most $k + \epsilon$, we know that $|E[X_t \cup Y]| \leq (k+\epsilon)(|X_t \cup Y| - 1)$, so

$$\frac{1}{1-\epsilon}(|E[X_t]| + d(X_t, Y) + E[Y]) = \frac{1}{1-\epsilon}|E[X_t \cup Y]| \leq \frac{k+\epsilon}{1-\epsilon}(|X_t \cup Y| - 1).$$

If we subtract $\frac{\epsilon}{1-\epsilon}$ times the inequality (13) from this, we get that

$$\begin{aligned} & \frac{1}{1-\epsilon}|E[X_t]| + d(X_t, Y) + \frac{1-2\epsilon}{1-\epsilon}|E[Y]| \\ \leq \; & \frac{k+\epsilon}{1-\epsilon}(|X_t \cup Y| - 1) - \frac{(k+1)\epsilon}{1-\epsilon}|Y| \\ = \; & \left(k + \frac{(k+1)\epsilon}{1-\epsilon}\right)(|X_t \cup Y| - 1) - \frac{(k+1)\epsilon}{1-\epsilon}|Y| \\ = \; & k(|X_t \cup Y| - 1) + \frac{(k+1)\epsilon}{1-\epsilon}(|X_t| - 1). \end{aligned}$$

Next we add inequality (14):

$$\begin{aligned} & \frac{1}{1-\epsilon}|E[X_t]| + d(X_t, X \setminus X_t) + \frac{1-2\epsilon}{1-\epsilon}|E[Y]| + E[X \setminus (Y \cup X_t)] \\ \leq \quad & k(|X_t \cup Y| - 1) + \frac{(k+1)\epsilon}{1-\epsilon}(|X_t| - 1) + k|X \setminus (Y \cup X_t)| \\ = \quad & k(|X| - 1) + \frac{(k+1)\epsilon}{1-\epsilon}(|X_t| - 1). \end{aligned}$$

This implies the inequality in the Lemma because

$$\frac{1-2\epsilon}{1-\epsilon}|E[Y]| + E[X \setminus (Y \cup X_t)] \geq \frac{1-2\epsilon}{1-\epsilon}|E[X \setminus X_t]|.$$

□

By Lemma 3.4, inequality (12) is true, since the inequality holds for the integrands for any value $t \in [0, \alpha]$. This concludes the proof of Theorem 3.1, hence also the proof of Theorem 2.1. Using the degree bounded matroid result described in Section 2, we obtain Theorem 1.3.

Remark. As we have already mentioned, Theorems 2.1 and 3.1 are true also for $\frac{1}{2} \leq \epsilon < 1$. We now sketch the proof. The overall structure of the proof is similar, but we remove the term $\frac{1-2\epsilon}{1-\epsilon}|E[X \setminus X_t]|$ from the bound in Claim 3.3. Therefore Lemma 3.4 should be modified: the inequality

$$\frac{1}{1-\epsilon}|E[X_t]| + d(X_t, X \setminus X_t) \leq k(|X| - 1) + \frac{(k+1)\epsilon}{1-\epsilon}|X_t|$$

should hold for any $0 \leq t \leq \alpha$ and $X \in \mathcal{L}_t$. The proof of this is simpler than the proof of Lemma 3.4: instead of considering the components of $X \setminus X_t$, we define Y as the set of nodes of $X \setminus X_t$ for which $d(v, X_t) \geq k+1$. Using the fact that $|E[X_t \cup Y]| \leq (k+\epsilon)(|X_t \cup Y| - 1)$ and the fact that $d(v, X_t) \leq k$ for any $v \notin X_t \cup Y$, we obtain the desired bound.

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References

[1] O.V. Borodin, A.V. Kostochka, N.N. Sheikh, G.Yu, *Decomposing a planar graph with girth 9 into a forest and a matching*, European J. Combin. 29 (2008), 1235-1241.

[2] D. Goncalves, *Covering planar graphs with forests, one having bounded maximum degree*, J. Combin. Theory Ser. B, 99, (2009), 314-322.

[3] T. Kaiser, M. Montassier, A. Raspaud, *Covering a graph by forests and a matching*, arXiv:1007.0316v1, 2010.

[4] S.-J. Kim, A.V. Kostochka, D.B. West, H. Wu, and X. Zhu, *Decomposition of sparse graphs into forests and a graph with bounded degree*, submitted (2010).

[5] T. Király, L.C. Lau, and M. Singh, *Degree bounded matroids and submodular flows*, Proceedings of 13th International Conference IPCO 2008, LNCS 5035 (2008), 259-272.

[6] M. Montassier, P. Ossona de Mendez, A. Raspaud, and X. Zhu, *Decomposing a graph into forests*, Manuscript (2010).

[7] M. Montassier, A. Pecher, A. Raspaud, D.B. West, X. Zhu, *Decomposition of sparse graphs, with application to game coloring number*, Discrete Mathematics, 310, (2010), 1520-1523.

[8] C.St.J.A. Nash-Williams, *Decomposition of finite graphs into forests*, J. London Math. Soc. 39 (1964), 12.