PPAD-completeness of polyhedral versions of Sperner's Lemma

Tamás Király[a,*], Júlia Pap[a]

[a] MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Budapest.

Abstract

We show that certain polyhedral versions of Sperner's Lemma, where the colouring is given explicitly as part of the input, are PPAD-complete. The proofs are based on two recent results on the complexity of computational problems in game theory: the PPAD-completeness of 2-player Nash, proved by Chen and Deng, and of Scarf's Lemma, proved by Kintali. We give a strengthening of the latter result, show how colourings of polyhedra provide a link between the two, and discuss a special case related to vertex covers.

Keywords: computational complexity, colouring of polyhedra, vertex cover

1. Introduction

Sperner's Lemma on the existence of a panchromatic triangle in a suitable colouring of a triangulation has many versions and generalizations. The following is a variant formulated in terms of colourings of n-dimensional polytopes, which can be seen as the usual multidimensional Sperner Lemma applied to a subdivision of the Schlegel diagram of a polytope; see [9] for a direct proof. Given a colouring of the vertices of a polytope by n colours, a facet is called *panchromatic* if it contains vertices of each colour.

Theorem 1. *Let P be an n-dimensional polytope, with a simplex facet F_0. Suppose we have a colouring of the vertices of P by n colours such that F_0 is panchromatic. Then there is another panchromatic facet.*

This leads to the problem of finding a panchromatic facet other than F_0.

Polytopal Sperner

Input: vectors $v^i \in \mathbb{Q}^n$ $(i = 1, \dots, m)$ whose convex hull is a full-dimensional polytope P; a colouring of the vertices by n colours; a panchromatic simplex facet F_0 of P.

*Corresponding author
Email addresses: tkiraly@cs.elte.hu (Tamás Király), papjuli@cs.elte.hu (Júlia Pap)

Output: vectors $v^{i_1}, \dots, v^{i_n}$ with different colours which define a facet of P different from F_0.

By the *polar* of a polyhedron P we mean the polyhedron $P^\Delta := \{c \in \mathbb{R}^n : cx \le 1 \text{ for all } x \in P\}$. By taking the polar after translating the polytope so that the origin lies in its interior, we get the following polar version of Theorem 1. A vertex of an n-dimensional polyhedron is *simple* if it lies on exactly n facets. For a colouring of the facets, a vertex is *panchromatic* if it lies on facets of every colour.

Theorem 2. *Let P be an n-dimensional polytope, with a simple vertex v_0. Suppose we have a colouring of the facets of P with n colours such that v_0 is panchromatic. Then there is another panchromatic vertex.*

The corresponding computational problem, where the polytope is given by a linear inequality system, is equivalent with Polytopal Sperner.

A related but slightly different Sperner-like theorem was introduced by the authors in [9]. Recall that the *extreme directions* of a polyhedron are the extreme rays of its characteristic cone. For a vector v, by *direction* v we mean the direction $\{\lambda v : \lambda \ge 0\}$.

Theorem 3 ([9])**.** *Let P be an n-dimensional pointed polyhedron whose characteristic cone is generated by n linearly independent vectors. If we colour the facets of the polyhedron by n colours such that facets having the i-th extreme direction do not get colour i, then there is a panchromatic vertex.*

The advantage of this theorem over the standard Sperner Lemma is that it enables short and graphic proofs of several combinatorial and game theoretic results about stable sets and matchings, see [8, 9, 11]. An example of this is the proof of Theorem 9, a new result about vertex covers of graphs. The computational problem corresponding to Theorem 3 is the following.

Extreme direction Sperner

Input: matrix $A \in \mathbb{Q}^{m\times n}$ and vector $b \in \mathbb{Q}^m$ such that $P = \{x : Ax \le b\}$ is a pointed polyhedron whose characteristic cone is generated by n linearly independent vectors; a colouring of the facets by n colours such that facets having the i-th extreme direction do not get colour i.

Output: a panchromatic vertex of P.

In this note we show, using recent developments on the computational complexity of problems in game theory, that the following two natural special cases of this problem are already PPAD-complete.

0-1 extreme direction Sperner

Input: matrix $A \in \{0,1\}^{m\times n}$ with no all-0 column; a colouring of the facets of $P = \{x : Ax \le \mathbf{1}, x \le \mathbf{1}\}$ by n colours such that facets with extreme direction $-e_i$ do not get colour i.

Output: a panchromatic vertex of P.

Extreme direction Sperner with $2n$ facets

Input: a matrix $A \in \mathbb{Q}_+^{n \times n}$; a colouring of the facets of $P = \{x : Ax \leq \mathbf{1}, x \leq \mathbf{1}\}$ by n colours such that facets with extreme direction $-e_i$ do not get colour i and every colour appears exactly twice.

Output: a panchromatic vertex of P.

We also show that extreme direction Sperner provides a link between the complexity of Scarf's Lemma and that of finding Nash equilibria in 2-player games. In particular, extreme direction Sperner with $2n$ facets can be considered as a special case of the computational version of Scarf's Lemma.

The structure of the paper is as follows. The remainig part of this section introduces the complexity class PPAD. In Section 2 we show that our problems belong to this class. Then in Section 3 we use the results of Kintali [7] to show that 0-1 extreme direction Sperner is PPAD-complete even in the case when each row of A contains at most three 1s. In contrast, the problem is solvable in polynomial time if each row contains at most two 1s. If arbitrary left sides are allowed, then we obtain Theorem 9 on vertex covers. Finally, in Section 4 we prove using the result of Chen and Deng [4] that extreme direction Sperner with $2n$ facets is PPAD-complete. We also show that this problem is in fact a special case of Scarf.

1.1. The class PPAD and PPAD-completeness

The complexity class PPAD is defined as the set of total search problems which are Karp-reducible to the following prototypical problem:

End of the line

Input: an algorithm that describes a directed graph on $\{0,1\}^n$, with running time polynomial in n. The digraph has in- and out-degrees at most 1, and $\mathbf{0}$ has in-degree 0 and out-degree 1. The algorithm outputs the out-neighbour and in-neighbour of a given node.

Output: any node in $\{0,1\}^n \setminus \{\mathbf{0}\}$ that has degree 1 (where the degree is the in-degree plus the out-degree).

A problem in PPAD is called *PPAD-complete* if every other problem in PPAD is Karp-reducible to it. The class PPAD was introduced by Papadimitriou [12], who proved among other results that a computational version of 3D Sperner's Lemma is PPAD-complete. Later Chen and Deng [2] proved that the 2-dimensional problem is also PPAD-complete. The input of these computational versions is the description of a polynomial algorithm that computes a legal colouring, while the number of vertices to be coloured is exponential in the input size. This is conceptually different from the computational problems that we consider, where the input explicitly contains the vertices or facets of a polyhedron and their colouring. Our problems are solvable in polynomial time in fixed dimension since then the number of facets and vertices is polynomial.

For a long time it had been open to find natural PPAD-complete problems that do not have a description of a Turing machine in their input. In 2006, Daskalakis, Goldberg and Papadimitriou [1] proved that approximating Nash-equilibria in 4-player games is PPAD-hard. Building on their work, Chen and Deng [4] managed to prove the same for 2-player Nash-equilibria, which is considered a breakthrough result in the area. In another line of research, Kintali [7] proved that the computational version of Scarf's Lemma (Theorem 13) is PPAD-complete, along with other related problems, see [6].

2. Membership in PPAD

Proposition 4. POLYTOPAL SPERNER *is in PPAD.*

Proof. We reduce it to the problem END OF THE LINE. First we compute a perturbation of the vertices in the input such that every facet of the convex hull of the perturbed vertices is a simplex, and every facet (as a vertex set) is a subset of an original facet. This can be done in polynomial time e.g. using the ε-perturbation method described in [10] for the polar of the polytope, and taking the polar again. Assume that the set of colours is $[n]$. We define a digraph whose nodes are the facets that contain all colours in $[n-1]$ (formally, we may associate a node to each n-tuple of vertices, all other nodes being isolated). Each $(n-2)$-dimensional face with all colours in $[n-1]$ is in exactly two facets. We can designate one to be on the left side of the face and the other to be on the right side in the following way. We compute the sign of the two determinants of the vectors going from a fixed inner point of P to the $n-1$ vertices of the $(n-2)$-dimensional face (in the order according to the colours) and the n-th vertex of the two facets; the facet on the left side is the one whose determinant has the same sign as the determinant of the matrix derived from the given panchromatic simplex facet F_0. For each such $(n-2)$-dimensional face, we introduce an arc from the node corresponding to the facet on the left side to the node corresponding to the facet on the right side.

This digraph has in-degree and out-degree at most 1 in every node, and the neighbours can be computed in polynomial time. A node has degree 1 if and only if the corresponding facet is panchromatic. We may assume w.l.o.g. that the node corresponding to F_0 is a source, so the solution of END OF THE LINE for this digraph corresponds to finding a panchromatic facet different from F_0. □

Proposition 5. EXTREME DIRECTION SPERNER *is in PPAD.*

Proof. We prove that EXTREME DIRECTION SPERNER is Karp-reducible to POLYTOPAL SPERNER. Suppose that matrix A and vector b are an instance of EXTREME DIRECTION SPERNER and let $P = \{x : Ax \leq b\}$. We can translate P so that it contains the origin in its interior. In this case its polar P^{Δ} is a polytope whose vertices can be obtained from A and b. The colouring of P defines a colouring of the vertices of P^{Δ} except for the origin which corresponds to the infinite facet of P. Let us cut off the origin with a hyperplane H – such a hyperplane can be computed in polynomial time. This way, since the origin is a simple vertex of P, we introduce exactly n new vertices and a simplex facet $P^\Delta \cap H$, which we take as F_O. The i-th new vertex lies on the facets that correspond to all but the i-th extreme direction of P; let the colour of it be i. We obtained a colouring of $P^\Delta \cap H^+$ (where H^+ is the halfspace bounded by H not containing the origin) which satisfies the criteria, so it is an instance of POLYTOPAL SPERNER. A panchromatic facet of $P^\Delta \cap H^+$ which is different from $P^\Delta \cap H$, corresponds to a panchromatic vertex of P. □

3. PPAD-completeness of 0-1 EXTREME DIRECTION SPERNER

The proof is similar to the proof of PPAD-completeness of SCARF by Kintali [7], and builds on his result that the problem 3-STRONG KERNEL defined below is PPAD-complete. A digraph $D = (V, E)$ is called *clique-acyclic* if for each clique K, there is a node $v \in K$ whose closed out-neighbourhood contains K (the node itself is included in the closed out-neighbourhood). A *strong fractional kernel* of D is a vector $x \in \mathbb{R}^V_+$ such that $x(K) \leq 1$ for every clique K, and for each node v there is at least one clique K in the closed out-neighbourhood of v such that $x(K) = 1$ (here $x(K)$ is shorthand for $\sum_{v \in K} x_v$).

3-STRONG KERNEL
Input: A clique-acyclic digraph D with maximum clique size at most 3.
Output: A strong fractional kernel of D.

Theorem 6 (Kintali [7])**.** 3-STRONG KERNEL *is PPAD-complete.*

Kintali used this theorem to prove the PPAD-completeness of the computational version of Scarf's Lemma, which we will define in Section 4. Here we use it to show the following.

Theorem 7. 0-1 EXTREME DIRECTION SPERNER *is PPAD-complete, even when every row of A contains at most three 1s.*

Proof. To reduce 3-STRONG KERNEL on digraph $D = (V, E)$ to 0-1 EXTREME DIRECTION SPERNER, we assume that $V = [n]$, and consider the polyhedron $P = \{x \in \mathbb{R}^n : x(K) \leq 1 \text{ for every clique } K \text{ of } D\}$. Since every clique has size at most 3, the number of cliques is polynomial in n. The extreme directions of P are $-e_j$ ($j \in [n]$). As the set of colours, we use $[n]$. Let the colour of the facet $x(K) = 1$ be a node of K whose closed out-neighbourhood contains K. This colouring satisfies the criterion in Theorem 3, so we have a valid input for 0-1 EXTREME DIRECTION SPERNER, and furthermore every row of the describing system contains at most three 1s. Let x^* be a panchromatic vertex. For each node v, there is a clique K such that the facet $x(K) = 1$ contains x^* and has colour v, hence K is in the closed out-neighbourhood of v. This means that x^* is a strong fractional kernel. □

Together with the proof of Proposition 5, the theorem implies that POLYTOPAL SPERNER is also PPAD-complete. Next we show that three 1s in a row is best possible.

Proposition 8. 0-1 EXTREME DIRECTION SPERNER *can be solved in polynomial time if every row of A contains at most two 1s.*

Proof. Let A be the matrix in the input. We may assume that every row of A contains exactly two 1s. Consider the graph on node set $[n]$ whose edge-node incidence matrix is A. The colouring of the facets corresponding to the rows of A determines an orientation of this graph: let the head of each edge be the colour of the corresponding facet. Let D denote the resulting directed graph. The goal is to find a vertex of the polyhedron $P = \{x \in \mathbb{R}^n : \ Ax \leq \mathbf{1}, \ x \leq \mathbf{1}\}$ for which for every $i \in [n]$ we have $x_i = 1$ or there is an arc ji of D which is saturated, that is, $x_j + x_i = 1$.

If each node has an incoming arc, then the vector $z = \frac{1}{2}\mathbf{1}$ is a panchromatic element of P, because $Az = \mathbf{1}$, that is, all arcs are saturated. Thus we can find a vertex x^* of P for which $Ax^* = \mathbf{1}$, which is therefore panchromatic.

If there is a source node i of D, then x_i has to be 1, because the only inequality with colour i is $x_i \leq 1$. If j is an out-neighbour of i, then to make arc ij saturated, let x_j be 0. This guarantees that x lies on facets of every colour in the closed outneighbourhood of i. We can delete the closed outneighbourhood of i and repeat the above, until we get a graph with no source node, and set the remaining variables as we did when each node had an incoming arc. $\square$

The above proof works only when the right side of every inequality is 1. If we remove this restriction, then we obtain an interesting problem on vertex covers. For an undirected graph $G = ([n], E)$ and a vector $w \in \mathbb{N}^E$, a vector $x \in \mathbb{N}^n$ is called a *w-cover* if $x_i + x_j \geq w_{ij}$ for every $ij \in E$.

Theorem 9. *Let $D = ([n], E)$ be a directed graph and let $w \in \mathbb{N}^E$. Then there is a $2w$-cover x of the underlying undirected graph of D such that for every node i with $x_i > 0$ there is an arc ji with $x_j + x_i = 2w_{ji}$.*

Proof. Let A be the edge-node incidence matrix of the underlying undirected graph and consider the polyhedron $P = \{x \in \mathbb{R}^n : \ Ax \geq 2w, \ x \geq \mathbf{0}\}$. Let us colour an inequality corresponding to an arc ji with colour i and an inequality $x_j \geq 0$ with colour j. Using Theorem 3 there is a panchromatic vertex x^* of P. By a result of Gallai [5], P is an integer polyhedron. Therefore x^* is the characteristic vector of a $2w$-cover which by panchromaticness has the desired properties. $\square$

It is open whether the $2w$-cover guaranteed by the above theorem can be found in polynomial time.

Proposition 10. *The $2w$-cover in Theorem 9 can be found in polynomial time in the case when each node has in-degree 1 in D.*

Proof. Let us first assume that D is a directed cycle, and the nodes are indexed according to the cyclic order. We can check in polynomial time if there is a nonnegative $2w$-cover x where every arc is tight, that is, $x_i + x_{i+1} = 2w_{i,i+1}$ for every $i \in [n]$. If there is no such $2w$-cover, then for every $2w$-cover x there must be a node i where $x_i = 0$. We claim that if $x_i = 0$, then this uniquely determines the next node j in the order where $x_j = 0$. Suppose for convenience that $x_0 = 0$. Then x_1 has to be $2w_{0,1}$. Thus x_2 has to be the minimum of 0 and $2w_{1,2} - 2w_{0,1}$, and so forth, x_i has to be $w_{i-1,i} - w_{i-2,i-1} \pm \cdots \pm w_{0,1}$, as long as these values are positive. If we reach a node i where this value is negative or 0, we have to set x_i to 0, and then repeating the above we get the values of the forthcoming nodes. If we determined all the x_i values, then the arc $n1$ is either covered, in which case we are done, or not. Since Theorem 9 guarantees a solution, thus by trying all possible starting points we will find a solution.

In the general case each component of D contains one directed cycle and some arborescences rooted on nodes of the cycle. First we solve the problem restricted to the cycle, then we can traverse the arborescences starting from the root; the values are uniquely determined and we get a solution. □

We note that the prescription of in-degree 1 means that the corresponding polyhedron has $2n$ facets and each colour appears exactly twice. This leads us to the topic of the next section.

4. PPAD-completeness of EXTREME DIRECTION SPERNER WITH $2n$ FACETS

It is a well-known result in game theory that finding a symmetric Nash equilibrium in a symmetric finite 2-player game is as hard as finding a Nash equilibrium in a not necessarily symmetric 2-player game. A useful property of symmetric games is that symmetric Nash equilibria can be characterized as vertices of a polyhedron having a certain complementarity property. The search problem can be described as follows.

SYMMETRIC 2-NASH

Input: a matrix $A \in \mathbb{Q}_+^{n\times n}$, such that the polyhedron $P = \{x : Ax \leq \mathbf{1}, x \geq 0\}$ is bounded and full-dimensional.

Output: a nonzero vertex v of P such that $a_i v = 1$ whenever $v_i > 0$, where a_i is the i-th row of A.

Theorem 11 (Chen and Deng, [4]). SYMMETRIC 2-NASH *is PPAD-complete.*

We can observe that SYMMETRIC 2-NASH is a special case of the polar version of POLYTOPAL SPERNER: if we colour by colour i both the facet corresponding to the i-th row of A and the facet $x_i \geq 0$, then $\mathbf{0}$ is a simple and panchromatic vertex, and a nonzero panchromatic vertex v clearly satisfies that $a_i v = 1$ whenever $v_i > 0$. Let us call this the *Nash colouring.*

In the following, we prove that SYMMETRIC 2-NASH is Karp-reducible to a special case of EXTREME DIRECTION SPERNER WITH $2n$ FACETS, which turns out to be a special case of SCARF.

Theorem 12. EXTREME DIRECTION SPERNER WITH $2n$ FACETS *is PPAD-complete.*

Proof. We reduce SYMMETRIC 2-NASH to it. Let $A \in \mathbb{Q}_+^{n\times n}$ define an instance of SYMMETRIC 2-NASH, and let $P = \{x : Ax \leq \mathbf{1}, x \geq 0\}$. The vertex $\mathbf{0}$ is simple, so it has n neighbouring vertices $v^1, \ldots, v^n$, which furthermore have the form $v^i = \lambda_i e_i$ for some $\lambda_i > 0$. These vertices can be computed in polynomial time, and we can check if one of them satisfies the conditions. We can assume that none of them does, that is, $a_i v^i < 1$ $(i \in [n])$.

Let $P_0 = \{x \in P : \sum_{j=1}^n \frac{1}{\lambda_j} x_j \geq 1\}$, which is the convex hull of the vertices of P except for the origin. The new facet F_0 contains the vertices $v_1, \ldots, v_n$. We can translate P_0 so that it contains the origin in its interior. In this case its polar P_0^Δ is a polytope; its vertices can be computed. Let w_0 be the vertex of P_0^Δ corresponding to F_0, and let $F_1, \ldots, F_n$ denote the facets of P_0^Δ corresponding to the vertices $v^1, \ldots, v^n$.

Let $w^1, \ldots, w^n$ be the vertices of P_0^Δ adjacent to w^0, indexed such that w_i is not on facet F_i. We can apply an affine transformation that takes w^0 to $\mathbf{0}$ and w^i to e_i $(i \in [n])$; let Q be the resulting polytope. The polar Q^Δ is a polyhedron of the form $\{x : Ax \leq \mathbf{1}, x \leq 1\}$. Let us take the Nash colouring of P. This induces a colouring of the facets of Q^Δ according to the two polarities. Clearly every colour appears exactly twice. We claim that the facets with extreme direction $-e_i$ do not get colour i. A facet of Q^Δ with extreme direction $-e_i$ corresponds to a vertex of P_0^Δ on facet F_i, which in turn corresponds to a facet of P containing vertex v^i. Since $v_i^i > 0$ and $a_i v^i < 1$, no facet containing v^i has colour i.

Suppose that we can find a panchromatic vertex v of Q^Δ. By a similar argument as above, this corresponds to a panchromatic vertex of P, which completes the proof. □

We conclude this section by showing a relation between EXTREME DIRECTION SPERNER WITH $2n$ FACETS and Scarf's Lemma [13]. In Scarf's Lemma we consider a bounded polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$, where A is an $m \times n$ nonnegative matrix (with non-zero columns) and $b \in \mathbb{R}^m$ is a positive vector. In addition, for every row a_i of A $(i \in [m])$, a total order $<_i$ of $\mathrm{supp}(a_i)$ (the support of the vector a_i) is given. If $j \in \mathrm{supp}(a_i)$ and $K \subseteq \mathrm{supp}(a_i)$, we use the notation $j \leq_i K$ as an abbreviation for "$j \leq_i k$ for every $k \in K$".

A vertex x^* of P *dominates* column j if there is a row i where $a_i x^* = b_i$ and $j \leq_i \mathrm{supp}(x^*) \cap \mathrm{supp}(a_i)$ (this implies that $j \in \mathrm{supp}(a_i)$). A vertex x^* of P is *maximal* if by increasing any coordinate of x^* we leave P (or formally, $(\{x^*\} + \mathbb{R}_+^n) \cap P = \{x^*\}$).

Theorem 13 (Scarf's Lemma [13]). *Let $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$ and let $<_i$ be a total order on supp(a_i) $(i \in [m])$, where a_i is the i-th row of A. Then P has a maximal vertex that dominates every column.*

It was shown by Kintali [7] that the following computational version of Scarf's Lemma is PPAD-complete.

SCARF

Input: a matrix $A \in \mathbb{Q}_+^{m\times n}$ and a vector $b \in \mathbb{Q}_+^m$; a total order $<_i$ on $\mathrm{supp}(a_i)$ for every $i \in [m]$.

Output: a maximal vertex of $P = \{x \in \mathbb{R}^n : Ax \le b,\ x \ge 0\}$ that dominates every column.

Proposition 14. Extreme direction Sperner with $2n$ facets *is a special case of* Scarf.

Proof. Let us consider an instance (A, c) of extreme direction Sperner with $2n$ facets, where c_i is the colour of the facet determined by the i-th row of A. We can assume without loss of generality that all vertices of $P = \{x : Ax \le \mathbf{1}, x \le \mathbf{1}\}$ are strictly positive, since we can get an equivalent problem of the same form by scaling from center $\mathbf{1}$. We can transform this into an instance $(A', b', <)$ of Scarf by setting $A' = \binom{A}{I}$, $b' = \mathbf{1}$, and defining $<_i$ to be an arbitrary total order on $\mathrm{supp}(a_i)$ whose smallest element is c_i (the order of the other elements does not matter). Let $P' = \{x \in \mathbb{R}^n : A'x \le b',\ x \ge 0\}$.

If v is a panchromatic vertex of P, then for every $j \in [n]$, either $v_j = 1$, or there is an index i such that $a_i v = 1$, $j \in \mathrm{supp}(a_i)$ and $c_i = j$. Thus v dominates every column according to $<$. It is also a maximal vertex of P' since it is a vertex of P. It is easy to check (using that every vertex of P is strictly positive) that the reverse also holds: any dominating maximal vertex of P' is a panchromatic vertex of P. □

Note that we obtain a special case of Scarf where only the smallest elements of the total orders $<_i$ play a role. It is natural to ask whether the problem remains PPAD-complete if we restrict it to 0-1 matrices. We pose this as an open question.

Question. Is 0-1 extreme direction Sperner PPAD-complete in the special case when A is an $n \times n$ matrix and every colour appears exactly twice?

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References

[1] Daskalakis, C., Goldberg, P., Papadimitriou, C.: The complexity of computing a Nash equilibrium. In STOC 06: Proceedings of the 38th Annual ACM Symposium on Theory of Computing (2006), pp. 71–78.

[2] Chen, X., Deng, X.: On the complexity of 2d discrete fixed point problem. Automata, Languages and Programming, LNCS, Vol. 4051 (2006), pp. 489-500.

[3] Chen, X., Deng, X.: Settling the complexity of computing two-player Nash equilibrium. Proceedings of 47th FOCS (2006).

[4] Chen, X., Deng, X., Teng, S.-H.: Settling the complexity of computing two-player Nash equilibria. Journal of the ACM, Volume 56 Issue 3, May 2009.

[5] Gallai, T.: Maximum-minimum Sätze über Graphen. Acta Math. Acad. Sci. Hungaricae 9 (1958), pp. 395–434.

[6] Kintali, S., Poplawski, L.J., Rajaraman, R., Sundaram, R., Teng, S.: Reducibility Among Fractional Stability Problems. Proceedings of 50th FOCS (2009).

[7] Kintali, S.: SCARF is PPAD-complete. arXiv preprint 0812.1601.

[8] Király, T., Pap, J.: Kernels, stable matchings, and Scarf's Lemma. Combinatorial Optimization and Discrete Algorithms, RIMS Kôkyûroku Bessatsu B23 (2010), pp. 131–145.

[9] Király, T., Pap, J.: A note on kernels and Sperner's Lemma. Discrete Applied Mathematics, Vol. 157, No. 15 (2009), pp. 3327–3331.

[10] Megiddo, N., Chandrasekaran, R.: On the ε-perturbation method for avoiding degeneracy. Operations Research Letters, Vol. 8, No. 6 (1989), pp. 305–308.

[11] Pap, J.: Integrality, complexity and colourings in polyhedral combinatorics. PhD thesis, Eötvös University, Budapest.

[12] Papadimitriou, C.H.: On the Complexity of the Parity Argument and Other Inefficient Proofs of Existence. J. Comput. Syst. Sci., Vol. 48 No. 3 (1994), pp. 498–532.

[13] Scarf, H.E.: The core of an n person game. Econometrica 35 (1967), pp. 50–69.