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Kernels, stable matchings, and Scarf's Lemma

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Abstract

Scarf's Lemma originally appeared as a tool to prove the non-emptiness of the core of certain NTU games. More recently, however, several applications have been found in the area of graph theory and discrete mathematics. In this paper we present and extend some of these applications. In particular, we prove results on the existence of kernels in orientations of h-perfect graphs. We describe a new direct link between Scarf's Lemma and Sperner's Lemma giving a new proof to the former.

1 Introduction

In one of his fundamental papers in game theory [16], Scarf proved that a balanced n-person game with non-transferable utilities (NTU) always has a non-empty core. The proof is based on a theorem on the existence of a dominating vertex in certain polyhedra, which became known as "Scarf's Lemma". The proof he gave is based on a finite (not necessarily polynomial) algorithm.

The interest in the lemma has been renewed in combinatorics when Aharoni and Holzman used it to give a short and elegant proof of the kernel-solvability of perfect graphs. This problem, previously known as the Berge-Duchet conjecture, was first solved by Boros and Gurvich [5] using fairly complicated game-theoretical arguments. In contrast, the proof of Aharoni and Holzman is surprisingly simple and clear.

The relation of Scarf's Lemma and Sperner's Lemma has already been mentioned in Scarf's original paper [16], and later it has been studied by other authors (see for example [14]). In Section 2 we present a new proof of Scarf's Lemma based on a polyhedral version of Sperner's Lemma, and show a strong link between the two theorems: essentially, Scarf's lemma for a polyhedron P corresponds to Sperner's Lemma for the polyhedron $P - \mathbb{R}^n_+$.

In Section 3 we briefly review some applications of the lemma in game theory and in graph theory, including the kernel-solvability of perfect graphs. This application is extended to h-perfect graphs in Section 4.

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There are several questions remaining related to Scarf's Lemma and kernels. The lemma states the existence of a certain dominating vertex; it would be useful to know classes of polyhedra where only one dominating vertex exists, since this could lead to characterizations of some classes of kernel-less graphs. Questions and conjectures about this topic are presented in Section 5.

2 Scarf's Lemma and Sperner's Lemma

In Scarf' Lemma we consider a bounded polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$ where A is an $m \times n$ non-negative matrix (with non-zero columns) and $b \in \mathbb{R}^m$ is a positive vector. In addition, for every row $i \in \{1, 2, \dots m\}$ of A, a total order $<_i$ of the columns (or a subset of them) is given. We denote the domain of $<_i$ by $\mathrm{Dom}(<_i)$. If $j \in \mathrm{Dom}(<_i)$ and $J \subseteq \mathrm{Dom}(<_i)$, we use the notation $j \leq_i J$ as an abbreviation for "$j \leq_i j'$ for every $j' \in J$". For a non-negative vector $x \in \mathbb{R}^n$, $\mathrm{supp}(x)$ denotes $\{j \in \{1, \dots n\} : x_j > 0\}$.

The central notion in Scarf's lemma is that of a dominating vertex.

Definition 2.1. A vertex x^* of P *dominates* column j if there is a row i where $a_i x^* = b_i$ and $j \leq_i \mathrm{supp}(x^*) \cap \mathrm{Dom}(<_i)$ (this implies that $j \in \mathrm{Dom}(<_i)$).

Theorem 2.2 (Scarf's Lemma)**.** *Let P be as above and let $<_i$ be a total order on $\{1, \dots, n\}$ $(i = 1, \dots, m)$. Then P has a nonzero vertex that dominates every column.*

We state another version, which will be more convenient to prove and also to apply. A vertex x^* of P is *maximal* if by increasing any coordinate of x^* we leave P (or formally, $(\{x^*\} + \mathbb{R}^n_+) \cap P = \{x^*\}$, where $\mathbb{R}^n_+$ is the set of all non-negative vectors, and for two polyhedra Q and Q' we use the notation $Q + Q' = \{x + y :\ x \in Q,\ y \in Q'\}$).

Theorem 2.3 (Scarf's Lemma, alternate version)**.** *Let P be as above and let $<_i$ be a total order on $\mathrm{supp}(a_i)$ $(i = 1, \dots, m)$, where a_i is the i-th row of A. Then P has a maximal vertex that dominates every column.*

Note that in Theorem 2.2 we cannot guarantee the maximality of the dominating vertex. Consider the following two dimensional example:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ b = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ 1 <_1 2,\ 1 <_2 2.$$

Here the only vertex that dominates every column is $(0, 1)$, which is not maximal since $(1, 1)$ is also a vertex.

On the other hand, Theorem 2.2 follows fairly easily from Theorem 2.3 by changing the 0 coefficients in the matrix A to some small positive values such that the facet-defining inequalities remain the same and the vertex sets of the facets remain also the same except for possible fission.

Next we show that Theorem 2.3 follows from the following polyhedral version of Sperner's Lemma.

Definition 2.4. For a colouring of the vertices of a polytope Q, a facet of Q is *multi-coloured* if it contains vertices of every colour. For a colouring of the facets of Q, a vertex of Q is *multi-coloured* if it lies on facets of every colour.

Theorem 2.5. *Let Q be an n-dimensional polytope, with a simplex facet F_0. Suppose we have a colouring of the vertices of Q with n colours such that F_0 is multi-coloured. Then there is another multi-coloured facet.*

Proof. Let us divide the non-simplex facets of Q into simplices. We need to show that there is a multi-coloured simplex. Let C be the set of all colours and let red be one of them.

Define a graph whose nodes are the simplices in the division and there is an edge between two simplices if and only if they share an ($n-2$ dimensional) facet whose vertices use each colour in $C \setminus \{\text{red}\}$ exactly once. It is easy to see that the multi-coloured simplices are of degree one in this graph, the simplices whose vertices use all colours in $C \setminus \{\text{red}\}$, and one of them twice are of degree two and the other simplices are of degree zero, hence the graph is the disjoint union of paths. The assumption implies that F_0 is a node of degree one, so there has to be another node of degree one which gives a multi-coloured simplex. □

By polarity, the following theorem is also true.

Theorem 2.6. *Let Q be an n-dimensional polytope, with a simplicial vertex v_0. Suppose we have a colouring of the facets of Q with n colours such that v_0 is multi-coloured. Then there is another multi-coloured vertex.*

The above results can be generalized to unbounded pointed polyhedra, which will be useful for our proof of Scarf's lemma. For this, we extend the notion of vertices.

Definition 2.7. For a pointed polyhedron Q, its *ends* are the vertices of Q and the extreme rays of Q (an *extreme ray* of a polyhedron is an extreme ray of its recession cone).

We extend also the incidences between facets and vertices to ends in the natural way. In addition, if a polyhedron has n linearly independent extreme rays then we consider the extreme rays as being on a "facet in infinity".

Definition 2.8. We call two polyhedra combinatorially equivalent if there is a bijection between their facets and their ends which preserves the incidences. We call two polyhedra combinatorially polar if there is a bijection between the facets of one and the ends of the other and vice versa which reverses the inclusion relation.

We claim that if Q is a pointed full-dimensional polyhedron then there exists a polytope which is combinatorially equivalent to it. This is because if we move Q so that the origin is in its interior and then take its polar, it will be a polytope which is combinatorially polar to Q. If we do the same a second time, we get a polytope which is combinatorially equivalent to Q. Now we can state a version of Theorems 2.5 and 2.6 to unbounded polyhedra.

Corollary 2.9. *Let Q be an n-dimensional pointed polyhedron with n linearly independent extreme rays (and no other).*

- *If we colour the vertices and the extreme rays by n colours such that the extreme rays receive different colours, then there is a multi-coloured facet.*
- *If we colour the facets of the polyhedron by n colours such that facets containing the i-th extreme ray do not get colour i, then there is a multi-coloured vertex.*

Proof. Let us take a polytope Q' which is combinatorially equivalent to Q, and let F be the face of Q' which corresponds to the infinite face of Q. So F is a multi-coloured simplex facet. For the first part, we can apply Theorem 2.5.

For the second part, we add a simplex to Q' on face F, and colour the new faces so that the face opposite (in the simplex) to the vertex corresponding to the i-th extreme ray gets colour i. Applying Theorem 2.6 we get that there is another multi-coloured vertex of Q' (besides the new vertex of the simplex) and from the assumption it follows that this corresponds to a vertex of Q. □

We now show that Scarf's Lemma (Theorem 2.3) follows from Corollary 2.9.

Proof. Let $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$ be the polyhedron as in Scarf's Lemma, and consider the polyhedron $Q = P - \mathbb{R}^n_+ = \{x - y :\ x \in P,\ y \in \mathbb{R}^n_+\}$. Because P is bounded, the recession cone of Q is $-\mathbb{R}^n_+$, so Q has n extreme rays: $-e_j$ $(j = 1, \ldots, n)$. Since A and b are non-negative, the vertices of Q are the maximal vertices of P, and the inequalities which define Q are of the form $a_i^J x \leq b_i$, where $a_i^J := a_i \chi_J$ for an index set J, and we can assume that $J = \mathrm{supp}(a_i^J)$.

Let us colour a face which is defined by inequality $a_i^J x \leq b_i$ with the index $j \in J$ which is the smallest in the ordering $<_i$. If a facet contains the extreme ray $-e_l$ for some l, then the l-th component of its defining inequality is zero, so the colour of the facet is different from l. So we can apply the second part of Corollary 2.9, and get that there is a vertex x^* of Q (thus a maximal vertex of P) which is multi-coloured. We have to show that x^* satisfies the criteria of Scarf's Lemma. If j is an arbitrary index then there is a j-coloured facet $a_i^J x = b_i$ containing x^*, which means that $j \leq_i \mathrm{supp}(a_i^J) = J$. Since x^* is also a vertex of P, it is non-negative, so $a_i x^* \geq a_i^J x^* = b_i$, but we know that $a_i x^* \leq b_i$, thus the facet $a_i x = b_i$ of P contains x^*. On the other hand this implies also that $\mathrm{supp}(x^*) \cap \mathrm{supp}(a_i) \subset J$ which with $j \leq_i J$ means that $j \leq_i \mathrm{supp}(x^*) \cap \mathrm{supp}(a_i) = \mathrm{supp}(x^*) \cap \mathrm{Dom}(<_i)$. Thus x^* dominates column j. □

3 Applications of Scarf's Lemma

3.1 Fractional core of NTU games

The role of the lemma in game theory can be described in several different ways. Here we use a combinatorial approach that does not require the definition of all the basic terms of game theory.

A possible definition of a *finitely generated non-transferable utility (NTU) game* is as follows. There are m *players*, and a finite multiset of *basic coalitions* $S_j \subseteq \{1,\dots,m\}$ $(j = 1,\dots,n)$. We may interpret a coalition as a possible action performed by a set of players; thus several different coalitions may be formed by the same set of players. Each player i has a total ordering $<_i$ of the basic coalitions that he participates in; $S_{j_1} <_i S_{j_2}$ means that the player i prefers coalition S_{j_2} to coalition S_{j_1}. We assume that every player is in at least one coalition.

A set $\mathcal{S}$ of basic coalitions is said to be in the *core* of the game if they are disjoint and for each basic coalition S' not in $\mathcal{S}$ there is a player $i \in S'$ and a basic coalition $S \in \mathcal{S}$ such that $S' <_i S$. In other words, an element of the core is a subpartition formed of basic coalitions, such that every basic coalition S' not in the subpartition has a player who is in a member of the subpartition and prefers this member to S'.

A related concept is the *fractional core* of the game: a vector $x : \{1,\dots,n\} \to \mathbb{R}_+$ is in the fractional core if for each player i,

$$\sum_{j:i\in S_j} x(j) \le 1,$$

and for each $j \in \{1,\dots,n\}$ there is a player i in S_j such that

$$\sum_{j:i\in S_j} x(j) = 1$$

and $S_j \leqslant_i S_{j'}$ whenever $i \in S_{j'}$ and $x(j') > 0$.

To motivate this definition, we can imagine that the action performed by each basic coalition can have an intensity (between 0 and 1), and the condition is that the sum of the intensities of the actions that a given player participates in is at most 1. Such a vector of intensities is in the fractional core if there is no basic coalition where every member wants to increase its intensity. It is an easy observation that integer-valued elements in the fractional core are exactly the elements of the core.

Let us call a vector $x : \{1,\dots,n\} \to \mathbb{R}_+$ *admissible* if

$$\sum_{j:i\in S_j} x(j) \le 1$$

for every player i. A corollary of Scarf's lemma is the following.

Corollary 3.1. *The fractional core of a finitely generated NTU-game is always non-empty. If the polyhedron of admissible vectors is integral, then the core is also non-empty.*

Proof. Consider a finitely generated NTU-game with coalitions $S_j \subseteq \{1,\dots,m\}$ $(j = 1,\dots,n)$, and total orders $<_i$ $(i = 1,\dots,m)$. Let A be the incidence matrix of the basic coalitions, with m rows and n columns, and let b be the all-1 vector. If we apply Theorem 2.3 with the total orders $<_i$ on the supports of the rows, we obtain that the polyhedron $\{x : Ax \le b, x \ge 0\}$ has a (maximal) vertex x^* that dominates every column. The property that x^* is in the polyhedron means that it is an admissible vector for the game, and the property that it dominates every column is equivalent to saying that it is in the fractional core.

If the polyhedron $\{x : Ax \leq b, x \geq 0\}$ is integral, then we obtain an integral element in the fractional core, which is in the core. $\square$

It is known by a theorem of Lovász [13] that the polyhedron of admissible vectors is integral if and only if the hypergraph whose edges are the basic coalitions is normal. This gives the following corollary, which was first proved by Boros, Gurvich and Vasin [7].

Corollary 3.2. *If the hypergraph defined by the basic coalitions is normal, then the core of the game is non-empty.*

An equivalent formulation of these results is in terms of hypergraphic preference systems and stable fractional matchings (for details, see [2]). We describe only the graphic case which corresponds to stable matchings and stable half-matchings.

3.2 Stable half-matchings

The traditional interpretation of stable matchings in a graph is the so-called stable roommates problem, where we want to assign pairs of students to college rooms so that there are no two students who prefer each other to their assigned roommates. Formally, let $G = (V, E)$ be an undirected graph, possibly with parallel edges, but no loops. For every $v \in V$ we are given a total order $<_v$ of the edges incident to v, where $uv <_v wv$ means that v prefers w to u. The set of these total orders is denoted by $\mathcal{O}$, and the pair $(G, \mathcal{O})$ is called a *graphic preference system.* For two edges e and f with a common endnode v, the notation $e \leqslant_v f$ is used if $e <_v f$ or $e = f$.

Definition 3.3. A *stable matching* of the preference system $(G, \mathcal{O})$ is a matching M of G with the property that every edge $e \in E$ has an endnode v that is covered by a matching edge $vw \in M$ for which $e \leqslant_v vw$.

A *stable half-matching* is a vector $x : E \to \{0, 1/2, 1\}$, for which

- $\sum_{v:uv \in E} x(uv) \leq 1$ for every $u \in V$,
- every edge $e \in E$ has an endnode v where $\sum_{f \geqslant_v e} x(f) = 1$.

In their celebrated paper [11], Gale and Shapley proved that every bipartite preference system has a stable matching, and they provided an efficient algorithm. It is easy to see that not every graphic preference system has a stable matching. However, Irving [12] showed a polynomial algorithm that decides if there is a stable matching, and, relying on this, Tan [18] observed the following.

Theorem 3.4 ([18])**.** *Every preference system has a stable half-matching.*

Proof. We prove the theorem using Scarf's Lemma (this is not Tan's original proof). Let A be the node-edge incidence matrix of the graphs, with the rows indexed by the nodes. Let b be the all-ones vector. We can define a total order $<_i$ on $\mathrm{supp}(a_i)$ by considering the node v corresponding to row a_i and using the total order $<_v$.

By Theorem 2.3 the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$ has a maximal vertex x^* that dominates every column. By a result of Balinski [3], the polyhedron P is half-integral, so x^* is half-integral. It is easy to see that the property that x^* dominates every column means that it is a stable half-matching. $\square$

Stable half-matchings that are maximal vertices of P have an interesting property that seems to be peculiar to this problem (it does not hold for other applications of Scarf's Lemma): all of them are non-integer on the same set of edges. More precisely, the graph has a given set of disjoint odd cycles so that every stable half-matching that is a vertex of P has value $\frac{1}{2}$ on exactly the edges of these cycles. This immediately gives the following corollary.

Corollary 3.5 ([18]). *Let x^* be a stable half-matching that is a vertex of P. Then x^* is integral if and only if the preference system has a stable matching.*

3.3 Kernel solvability of perfect graphs

Let $D = (V, A)$ be a directed graph. The *out-neighbourhood* $O_D(v)$ of a node $v \in V$ is the set of nodes consisting of v and the nodes $w \in V$ for which $vw \in A$. A subset X of nodes is said to *dominate* a node $v \in V$ if $X \cap O_D(v) \neq \emptyset$. X is called *dominating* if it dominates every node. A *kernel* of D is a dominating stable set of nodes. Kernels have several applications in combinatorics and game theory, and there has been extensive work on the characterization of digraphs that have kernels. See [6] for a survey on the topic.

One approach is to identify undirected graphs for which every "nice" orientation has a kernel. Let $G = (V, E)$ be an undirected graph. A *superorientation* of G is a directed graph obtained by replacing each edge uv of G by an arc uv or an arc vu or both. A *proper directed cycle* in a superorientation is a directed cycle consisting of arcs that are not present reversed in the digraph.

A superorienation is *clique-acyclic* if no oriented clique contains a proper directed cycle. A graph G is *kernel solvable* if every clique-acyclic superorientation of G has a kernel.

Boros and Gurvich [5] proved the following conjecture of Berge and Duchet.

Theorem 3.6 ([5]). *Every perfect graph is kernel solvable.*

Let $(G, \mathcal{O})$ be a graphic preference system, and let D be the line graph of G oriented according to the preferences at the nodes. Then stable matchings of $(G, \mathcal{O})$ correspond to kernels of D. This means that stable matching problems can be formulated as kernel problems in line graphs. Since line graphs of bipartite graphs are perfect, the Gale-Shapley theorem follows from Theorem 3.6.

In the following we describe the proof of Aharoni and Holzman [1] for Theorem 3.6 which is a simple and elegant use of Scarf's Lemma.

Proof. Let $G = (V, E)$ be a perfect graph and let D be a clique-acyclic superorientation of G. Let $K_1, \dots K_m$ denote the maximal cliques in G. Let $A \in \{0,1\}^{m \times n}$ be the incidence matrix of the maximal cliques, i.e. the i-th row, a_i is the characteristic vector of K_i $(1 \le i \le m)$. Finally let $b \in \mathbb{R}_+^m$ be the all-ones vector. Since G is perfect, the polyhedron $P = \{x \in \mathbb{R}^n : \ x \ge 0, \ Ax \le b\}$ is the convex hull of the characteristic vectors of the stable sets of G.

Because D is clique-acyclic, every maximal clique K_i has an ordering of its nodes with the property that there is no edge in K_i which is oriented only backwards. Let $<_i$ be this ordering.

Applying Theorem 2.3 for this instance we get that there is a maximal vertex x^* of P with the property that for each node $v \in V$ there is a maximal clique $K_{i(v)}$ containing v such that $a_{i(v)}x^* = 1$ and $x^*_{v'} = 0$ for every $v' <_{i(v)} v$. By the properties of P, x^* is the characteristic vector of a maximal stable set S.

We now prove that S is a kernel. Let v be a node of V. The above implies that $w \ge_{i(v)} v$ holds for every $w \in K_{i(v)} \cap S$. Because of $a_{i(v)}x^* = 1$, there is a node w in $K_{i(v)} \cap S$, so $w \ge_{i(v)} v$ implies $w \in O_D(v)$ by the definition of the ordering $<_{i(v)}$. Thus v is dominated by S. □

Note that it follows easily from the Strong Perfect Graph Theorem [9] that non-perfect graphs are not kernel solvable. One needs the observations that *a)* odd holes and odd antiholes are not kernel solvable, and *b)* induced subgraphs of kernel solvable graphs are kernel solvable. However, no proof is known that does not rely on the SPGT.

4 Kernels in h-perfect graphs

4.1 h-perfect graphs

Sbihi and Uhri [15] introduced the class of *h-perfect graphs* as the graphs for which the stable set polytope is described by the following set of inequalities:

$$x_v \ge 0 \qquad \text{for every } v \in V, \tag{1}$$
$$x(C) \le 1 \qquad \text{for every maximal clique } C, \tag{2}$$
$$x(Z) \le \frac{|Z| - 1}{2} \qquad \text{for every odd hole } Z. \tag{3}$$

In addition to perfect graphs, it is known that the class of h-perfect graphs includes

- all graphs containing no odd-K_4-subdivision (see [10]),
- all near-bipartite graphs containing no odd wheel and no prime antiweb except for cliques and odd holes (this is implicitly in [17]),
- line graphs of graphs that contain no odd subdivision of $C_5 + e$ (see [8]).

It follows from the Strong Perfect Graph Theorem that the property in Theorem 3.6 does not hold for non-perfect graphs. To extend the theorem to h-perfect graphs, let us call a superorientation of a graph *odd-hole-acyclic* if no oriented odd hole is a proper directed cycle. Obviously a superorientation of a perfect graph is always odd-hole-acyclic. Our result is as follows.

Theorem 4.1. *If G is an h-perfect graph then every clique-acyclic and odd-hole-acyclic superorientation of G has a kernel.*

Our proof is a slight modification of the proof of Aharoni and Holzman for Theorem 3.6 [1].

Proof. Let G be an h-perfect graph and D a clique-acyclic and odd-hole-acyclic superorientation of G. Let c and o denote the number of maximal cliques and odd holes in D, respectively. Let $C_1, \dots C_c$ denote the maximal cliques in D and $C_{c+1}, \dots C_{c+o}$ the odd holes in D. Let A be the matrix of size $(c+o) \times n$ whose i-th row, a_i is the characteristic vector of C_i ($1 \le i \le c+o$). Finally let $b \in \mathbb{R}_+^{(c+o)}$ be the vector whose i-th component is 1 if $i \le c$ and $\frac{|C_i|-1}{2}$ if $i > c$. Since G is h-perfect, the polyhedron $P = \{x \in \mathbb{R}^V : \ x \ge 0, \ Ax \le b\}$ is the convex hull of the stable sets of G.

Because D is clique-acyclic and odd-hole-acyclic, if C_i is a maximal clique or an odd hole, its nodes have an order with the property that there is no edge in C_i which is oriented only backwards. Let $<_i$ be this ordering of C_i.

Applying Theorem 2.3 for this instance we get that there is a vertex x^* of P with the property that for each node $v \in V$ there is a maximal clique or odd hole $C_{i(v)}$ containing v such that $a_{i(v)}x^* = b_{i(v)}$ and $x^*_{v'} = 0$ for every $v' <_{i(v)} v$.

The vector x^* is the characteristic vector of a stable set S because it is a vertex of P. We want to show that it is the characteristic vector of a kernel.

Let v be a node not in S. Scarf's lemma implies that if $w \in C_{i(v)} \cap S$, then $w \ge_{i(v)} v$ holds. If $C_{i(v)}$ is a clique, then because of $a_{i(v)}x^* = b_{i(v)} = 1$, there is a node w in $C_{i(v)} \cap S$. Hence $w \ge_{i(v)} v$ implies that $w \in O_D(v)$.

If $C_{i(v)}$ is an odd hole, then $a_{i(v)}x^* = b_{i(v)} = \frac{|C_{i(v)}|-1}{2}$ implies that S contains every second node in $C_{i(v)}$, except two consecutive nodes not in S. This means that v has at least one neighbour w on the circuit which is in S. Like above, $w \ge_{i(v)} v$, so w must be in $O_D(v)$. This concludes the proof of the theorem. □

A stronger version of the theorem can also be proved with the same method. For an undirected graph $G = (V, E)$, let $\mathrm{STAB}(G)$ denote the convex hull of the characteristic vectors of stable sets. We say that a digraph is *acyclic* in a subset of nodes if there is no proper directed cycle in the subset.

Theorem 4.2. *If $\{x \in \mathbb{R}_+^V : Ax \le b\} = \mathrm{STAB}(G)$ for an undirected graph $G = (V, E)$ and D is a superorientation of G which is acyclic in* $\mathrm{supp}(a_i)$ *for every row a_i of A, then there is a kernel in D.*

Proof. We can assume that every inequality is facet-defining in the system $\{x \in \mathbb{R}_+^V : Ax \le b\}$. Then A is non-negative and b is positive.

For a row a_i of A, let $<_i$ be a total order of the elements of $\mathrm{supp}(a_i)$ given by a topological order of the one-way edges. Theorem 2.3 implies that there exists a vertex x^* of STAB(G) such that for every $v \in V$ there is a row $a_{i(v)}$ of A for which

(i) $a_{i(v)}x^* = b_i$, and

(ii) if $w \in \mathrm{supp}(x^*) \cap \mathrm{supp}(a_{i(v)})$ then $w \geqslant_{i(v)} v$.

Since the system describes STAB(G), x^* is the characteristic vector of some stable set S. We want to show that S dominates every node v. Let $v \in V \setminus S$; then v is in $\mathrm{supp}(a_{i(v)})$. Moreover, (i) implies that there is a node $w \in S \cap \mathrm{supp}(a_{i(v)}) \cap N_G(v)$ (where $N_G(v)$ denotes the neighbourhood of v in G with v) because otherwise $(S \cap \mathrm{supp}(a_{i(v)})) \cup \{v\}$ would be a stable set which violates the inequality of $a_{i(v)}$. From (ii) w is an out-neighbour of v, so v is dominated by S. □

It is a well-known result in the theory of stable matchings that a clique-acyclic and odd-hole-acyclic *orientation* of a line graph always has a kernel (it follows for example from the stable roommates algorithm of Irwing [12]). However, this is not true for *superorientations*, as the superorientation of the line graph of $C_5 + e$ on Figure 1 shows.

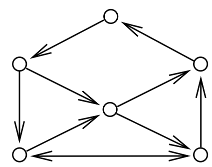

Figure 1: A kernel-less superorientation of the line graph of $C_5 + e$

5 Counterexamples and open questions

5.1 A conjecture on the characterization of h-perfect graphs

We have mentioned that the reverse direction of Theorem 3.6 is also true, due to the Strong Perfect Graph Theorem. The same does not hold for Theorem 4.1, and a counterexample is given here. The graph on Figure 2 is not h-perfect (this follows from the results of Barahona and Mahjoub [4]), but it can be seen by case analysis that every clique- and odd-hole-acyclic superorientation of it has a kernel.

Nevertheless, one may hope for a stronger theorem where the reverse direction also holds. We give here a less elegant but stronger theorem for which we conjecture that this is the case.

Let G be an h-perfect graph, and let D be a clique-acyclic superorientation of G. Some odd holes of G may become proper directed cycles; let us denote these by $Z_1, \dots, Z_k$. Let us select nodes $v_1, \dots, v_k$ such that $v_i \in Z_i$ for $i = 1, \dots, k$ (the selected nodes need not be distinct). We call this a *superorientation with special nodes*. An *almost-kernel* for a superorientation with special nodes is a stable set S with the following property:

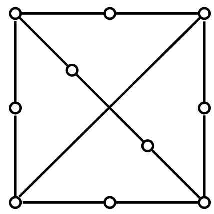

Figure 2: A non-h-perfect graph whose clique- and odd-hole-acyclic superorientations all have kernels

If a node v is not dominated by S, then $v = v_i$ for some i and $|Z_i \cap S| = (|Z_i| - 1)/2$.

Theorem 5.1. *If G is an h-perfect graph then every clique-acyclic superorientation with special nodes has an almost-kernel.*

Proof. We use Scarf's Lemma in a similar way as in the proof of Theorem 4.1. The orderings $<_i$ associated to the lines of the matrix can be defined the same way as there, except for the odd holes which are proper directed cycles. For these, we can define the ordering so that the special node is the smallest node of the ordering, and the only edge oriented backwards is the one entering the special node.

Using Scarf's lemma as in the proof of Theorem 4.1, we get that the only possible case when a node v is not dominated by the stable set S corresponding to x^* is when $C_{i(v)}$ is an odd hole which is a proper directed cycle, v is its special node, and $|S \cap C_{i(v)}| = (|C_{i(v)}| - 1)/2$. This implies that S is an almost-kernel. □

Note that this theorem is stronger than Theorem 4.1 since every almost-kernel in a clique-acyclic and odd-hole-acyclic orientation is a kernel. We conjecture that here the converse also holds:

Conjecture 5.2. *A graph G is h-perfect if and only if every clique-acyclic superorientation with special nodes has an almost-kernel.*

5.2 Possible converse of Scarf's Lemma

It would be tempting to formulate a more general conjecture, which is a kind of converse to Scarf's Lemma.

Question 5.3. Let A be a non-negative $m \times n$ matrix and let $b \in \mathbb{R}^m$ be a positive vector so that the polyhedron $P = \{x : \ Ax \leq b, \ x \geq 0\}$ is bounded. Let x^* be a maximal vertex of P. Is it true that for each row a_i of A we can give a total order on $\mathrm{supp}(a_i)$, so that x^* is the only maximal vertex of P that dominates every column?

We now show that the answer to this question is 'No'. Let us first formulate a similar question about colourings of vertices of polytopes.

Question 5.4. Let P be a d-dimensional polytope, and let x^1 and x^2 be two distinct vertices of P, where x_1 is simplicial. Is it true that the facets of P can be coloured by d colours so that x^1 and x^2 are precisely the vertices that are incident to facets of all colours?

This is true in 3 dimensions: the skeleton of P contains 3 vertex-disjoint paths between x^1 and x^2; these paths partition the set of facets into 3 classes, and the colouring given by these 3 colour classes satisfies the conditions. However, it turns out to be false in 4 dimensions, as the following polyhedron shows:

Facets:
$$\begin{aligned} -x_1 - x_3 + x_4 &\leq 1 \\ x_1 + x_2 + x_4 &\leq 1 \\ x_2 - x_3 + x_4 &\leq 1 \\ -x_1 - x_2 - x_3 + x_4 &\leq 1 \\ x_1 - x_2 - x_3 - x_4 &\leq 1 \\ -x_1 - x_3 - x_4 &\leq 1 \\ -x_1 - x_4 &\leq 1 \\ -x_1 - x_2 + x_3 - x_4 &\leq 1 \end{aligned}$$

Vertices:

(0,0,0,1), (0,0,0,-1),
(-2,2,2,1), (2,2,2,-3),
(1,2,0,-2), (2/3,4/3, -2/3,-1),
(0,2,0,-1), (-1,1,0,0),
(2,-3,2,2), (2/3,-1/3,-2/3,2/3),
(-1,0,2,2), (-1/3,-2/3,-1/3,-1/3),
(0,0,-1,0), (-1,0,0,0).

The first four facets of this polyhedron are incident to the vertex $x^1 = (0, 0, 0, 1)$, while the last four facets are incident to the vertex $x^2 = (0, 0, 0, -1)$. It can be shown by case analysis that no matter how we colour the first four facets by four different colours and the last four facets by the same four colours, there will be another vertex incident to facets of all four colours.

Now we show that this counterexample can be transformed into a counterexample for Question 5.3 using the technique in Section 2. Let P be the polytope defined above. First, we cut off the vertex x^1 with a hyperplane to obtain a simplex facet F_0. Then we take the polar of this polytope and affinely transform it into a polytope P' so that the image of F_0 is the origin and the facets containing it are $\{x \in P' : x_i = 0\}$ $(i = 1, \dots, 4)$. If we now take the polar from the origin, we obtain a polyhedron whose extreme rays are $-e_i$ $(i = 1, \dots, 4)$; we can translate this to a polyhedron P'' whose vertices are all in $\mathbb{R}^4_+$. Let x^* be the image of x^2; we know that this is a maximal vertex of $P''' := P'' \cap \mathbb{R}^4_+$. We claim that P''' and x^* give a counterexample for Question 5.3. Suppose we have total orders $<_i$ on the supports of the rows such that x^* dominates every column. These can be transformed into a colouring on the facets of $P'' = P''' - \mathbb{R}^4_+$ as in the proof of Theorem 2.3, such that x^* is multi-coloured. Furthermore, such a colouring of the facets of P'' defines a colouring of the facets of P where x^1 and x^2 are multi-coloured. Since P is a counterexample for Question 5.4, there is a third multi-coloured vertex x^3. The polyhedron P''' has a corresponding maximal vertex, and this vertex dominates every column by the construction.

It may be interesting to know special classes of polyhedra where the answer to Question 5.3 is affirmative. We have no counterexamples for the following conjecture.

Conjecture 5.5. *Let A be an $m \times n$ matrix with $0-1$ coefficients and let $b \in \mathbb{R}^m$ be a positive vector so that the polyhedron $P = \{x : Ax \leq b, x \geq 0\}$ is bounded. Suppose that P has a non-integer maximal vertex. Then for each row a_i of A we can give a total order on supp(a_i) so that for every $0-1$ vertex x' of P there is a column that it does not dominate.*

To see that Conjecture 5.2 follows from Conjecture 5.5, consider a non-h-perfect graph G. The polyhedron P defined by inequalities (1) $-$ (3) has a non-integral vertex, hence it has a non-integral maximal vertex x^*. Let $<_i$ $(i = 1, \ldots, m)$ denote the total orders given by Conjecture 5.5. These total orders define a clique-acyclic superorientation with special vertices:

- For each maximal clique, we orient the edges of the clique according to the total ordering of the clique. (An edge may appear in two cliques and its endpoints may be in different order in the two total orders; in this case, we orient the edge in both directions.) This defines the superorientation.
- If an odd hole is a proper directed cycle in this superorientation, we define its special node to be the smallest node in its total order.

Let S be an arbitrary stable set of G. The characteristic vector of S is a $0-1$ vertex of the polyhedron P. By the properties of the partial orders, there is a node $v \in V$ with the following properties:

- If there is a maximal clique K_i with $|K_i \cap S| = 1$ and $v \in K_i$, then there is a node $u \in K_i \cap S$ with $u <_i v$.
- If there is an odd hole Z_i with $|Z_i \cap S| = (|Z_i| - 1)/2$ and $v \in Z_i$, then there is a node $u \in Z_i \cap S$ with $u <_i v$.

The first property means that $v \notin S$ and the out-neighbours of v in the superorientation are not in S, so v is not dominated by S. The second property implies that if v is the special node of an odd hole Z (i.e. it is the smallest node in the total order) then $|Z \cap S| < (|Z| - 1)/2$. Therefore the existence of v proves that S is not an almost-kernel.

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